THE ERNST MODEL FOR CABLE BENDING ANALYSIS
An early stick-slip model

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Presentation and translation of a Ph.D. thesis theoretical part by H. Ernst (1933). This work is probably the first presentation of the stick-slip model principles for cable bending analysis, in which inner friction is considered through Coulomb’s law. It laid the ground for later developments on that subject, such as those of Lehanneur (1949) and Papailiou (1995).
The Ernst model for cable bending analysis

By Alain Cardou, Ph.D.

INTRODUCTION

This report is a follow-up to two previous publications by the same author:

*Stick-slip mechanical models for overhead electrical conductors in bending (with Matlab applications)*, 2013, 97 pages. ISBN 978-2-9812337-2-1

*The Lehanneur-Rebuffel model for cable bending analysis*, 2013, 46 pages.

Both of them are currently available on-line on the author’s iMechanica blog.

They were motivated by K.O. Papailiou’s 1995 Ph.D. thesis:

*Die Seilbiegung mit einer durch die innere Reibung, die Zugkraft und die Seilkrümmung veränderlichen Biegesteifigkeit (Cable bending, taking into account bending stiffness variation under the effect of inner friction, axial load and imposed curvature)*, Ph.D. Thesis. E.T.H. Zurich, Switzerland. 1995; 158 pages.

Both Papailiou (1995) and Lehanneur (1949) refer to an earlier work by H. Ernst, in 1933:


Apparently, Ernst’s thesis was the first serious attempt to derive a model for cable bending taking inner friction into account. His basic ideas and assumptions have been developed and applied by later authors.

CONTENTS OF ERNST’S THESIS

While the general topic of the thesis is about passenger carrying ropeways design standards, the contribution delves mostly, if not exclusively, on the problem of bending stress evaluation, in contrast with the primary stresses resulting from the axial load acting on a cable.

Introduction


Part 1

The equilibrium curve of a ropeway, taking into account friction stiffness
Study of taut cable bending under a local transverse force $V$. Author shows that, at first, cable behaves as a solid beam of inertia $J_1$. As $V$ reaches level $V_{\text{\tiny min}}$, impending slip (based on Coulomb’s law of friction) occurs under the force point of application between wires in outer layer and the adjacent layer (same layer wire contact is neglected). As $V$ is increased beyond $V_{\text{\tiny min}}$ slip region propagates along wire contact lines. Thus cable section inertia varies along cable axis, and it is not possible to obtain an exact solution for the curved cable equation. Ernst is led to simplify the problem by assuming a two-stage region with constant inertias $J_1$ (far from $V$ point of application), and $J_2$ (near that point of application). $J_1$ is the solid section inertia, while $J_2$ is a lower value, since part of the section is in the slip stage. In that slip region, an internal friction moment $K$ has to be introduced in the equilibrium equation. Geometric continuity conditions between regions allow the calculation of the boundary.

Another approximate three-region piecewise constant inertia solution is also proposed, with an intermediate inertia $J'$. 

**Part 2**

**The state of a section in a simple strand (spiral cable)**

In this section, Ernst establishes relationships between $V$ and $J$, concentrating on a discrete number of values of $V$: those corresponding to impending slip ($V_{\text{\tiny n, min}}$) and complete slip ($V_{\text{\tiny n, 0}}$) in each layer $n$, starting with the outer layer. Then, it is assumed that slip is sequential: layer 2 starts slipping after slip in layer 1 is complete etc. Also, the intermediate stages of slip are neglected, and it is assumed that in one given layer complete slip occurs instantaneously, being achieved for a value of $V$ taken as the average of $V_{\text{\tiny n, min}}$ and $V_{\text{\tiny n, 0}}$. Thus, section inertia $J_2$ remains constant between each of these values, as well as friction moment $K$. Increasing $V$ from zero, solution described in Part 1 may be applied step by step.

*The following more technologically oriented parts of the paper have not been translated.*

**PART 3** : Official recommendations for a carrying cable

**PART 4** : Official recommendations for hauling and balance cables

**PART 5** : Official recommendations for braking and arresting cables

**CONCLUSION**

It is easy to see that several of the hypotheses found in subsequent reports can be found in Ernst’s work. The main difference with the later contributions lays in the problem he tried to solve, that is a variable bending moment, compared with the uniform imposed curvature problem found in practically all later works. Papailiou (1995) used his uniform curvature results to solve numerically the variable case (also called “free bending” problem).
A CONTRIBUTION TO THE EVALUATION OF PASSENGER AERIAL ROPEWAY OFFICIAL STANDARDS

A Ph.D. thesis by Hellmut Ernst, Technischen Hochschule Dantzig

1933-34

FOREWORD

This paper presents a study of the loads acting on cables used in passenger aerial ropeways and on the corresponding official design standards. Because such means of transportation is recent, and because of its rapid expansion, there are still many important questions which should be clarified, either out of safety concerns or for economic considerations. While not claiming to be exhaustive, this work tries to bring some new contributions, being open to further developments, and in particular to new experimental data.

I wish to express my special thanks to Dr Eng. Rubin for his kind comments and suggestions as well as for his encouragements, and for giving me access to very helpful documents. I am also grateful to Prof. Cranz and Prof. Dr Eng. E.H. Petersen for their help as members in my evaluation committee.

The Author

INTRODUCTION

Among the various means of passenger transportation, aerial ropeways are one of the most recent. Their public use has been increasing after World War I and up to this day.

This development has resulted in the draft of several official design standards for passenger aerial ropeways. There are currently four such standards: in Italy, in Switzerland, in Bavaria and in Japan. The Japanese standards should be construed as obsolete and will not be discussed in this report. Bavarian standards are explicitly labeled as temporary, as well as the old 6-10-1926 Swiss standards, which were updated on the 1-1-1933, thus taking their current final shape. Austria and Czechoslovakia standards are defined in a case by case approach, according to each ropeway attribution contract. Such approach should be construed as temporary, though Austrian regulations are quite consistent. In other countries, such as France, Spain and Norway, sometimes they follow Italian and sometimes Bavarian regulations.

For the six aforementioned countries, definitions on several important points often differ, which shows that evolution of this technical field is not over yet. In particular, this applies to the wire ropes themselves; moreover as these regulations have an impact on passenger safety as well as on the

1 “Technical standards for public passenger aerial ropeways “ (9-3-1926)
2 “Standards for the design of aerial ropeways commissioned for passenger transportation” (1-1-1933)
3 “Proposed draft of design standards for passenger aerial ropeways “ (from the German Railway Co., Bavaria region, Private Line Supervisory Authority), January 1931 version.
4 “Recommendations for the design and use of passenger aerial ropeways, “ Regulation No 36, Ministry of Transportation (9-3-1927)
5 Author has been able to study regulations applying to the following ropeway concessions: Zugspitze, Raxalpe, Pfänder, Hahnenkanim, Obervellach. As for the Czech regulations, the Jeschkenbahn ones were made available to him.
economic viability of the ropeway installation, their importance cannot be overstated. For example, the
many discussions which have taken place after the war on hauling rope reliability have allowed the
practical development of passenger aerial ropeways. It thus seems to be of practical interest to examine
and compare the current regulations used to evaluate the strength of ropes, thus allowing a critical
appraisal of the various approaches. It is this report objective.

First, it seems essential to clarify the matter of the loads acting on wire ropes used in passenger aerial
ropeways, as well as related theoretical questions. This is covered in the report first two sections.
There, bending theory of a taut wire rope under a transverse force is revised in order to take friction
forces into account when evaluating the bending stiffness. This new approach is then applied to actual
ropeways. In Section 3, the carrying rope is studied. Section 4 considers the hauling and balance ropes.
The case of braking or stopping cables is studied in Section 5. Clearly, on several aspects, a purely
theoretical approach is not possible. Thus, in order to give this report as wide a scope as possible, older
results as well as the Author own industrial experience have been used. Such experience includes visits
he made to about twenty ropeway installations in Bavaria, Austria, Italy, Switzerland and France.

This work is limited to the various conditions found in passenger aerial ropeways. The question of
cable testing standards is not covered. The main recommendations have been summarized at the end of
this thesis.

SECTION 1

The equilibrium curve of a ropeway, taking into account friction stiffness

According to previous reports on wire and cable fatigue strength tests, considering lay parameters,
material and laying process, as well as pulley design and material, there are two main factors:

1. The axial load on cable
2. The small bending stiffness of cable when imposed some curvature

6 Rudeloff, “Report of data obtained from tests on wires and strands to study the influence of lay parameters on cable
   strength”, Mitteilungen aus den kgl. techn. Versuchanstalten zu Berlin 1897, pp. 137 etc.
   Bock, “Cable strength”, Dissertation, Hanover, 1909
   Speer, “Cable reliability in ropeways”, Dissertation, Braunschweig, 1912
   Sieglerschmidt, “Cable bending stiffness”, Z.V.D.I., 1927, pp. 517 etc.
7 Benoit, Glückauf, 1913, pp. 1329 etc.
   Benoit and Woernle, “The cable problem”, Karlsruhe, 1915
   Woernle, “Evaluation of aerial ropeways used for passenger transportation”, Karlsruhe, 1913
   Woernle, “Contribution to the appraisal of current cable calculation methods”, Karlsruhe, 1914
   Woernle, “Maschinenbau 1924, pp. 765 etc.
   Woernle, Z.V.D.I., 1929, pp. 417 etc. and 1623 etc.
   Woernle, Z.V.D.I., 1930, pp. 185 etc. and 1417 etc.
   Woernle, Z.V.D.I., 1931, pp. 206 etc. and 1485 etc.
   Woernle, Z.V.D.I., 1932, pp. 557 etc.
   Rubin, Z.V.D.I., 1926, pp. 1756 etc.
   Barth, Z.V.D.I., 1930, p. 381
For crane or elevator cables, such factors are immediately known, tensile load is given, and pulley or hoisting drum diameter, as the cable is generally wound over these elements. In contrast, in a ropeway, cable radius of curvature has to be larger than the one of the pulley or roller. When such conditions apply, cable will be called, for short, “a ropeway cable”. For such a cable, radius of curvature is unknown and has to be determined. While results from previous studies on the behaviour and fatigue strength of ropeways will be used, we shall strive to lay out the mathematical theory allowing the calculation of the cable equilibrium curve, that is the cable bending behavior.

Currently, the method used to determine the shape of a bent cable under the action of a transverse force is Isaachsen’s\(^8\) which considers the cable as an assemblage of frictionless parallel wires or else, as a solid beam having the cable cross-section\(^9\). As shown theoretically and experimentally by Woernle\(^10\) and Benoit\(^11\), the wire helical shape may be neglected in practical applications\(^12\). Validity of this hypothesis will be studied in the following development, taking into account the influence of inter-wire friction forces. While Isaachsen’s equations for both limit cases (frictionless cable and solid section) are identical, it will be shown that the hypothesis according to which cable inner friction has no incidence on stress levels should be rejected. The procedure is as follows.

Consider a taut cable under a transverse force \(V\) acting on a given section. Using a roller, application of \(V\) is assumed to be increasing monotonically, generating some local bending of the cable (Fig.1). Cable cross-section behaves as a solid up to the point where tangential force between a certain pair of contacting wires reaches its limit value. As pressure and friction forces vary within the cable cross-section, wire slip will not occur instantaneously over the whole section. Rather, it will propagate progressively with the imposed bending moment. Slip initiation occurs at that point where friction force limit is smallest and friction force value is highest. For example, in the case of a spiral (or single strand) cable, it occurs on the outer layer, at a wire section located on the cable cross-section neutral

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8 Isaachsen, “Die Beanspruchung von Drahtseilen” (Cables under load), Z.V.D.I., 1907, pp. 655 etc.  
9 Woernle, “On the calculation of passenger aerial ropeways”, pp. 46 etc.  
10 Woernle, “On the calculation of passenger aerial ropeways”, pp. 37 etc.  
11 Benoit and Woernle, “The cable question”  
12 This hypothesis is valid only for the calculation of the cable equilibrium shape taken globally. In order to determine the stresses and friction forces on an individual wire, its helical shape must of course be taken into account.
axis. The slip region extends progressively to inner layers and to wire sections located away from neutral axis. During this process, in which the slip region extends, bending moment of inertia of cable cross-section varies. Indeed, after slip has initiated at a particular wire section, while continuing to carry part of the bending moment, that section is now bending with respect to its own neutral axis.

After slip has initiated in the cable, bending stresses may be considered as resulting from two components: 1) The average stress, or “base stress”, acting at the wire section centroid, arising from cable bending with respect to its neutral axis; being transmitted from wire to wire through friction forces, this stress component can also be called “friction stress”; 2) A stress component arising from wire bending with respect to its own axis, which will be called “complementary stress”. These components, base stress and complementary stress are shown in Fig. 1, for a spiral cable made of circular section wires. In Fig. 1a (abb. 1a), stress state corresponds to the stick state, before any slip occurring in the cable cross-section, while Fig. 1b (abb. 1b) shows the stress state after slip has occurred in the outer layer. After slip starts there, increasing the bending moment does not bring any increase in wire base stress. Its effect is only on the complementary stress. Taking the moment resulting from all base stress components with respect to the conductor neutral axis yields what will be called the “friction moment”, which acts against the imposed moment. This “reaction moment” remains constant as long as the slip state in the cable cross-section remains the same. The same can be said of the section bending stiffness (its “moment of inertia” multiplied by Young’s modulus).

In the following development, it will be assumed that a cable, bent under the action of a roller, may experience two distinct slip stages. These two stages are characterized by inertias \( J_1 \) and \( J_2 \). Inertia \( J_1 \) corresponds to cross-sections in the zero slip, or solid stage. Inertia \( J_2 \) corresponds to sections in the full-slip stage (Fig. 2a). In this model, there are no intermediate stages. Results obtained using such model will be examined.

Thus, under this hypothesis, a cable under axial load \( S \), and subjected to a sufficiently large transverse force \( V \), will comprise two regions. In the first one, bending moment is determined using section inertia \( J_1 \), while in the second region, inertia \( J_2 \) is to be used. In that region, one must include in the calculation the “reaction moment” \( K \) arising from the “base stress”; the smaller inertia \( J_2 \) should be used in order to determine the deformed curve in that region.

Moment \( K \) is determined as follows: \( \sum f_i \) represents the total area of wire cross-sections in the slip state. \( h_i \) is the distance of a given wire section centroid from conductor section neutral axis. Stress \( \sigma_{bi} \) is the bending stress on this centroid at incipient slip. Thus:

\[
K = \sum f_i h_i \sigma_{bi} \tag{1}
\]

At incipient slip:

\[
\sigma_{bi} = \frac{M_1}{J_1} h_i \tag{2}
\]

Whence:

\[
K = \sum f_i h_i^3 \frac{M_1}{J_1} \tag{1a}
\]

In the sequel, \( M_1 \) is the bending moment at incipient slip. It will be determined in Part 2.
Besides, \( J_1 \) and \( J_2 \) are related through the following relationship:

\[
\sum f_i h_i^2 = J_1 - J_2 \tag{3}
\]

in which summation is taken over slipping wires. Thus:

\[
K = \frac{J_1 - J_2}{J_1} M_1 \tag{1b}
\]

In order to determine the deformed cable axis, the coordinate system is taken as shown in Fig. 2a. Origin is taken at the boundary between the no-slip and slip regions.

The x-axis is the asymptote to the curve, far away from the point of application of force \( V \). Material Young’s modulus is noted \( E \). The deformed curve is given by the following differential equations.

Region 1:

\[
\frac{d^2 y_1}{dx^2} = \frac{S y_1}{E J_1} \tag{4}
\]

Region 2:

\[
\frac{d^2 y_2}{dx^2} = \frac{S y_2 - K}{EJ_2} \tag{5}
\]

General solutions to these equations are:
Region 1:

\[ y_1 = C_1 e^{w_1 x} + C_2 e^{w_2 x} \quad (6) \]

with:

\[ w_1 = \sqrt{\frac{S}{EJ_1}} = -w_2 \quad (7) \]

Region 2:

\[ y_2 - \frac{K}{S} = C_3 e^{w_3 x} + C_4 e^{w_4 x} \quad (8) \]

with:

\[ w_3 = \sqrt{\frac{S}{EJ_2}} = -w_4 \quad (9) \]

Constants of integration are obtained using the following boundary conditions:

1) At \( x = -\infty \), one must have \( y_1 = 0 \), yielding:

\[ C_2 = 0 \quad (10) \]

In Region 1, the curve equation is thus given by:

\[ y_1 = C_1 e^{w_1 x} \quad (11) \]

At \( x = 0 \), bending moment on conductor cross-section, by hypothesis, has its limit value \( M_1 \), thus:

\[ C_1 = \frac{M_1}{S} \quad (12) \]

2) Function continuity at \( x = 0 \), \( y_1(0) = y_2(0) \), yielding:

\[ C_3 + C_4 + \frac{K}{S} = C_1 \quad (13) \]

And, using Eqs. (1b) and (12):

\[ C_3 + C_4 = \frac{J_2}{J_1} C_1 \quad (13a) \]

3) Slope continuity at \( x = 0 \), \( y_1'(0) = y_2'(0) \), yielding:
\[ C_3 w_3 + C_4 w_4 = C_1 w_1 \]  

(14)

And, using Eqs (9) and (7):

\[ C_3 - C_4 = \frac{J_2}{\sqrt{J_1}} C_i \]  

(14a)

Now, using Eqs (13a) and (14a), one gets:

\[ C_3 = \frac{C_1}{2} \left[ \frac{J_2}{J_1} + \frac{J_2}{\sqrt{J_1}} \right] \]  

(15)

and

\[ C_4 = \frac{C_1}{2} \left[ \frac{J_2}{J_1} - \frac{J_2}{\sqrt{J_1}} \right] \]  

(16)

In Region 2, shape of the deformed cable center line is given by:

\[ y_2 = \frac{C_1}{2} \left[ \left( \frac{J_2}{J_1} + \frac{J_2}{\sqrt{J_1}} \right) e^{w_1 x} + \left( \frac{J_2}{J_1} - \frac{J_2}{\sqrt{J_1}} \right) e^{w_2 x} \right] + \frac{K}{S} \]  

(17)

Cable global equilibrium, under forces S and V, requires that at \( x = a_2 \), point of application of transverse force V, where bending moment is a maximum, tangent to the deformed centerline must have a slope \( \beta \) such that:

\[ \tan \beta = \frac{V}{2 \sqrt{S^2 - (V/2)^2}} \]  

(18)

In normal conditions, term \((V/2)^2\) is small with respect to \(S^2\), yielding:

\[ \tan \beta \approx \frac{V}{2S} \]  

(18a)

Thus, one can write:

\[ \left. \frac{dy_2}{dx} \right|_{x=a_2} = C_3 w_3 e^{w_3 a_2} + C_4 w_4 e^{w_4 a_2} = \frac{V}{2S} \]  

(19)

If the curve \( y_1(x) \) is extended beyond \( x = 0 \), its slope at \( x = a_i \) is \((V/2S)\), yielding:

\[ \left. \frac{dy_1}{dx} \right|_{x=a_i} = C_1 w_1 e^{w_1 a_i} = \frac{V}{2S} \]  

(19a)
With \( w_4 = -w_3 \), Eqs (19) and (19a) yield:

\[
C_3 e^{w_4 w_2} = C_1 e^{w_1} \frac{w_4}{w_3} + C_2 e^{w_2}
\]

(19b)

At point \( x = a_2 \), function \( y_2(x) \) reaches its maximum. Thus:

\[
y_{2,\text{max}} = C_3 e^{w_2} + C_4 e^{w_2} + \frac{K}{S}
\]

(20)

And, using Eqs (19b) and (16):

\[
y_{2,\text{max}} = C_1 \sqrt{\frac{J_2}{J_1}} \left[ e^{w_1} + \left( \frac{J_2}{J_1} - 1 \right) e^{w_2} \right] + \frac{K}{S}
\]

(20a)

where \( C_3 e^{w_2} \) may be replaced by the expression from Eq. (19a). After some algebraic manipulations, one gets:

\[
y_{2,\text{max}} = \frac{V}{2S} \sqrt{\frac{EJ_2}{S}} \left[ 1 - \frac{1 - \sqrt{\frac{J_2}{J_1}}}{e^{w_2}} \right] + \frac{K}{S}
\]

(20b)

Finally, bending moment maximum value is obtained as:

\[
S y_{2,\text{max}} = \frac{V}{2} \sqrt{\frac{EJ_2}{S}} \left[ 1 - \frac{1 - \sqrt{\frac{J_2}{J_1}}}{e^{w_2}} \right] + K
\]

(20c)

In this expression, each of the two components may be interpreted as follows:

1. The \( K \) term is the “friction moment”. It is statically equivalent to the moment arising from the “base stress” on slipping wires;

2. The other term arises from the bending stresses in the section no-slip region, which still acts as a solid: \( \frac{V}{2} \sqrt{\frac{EJ_2}{S}} \left[ 1 - \frac{1 - \sqrt{\frac{J_2}{J_1}}}{e^{w_2}} \right] \) where \( \frac{V}{2} \sqrt{\frac{EJ_2}{S}} \) is the maximum bending moment induced by \( V \), assuming a uniform inertia \( J_2 \) over the whole cable.

The following correction factor applies on this maximum bending moment:
\[ \Phi = 1 - \frac{\sqrt{J_2}}{\sqrt{J_1}} \]

(21)

It includes terms \( e^{w_{1x_1}} \) and \( e^{w_{2x_2}} \), with limit values equal to 1 when \( a_1 = \infty \) and \( a_2 = \infty \), and \( \sqrt{J_2/J_1} \) when \( a_1 = 0 \) and \( a_2 = 0 \). In general, calculation of factor \( \Phi \) requires calculation of \( e^{w_{1x_1}} \) and \( e^{w_{2x_2}} \).

Once the calculation has been performed, moment \( M_1 \), and thus constant \( C_1 \), allow determination of the section position at which there is the inertia jump. Indeed, from Eq. (19a):

\[ e^{w_{1x_1}} = \frac{V}{2SC_1} \sqrt{\frac{EJ_1}{S}} \]

(21a)

Besides, combining Eq. (19) with Eqs (15) and (16) yields:

\[ e^{w_{2x_2}} = \frac{V}{2SC_1 \left( 1 + \sqrt{J_2/J_1} \right)} \sqrt{\frac{EJ_1}{S}} + \sqrt{\frac{V}{2SC_1 \left( 1 + \sqrt{J_2/J_1} \right)}} \left( 1 - \frac{\sqrt{J_2/J_1}}{1 + \sqrt{J_2/J_1}} \right) \]

(21b)

with an approximate maximum value of:

\[ e^{w_{2x_2}} \approx \frac{V}{SC_1 \left( 1 + \sqrt{J_2/J_1} \right)} \sqrt{\frac{EJ_1}{S}} = \frac{2}{1 + \sqrt{J_2/J_1}} e^{w_{1x_1}} \]

(21c)

Taking constant \( C_1 \) into account, product \( e^{w_{1x_1}} e^{w_{2x_2}} \) is practically proportional to the square of transverse force \( V \), and inversely proportional to the cube of the axial force \( S \) on the cable.

The problem now is to determine how factor \( \Phi \) varies when transition from inertia \( J_1 \) to inertia \( J_2 \) does not occur instantaneously but, more realistically, within a finite transition zone. In order to tackle this problem, it will be assumed as shown in Fig (2b) that there is an intermediate \( J' \) region, length \( a' \), between the \( J_1 \) and \( J_2 \) regions. Thus, the deformed curve is defined as a three-branch function, as follows:

Region 1, inertia \( J_1 \)
Region 2, inertia \( J' \), with friction moment \( K' \)
Region 3, inertia \( J_2 \), with friction moment \( K \)

Equations in these three regions are obtained as above:

Region 1:

\[ y_1 = C_1 e^{w_{1x}} + C_2 e^{w_{2x}} \]

(22)

with

\[ w_1 = \sqrt{\frac{S}{EJ_1}} = -w_2 \]

(22a)
Region 2:

\[ y' = C'e^{w'x} + C''e^{w''x} + \frac{K'}{S} \]  (23)

with

\[ w' = \sqrt{\frac{S}{EJ}} = -w'' \]  (23a)

Region 3:

\[ y_2 = C_3e^{w_1x} + C_4e^{w_2x} + \frac{K}{S} \]  (24)
Origin of the coordinate system is still taken at the boundary between regions 1 and 2 (Fig. 2b). In order to determine the six integration constants, the following six conditions must hold:

1) \[ x = -\infty \quad y_1 = 0 \] (25)

2) \[ x = 0 \quad y_1 = y' \] (26)

3) \[ x = 0 \quad dy_1/dx = dy'/dx \] (27)

4) \[ x = a' \quad y' = y_2 \] (28)

5) \[ x = a' \quad dy'/dx = dy_2/dx \] (29)

6) \[ x = a_2 \quad dy_2/dx = V/2S \] (30)

Finally, the following condition must hold:

\[ x = a_1 \quad dy_1/dx = V/2S \] (31)

Thus:

from Eq. (25):

\[ C_2 = 0 \] (32)

from Eqs (26) and (27):

\[ C' = \frac{C_1}{2} \left( \frac{J'}{J_1} + \sqrt{\frac{J'}{J_1}} \right) \] (33)

and:

\[ C'' = \frac{C_1}{2} \left( \frac{J'}{J_1} - \sqrt{\frac{J'}{J_1}} \right) \] (34)

from Eqs (28) and (29):

\[ C_3 = \frac{C' \left( 1 + \sqrt{J_2/J'} \right) e^{w_{a'}} + C'' \left( 1 - \sqrt{J_2/J'} \right) e^{w_{a'}} + (K' - K)/S}{2e^{w_{a'}}} \]

As in Eq. (1b), one has:

\[ \frac{K' - K}{S} = -\frac{J' - J_2}{J'} \left( C' e^{w_{a'}} + C'' e^{w_{a'}} \right) \]
Using Eqs (33) and (34), this equation yields:

\[ C_3 = \frac{C_1}{2} \left\{ \frac{J'}{J_1} \left( \frac{J_2}{J'} + \sqrt{\frac{J_1}{J'}} \right) e^{w_{1a}} + \frac{J'}{J_1} \left( \frac{J_2}{J'} - \sqrt{\frac{J_1}{J'}} \right) e^{w_{2a}} \right\} \]  \tag{35}

\[ C_4 = \frac{C_1}{2} \left\{ \frac{J'}{J_1} \left( \frac{J_2}{J'} + \sqrt{\frac{J_1}{J'}} \right) e^{w_{1a}} + \frac{J'}{J_1} \left( \frac{J_2}{J'} - \sqrt{\frac{J_1}{J'}} \right) e^{w_{2a}} \right\} \]  \tag{36}

Furthermore, using Eqs (30) and (31), one gets:

\[ y_{2\text{max}} = C_1 \frac{w_1}{w_3} e^{w_{1a}} + 2C_4 e^{w_{2a}} + \frac{K}{S} \]  \tag{37}

Using Eq. (36), and after some algebraic manipulations:

\[ y_{2\text{max}} = \frac{V}{2} \sqrt{\frac{EJ_2}{S^2} \Phi + \frac{K}{S}} \]  \tag{38}

Correction factor \( \Phi \) can now be expressed as:

\[ \Phi = 1 - \frac{e^{w_{1a}}}{2e^{w_{1a}} e^{w_{2a}}} \left[ \frac{J_1}{J_2} \left( \frac{J'}{J_1} + \sqrt{\frac{J_1}{J'}} \right) e^{w_{1a}} + \left( \frac{J'}{J_1} - \sqrt{\frac{J_1}{J'}} \right) e^{w_{2a}} \right] \]  \tag{39}

Furthermore, using \( a_2 = a' + a'' \) :

\[ \Phi = 1 + \frac{1}{2e^{w_{1a}} e^{w_{2a}}} \sqrt{\frac{J_1}{J_2}} \left[ \left( \frac{J'}{J_1} + \sqrt{\frac{J_1}{J'}} \right) e^{w_{1a}} + \left( \frac{J'}{J_1} - \sqrt{\frac{J_1}{J'}} \right) e^{w_{2a}} \right] \]  \tag{39a}

It should be noted that Eq. (39a) is similar to Eq. (21).

Here also, terms \( e^{w_{1a}} \) et \( e^{w_{2a}} \) depend on transverse force \( V \) and axial load \( S \) and, obviously, on moments of inertia. Besides, intermediate length \( a' \) may be given a physical interpretation. When \( a' = 0 \), it is easily checked that Eq. (39a) is reduced to Eq. (21). As \( a' \) increases, correction factor \( \Phi \) increases, as does the term \( e^{w_{1a}} \) and, when \( a' = \infty \), it becomes:
\[ \phi = 1 - \frac{1 - \sqrt{J_2/J'}}{e^{w_{J_2}^{a''}} e^{w'\alpha x'}} \]  

(39b)

where abscissa \( x = a' + a'' \) (Fig. 2b) corresponds to that point where tangent line in region 2 (\( J', K' \)), drawn beyond \( x = a' \) makes an angle \( \beta \) with the x-axis. Indeed, one has:

\[ C'e^{w(a'+a'')} - C''e^{w''(a'+a'')} = \frac{V}{2Sw''} \]

Or else:

\[ e^{w(a'+a'')} = \frac{V}{2SC(I/J_1 + \sqrt{J'/J_1})} \sqrt{\frac{EJ'}{S}} + \frac{V}{2SC_1(I'/J_1 + \sqrt{J'/J_1})} \sqrt{\frac{EJ}{S}} - \frac{1 - \sqrt{J'/J_1}}{1 + \sqrt{J'/J_1}} \]

(40)

Letting \( a' + a'' = \infty \), it yields:

\[ e^{w(a'+a'')} = \frac{V}{SC_1(I'/J_1 + \sqrt{J'/J_1})} \sqrt{\frac{EJ'}{S}} \]

(40a)

However:

\[ e^{w_{J_1}} = \frac{V}{2SC_1} \sqrt{\frac{EJ_1}{S}} \]

Thus, Eq. (39a) reduces to (39b).

Using Eq. (39b), similar to Eq. (21), it is possible to see that for large \( a' \), influence of the first transition, from \( J_1 \) to \( J' \), vanishes, leaving only the effect of second transition \( J' \) to \( J_2 \). The greater ratio \( J'/J_2 \) is with respect to \( J_1/J' \), the smaller is the first transition influence. In general, correction factor given by Eq. (39a) should be accurate enough. Indeed, considering the number of simplifying hypotheses, a more refined analysis of section inertia, with more than two regions, seems to be superfluous.

In order to use Eq. (39a), as well as Eq. (21), one has to know the moment with respect to the considered section, shifted by an unknown value, the deformed shape function \( y(x) \), which itself is dependent on the global section behavior\(^\text{13}\). For example, if constant \( C_1 \) is known, term \( e^{w_{J_1}} \) is determined using Eq. (21a). Term \( e^{w''} \) may be determined through Eq. (40) by letting \( a'' = 0 \), and for \( V \), the selected value will be the one such that inertia jumps from \( J' \) to \( J_2 \). Finally, \( e^{w_{J_2}} \) is obtained through Eqs (24) and (30):

\[ C_3 e^{w_{J_2}} - C_4 e^{w_{J_1}} = \frac{V}{2S} \sqrt{\frac{EJ_2}{S}} \]

\(^{13}\) These values will be determined in Part 2, where the state of a section will be taken into account.
or:

\[ C_3 e^{w_a t} e^{w_a t} - C_4 e^{w_a t} e^{w_a t} = \frac{V}{2S} \sqrt{\frac{EJ_2}{S}} \]

Thus:

\[ e^{w_a t} = \frac{V}{4SC_3 e^{w_a t}} \sqrt{\frac{EJ_2}{S}} + \left( \frac{V}{2SC_3 e^{w_a t}} \sqrt{\frac{EJ_2}{S}} \right)^2 \frac{C_4 e^{w_a t}}{C_3 e^{w_a t}} \]  

When factor \( \Phi \) is determined, one gets the following results.

Maximum of bending moment in the ropeway cable, stiffened by friction and loaded by transverse force \( V \), is given by:

\[ S_{y_{\text{max}}} = \frac{V}{2} \sqrt{\frac{EJ}{S}} + K \]  

in which \( J \) is the section inertia at the transverse force application point, and \( K \) is the friction moment at that point.

This external moment balances the internal moment corresponding to the stresses (see Fig. 1b), which includes moment \( K \) from base stress, moment from complementary stress, including those arising from wires in the slip state, as well as those in the stick state. Calling \( \sigma_b \) the bending stress at any point, and \( h \) the distance of that point with respect to the corresponding neutral axis, that is the wire neutral axis for those wires in the slip state, and the cable section neutral axis for those in the stick state, bending moment from internal stresses is given by:

\[ K + \sigma_b \frac{J}{h} \]

Writing the balance between external and internal moments yields:

\[ \sigma_b = \frac{V}{2} \sqrt{\frac{E}{SJ}} \frac{\Phi h}{h} \]  

Calling \( \delta \) the wire diameter, one gets the complementary stress in the section slip region:

\[ \sigma_b = \frac{V}{2} \sqrt{\frac{E}{SJ}} \frac{\delta}{\Phi 2} \]  

In the section stick region, \( \sigma_b \) must also include the base stress.

Calling \( \rho \) the cable radius of curvature at the transverse force point of application, the classical bending equation yields:
\[ \sigma_b = \frac{\delta E}{2\rho} \]  

Using Eq. (43a), it yields:

\[ \rho = \frac{2}{\sqrt{V\Phi}} \sqrt{EJS} \]  

In a bent cable (under a transverse force, thus at the point of application of that force), the smallest radius of curvature, taking internal friction into account, is dependent mainly, under a given axial load, on the moment of inertia of the corresponding section. That moment of inertia is dependent on the state of that section, and on a factor \( \Phi \) also dependent on the section state, on the transverse force, and also on the axial load carried by the cable. When the section is fully in the slip state, one gets the minimum radius of curvature, and it corresponds approximately to a bunch of parallel unconnected, frictionless, wires. This is even more so for a large transverse force and a small axial load.

Once the minimum radius of curvature is determined, it may be used to compare ropeway situations with those of cranes and elevators. In the case of a ropeway, the base stress, arising from friction, does depend on the free bending radius of curvature \( \rho \), while in an elevator cable, it depends on the drum radius \( \rho \). In both cases, the cable deformed curve is the same\(^{14} \). On the winding arc \( \alpha \) (Fig. 3), cable is fixed with respect to the pulley (or roller), and the load is a constant. The winding arc has no influence on the bending loads; only its effect on contact pressure between cable and roller will be studied.

Assuming that the cable cross-section is perfectly circular and non-deformable, the transverse force may be considered as a pointwise force, as in a ropeway cable, while in a hoisting cable, wound on a

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drum radius \( \rho \), there is a line contact, corresponding to a line force density over the winding arc given by the well-known formula:

\[
p = \frac{S}{\rho}
\]

In fact, the theoretical winding angle \( \alpha' \) (Fig. 3) is greater than the real angle \( \alpha \). Besides, equilibrium conditions require complementary forces between cable and roller. The cable deformed curve calculation yields an approximate evaluation of these forces. In Fig. 3, forces acting at the exit point are shown: force \( S \) on cable is the sum of axial force \( Z \) and transverse force \( Q \). There must also be a moment \( M = s \times S \). Along the winding arc, moment is a constant as radius of curvature is a constant. Thus, transverse force in all sections vanishes; force \( Q \) is thus a maximum at point \( a \). It is easily found that the transverse force is just half of the force \( V \) which is necessary to induce radius of curvature \( \rho \) to the cable in free bending.

Reasoning may go as follows. For a given free bending radius of curvature \( \rho \), contact pressure between cable and roller is higher than that required for winding on a pulley having the same radius. Indeed, in the first case, force \( V \) is theoretically a point force, while in the second case, one half only of force \( V \) acts at the exit point. Thus, in neither case equation

\[
p = \frac{S}{\rho}
\]

yields the maximum contact pressure. Rather, it is given by Eq. (44), through which \( V \) and \( \rho \) are connected.

**PART 2**

**The state of a section in a simple strand (spiral cable)**

The objective of this second part is to show that the bending load as well as the minimum value of the radius of curvature of a ropeway in contact with a roller can be determined based on the section inertia \( J \), which itself depends on the section state, that is, on the stick-slip state of individual wires. The following developments are an attempt to characterize how that state is influenced by the applied loads (axial and transverse). The simpler case of a spiral cable made of identical circular section wires will first be considered, allowing the presentation of the theory basic principles.

First, it is assumed that friction prevents any relative displacement between wires. Consider now a cable element between two parallel cross-sections, as well as the corresponding wire elements. When imposed a curvature, corresponding stresses vary from one wire to the other. The following two reasons may explain this variation:

1. The wire helical shape, meaning that a given wire will pass successively in the compression, and in the tension region of a cable section.

\[^{15}\text{Obviously, this is valid only within the given hypotheses. In practice, as with a ropeway cable, the cable cross-section will undergo some flattening and several points will be involved in the load transmission.}\]
2. Bending moment variation, whose maximum value will occur on both sides of the transverse force point of application, with a rapid decrease for sections away from that point (Fig. 4).

On a given wire, different sections are under different stress levels, thus different normal force, leading to tangential forces on the wire element. When these tangential forces reach the friction limit, impending slip occurs, which tends to equalize stresses in that element, while the element starts slipping and frees itself from the remaining solid part of the cable cross-section.

Consider those wires still in the solid part of the section. Previous hypotheses still hold and bending stress is still given by:

\[
\sigma_b = \frac{Sy - K}{J} h
\]  

(45)

Elementary stress variation over a wire element, and inducing a tangential force on the element:

\[
d\sigma_b = S \frac{dv}{dx} \frac{h}{J} dx + \frac{Sy - K}{J} dh
\]  

(45a)

It is easily seen that the tangential force will be a maximum in the section corresponding to the maximum bending moment, that is, at the transverse force point of application. Letting \(x = a_2\) in Eq. (30), and using Eq. (41) yields:

\[
d\sigma_b = \frac{V}{2J} h dx + \frac{V}{2J} \sqrt{\frac{E}{S}} dh
\]  

(45b)
A relationship between \( x \) and \( h \) arises from a wire helical shape. That shape may be considered kinematically as the superposition of a uniform circular motion and a uniform translation parallel to the helix axis. Let \( \alpha \) be the polar angle on the circle (Fig. 5), \( c \) the wire strand lay length, \( d \) the diameter of the lay cylinder on which the wire centerline is wound in a given layer, \( \omega \) the wire lay angle.

These parameters are related by:

\[
\frac{d\alpha}{dx} = \frac{2\pi}{c} = \frac{2\tan \omega}{d}
\]

Distance \( h \) of the particular wire section center with respect to the cable neutral axis is given by (Fig. 5):

\[
h = \frac{d}{2}\cos \alpha
\]

Hence:

\[
dh = -\frac{d}{2}\sin \alpha \, d\alpha
\]

Thus, Eqs (45b) and (47a) yield:

\[
d\sigma_b = \frac{Vd}{4} \left[ \frac{d}{2J\tan \omega} \cos \alpha - \phi \sqrt{\frac{E}{S}} \sin \alpha \right] d\alpha
\]

In Eq. (45c), terms within brackets add up in intervals \( \left[ \pi/2, \pi \right] \) and \( \left[ 3\pi/2, 2\pi \right] \). Now, instead of considering the cable on the left hand side of \( V \), one considers the right hand side (bending moment is now decreasing), terms add up in intervals \( \left[ 0, \pi/2 \right] \) and \( \left[ \pi, 3\pi/2 \right] \). Here, the objective is to determine the tangential force maximum value, either to the right or to the left of force \( V \). Thus, Eq. (45c) may be written as follows:
\[ d\sigma_b = \frac{Vd}{4} \left[ \frac{d}{2J \tan \phi} \cos \alpha \pm \Phi \sqrt{\frac{E}{SJ}} \sin \alpha \right] d\alpha \]  \hspace{2cm} (45d)

with the convention that the plus or minus sign has to be selected in such a way that both terms within the brackets add up.

Once the tangential force is obtained, there remains to determine the slip limit. In a spiral strand made of identical, circular section wires, and, except for the core wire, with typical lay angles\(^{16}\), same layer wires do not touch each other since, theoretically, they do not fill completely the circle on which is located their center\(^{17}\). For a wire element in the outer layer, the limit of friction will depend mainly on the contact forces with the adjacent layer. It will be determined by assuming that this second layer is a solid cylinder. Let \( t \) the wire tensile stress, \( \ell \) and \( u \) its components parallel and normal to strand axis (Fig. 5). Thus:

\[ dp = u \, d\alpha \]  \hspace{2cm} (48)

And, considering that \( u = \ell \tan \phi \), it yields:

\[ dp = \ell \tan \phi \, d\alpha \]  \hspace{2cm} (48a)

Hence, limit of friction is given by:

\[ dR = \mu dp = \ell \tan \phi \, \mu d\alpha \]  \hspace{2cm} (49)

Stress component \( \ell \) may be expressed as:

\(^{16}\) Findeis, Basic principles for the calculation of ropeways, pp. 6 and 7  
\(^{17}\) However, this applies only to unloaded cables, because of the cable deformation under an axial or transverse load. Cable extension yields lay angle decrease, thus an increase of same layer wire distance. But a contraction of cable would correspond to an increase of lay angle, a decrease of same layer wires, up to the point where they touch each other, with ensuing pressure along contact lines. Here, cables are always under a tensile force, thus yielding an increase in the gap between same layer wires. Under imposed bending, this gap increases on the tensile side, and decreases on the compression side. If imposed curvature is large enough, compressive stresses may even be greater than tensile stresses from the axial load.

On the compression side, if wires are contacting, one may try to determine the condition for impending slip in that layer. When this occurs, those wires for which the limit of friction has been reached start slipping on the lower adjacent layer, slip motion is parallel to a wire axis. In these conditions, the friction force component normal to the wire axis is practically negligible. Thus, lateral pressure between two neighboring wires cannot be substantial, if only for the reason that, in the tensile side, the gap between wires increases. Hence, in the bent cable, material displacement occurs, not only parallel to wire axis, but also in the normal direction.

When a cable is made of a large number of wires, it may happen that, in internal layers, starting with the 4th or the 5th, lateral contact between wires occurs before their slipping on the adjacent inner layer. Lateral pressure between wires is most probably much lower than radial pressure and it seems reasonable to neglect it. Exception should be made, however, for the innermost layer, in a strand where all wires are of the same diameter. It is easily shown that 6 parallel wires may be accommodated on a core wire of the same diameter. They are then in line contact; the 6 wires are just touching the core wire, leading to a small radial motion, a small flat contact strip and a lateral pressure. It is not possible, however, to say in which proportion radial and lateral forces are on a given wire. Besides, one should consider the pressure arising from bending of strand. As for the core wire, it must be emphasized that the calculated values, which neglect lateral forces, are but approximations.
\[ \ell = \sigma_z + \sigma_b \]  

(50)

where \( \sigma_z \) is the stress component parallel to strand axis arising from axial load, while \( \sigma_b \) is the bending stress component. Thus:

\[ dR = (\sigma_z + \sigma_b) \tan \mu d\alpha \]  

(49a)

On the inner layer, taking into account pressure coming from wires in the outer layer, one gets a similar relationship. Considering the alternate lay case, in which wires in adjacent layers are wound in alternate direction\(^{18}\), material displacements are in opposite direction. Thus, on the second layer, one has the approximate expression:

\[ dR = (3\sigma_z + 2\sigma_{b1} + \sigma_{b2}) \tan \mu d\alpha \]  

(49b)

where the index of the \( \sigma_b \) components indicate the corresponding layer. In the third layer, one has similarly:

\[ dR = (5\sigma_z + 2\sigma_{b1} + 2\sigma_{b2} + \sigma_{b3}) \tan \mu d\alpha \]  

(49c)

Displacement of a wire element with respect to the rest of the section may happen when the tangential force reaches the limit of friction, that is:

\[ d\sigma_b \geq dR \]  

(51)

From Eqs (45d) and (49a), one gets the outer layer limit condition:

\[ (\sigma_z + \sigma_{b1}) \mu \tan \omega d\alpha = \frac{Vd_1}{4} \left[ \frac{d_1}{2J_1 \tan \omega} \cos \alpha \pm \Phi_1 \sqrt{\frac{E}{SJ_1}} \sin \alpha \right] d\alpha \]  

(51a)

Stress component \( \sigma_{b1} \) is obtained from Eq. (43), thus:

\[ (\sigma_z + \Phi_1 \frac{V}{2} \sqrt{\frac{E}{SJ_1}} \frac{d_1}{2} \cos \alpha) \mu \tan \omega d\alpha = \frac{Vd_1}{4} \left[ \frac{d_1}{2J_1 \tan \omega} \cos \alpha \pm \Phi_1 \sqrt{\frac{E}{SJ_1}} \sin \alpha \right] d\alpha \]  

(51b)

Hence, the level of transverse force \( V \) necessary to induce slip on a given wire element is:

\[ V = \frac{4\mu \tan \omega \sigma_z}{d_1 \left( \frac{d_1}{2J_1 \tan \omega} \mu \tan \omega \sqrt{\frac{E}{SJ_1}} \cos \alpha \pm \Phi_1 \sqrt{\frac{E}{SJ_1}} \sin \alpha \right)} \]  

(52)

It is found that \( V \) is a minimum for:

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\(^{18}\) Benoit, The cable problem, p. 20
\[
\tan \alpha = \frac{\phi_i \sqrt{E}}{d_1 \sqrt{2J_1 \tan \omega - \mu \tan \phi_i \sqrt{E}}} \sqrt{S J_1}
\]

For the outer layer, taking \( \phi_i = 1 \), and the usual values for \( J_1, d_1 \) and \( S \), angle \( \alpha \) is found to have a value between \( 60^\circ \) and \( 90^\circ \), meaning that the wire element at impending slip is not too far from the strand section neutral axis. A good approximation is obtained by letting \( \alpha = 90^\circ \) in Eq. (52). Hence:

\[
V_{\min} = \frac{4\mu \tan \omega \sigma_z}{d_1 \sqrt{E \sqrt{S J_1}}} \quad (52a)
\]

Eq. (52a) yields the minimum value of transverse force \( V \) corresponding to outer layer impending slip with respect to the rest of strand solid section. For inner layers, the same approach can be used. In Eq. (51) the \( dR \) term is given by Eqs (49b), (49c) etc. The further from the outer layer, the more involved calculations become. However, considering the degree of uncertainty on \( \mu \) and \( \Phi \), an approximate evaluation is acceptable. Letting \( \alpha = 90^\circ \) in these equations, for layer \( n \)th, one gets:

\[
V_{\min n} = \frac{3 \times 4\mu \tan \omega \sigma_z}{\phi_2 d_2 \sqrt{E \sqrt{S J_2}}} \quad (52b)
\]

where \( d_2 \) is that layer diameter, \( J_2 \) the moment of inertia for that \( V_{\min} \) and \( \phi_2 \) the corresponding correction factor. This may be generalized for the \( n^{th} \) layer (counting from the outer layer):

\[
V_{\min n} = \frac{(2n - 1) \times 4\mu \tan \omega \sigma_z}{\phi_n d_n \sqrt{E \sqrt{S J_n}}} \quad (52c)
\]

Eq. (52c) yields the approximate value of \( V \) inducing impending slip in each layer. In the same fashion, one may determine the \( V \) value for which slip is complete in a given layer. Eq. (52c) cannot be used to answer this question, as it has been derived by only considering a wire element. It is based on the assumption that the neighboring element is still in the stick state. However, on a wire element, stresses acting on limiting sections must satisfy equilibrium conditions. Hence, stress states on neighboring elements are not stable. Such instability, combined with friction forces when impending slip is reached leads to an unlocking of the neighboring element. Equilibrium is again achieved if at any point of the wire loop, tangential force is smaller or just equal to the limit of friction. Initial distribution of bending stresses will depend on this change in equilibrium condition. Let \( f(\alpha) \) be the variation of stress with respect to angle \( \alpha \) (Fig. 5). An equilibrium condition is given by:

\[
\text{Friction forces } = \text{difference between forces on element limiting cross-sections}
\]

For the first layer, and in mathematical form, it yields:
\[-(\sigma_z + f(\alpha))\tan \omega \mu d\alpha = d(f(\alpha))\]  

(53)

After integration, and e being the base of natural logarithms, one gets:

\[\sigma_z + f(\alpha) = Ce^{-\tan \omega \mu \alpha}\]  

(53a)

At \(\alpha = \pi/2\), that is on the cable section neutral axis, the following condition is practically met:

\[f(\alpha)\bigg|_{\alpha = \pi/2} = 0\]

Hence:

\[C = \sigma_z e^{\tan \omega \mu \pi/2}\]

\[\sigma_z + f(\alpha) = \sigma_z e^{\tan \omega \mu (\pi/2 - \alpha)}\]  

(53b)

On any two wire sections, the maximum difference between corresponding stresses is given by the difference taken by \(f(\alpha)\) on these two sections. Take, for example, a wire element between \(\alpha = \pi/2\) and \(\alpha = \pi\), the maximum stress difference on the corresponding sections is:

\[\left[f(\alpha)\right]_{\alpha = \pi/2}^{\alpha = \pi} = \sigma_z \left(e^{-\tan \omega \mu (\pi/2)} - 1\right)\]  

(54)

And for the quarter-loop from \(\alpha = 0\) to \(\alpha = \pi/2\):

\[\left[f(\alpha)\right]_{\alpha = 0}^{\alpha = \pi/2} = \sigma_z \left(1 - e^{\tan \omega \mu (\pi/2)}\right)\]  

(55)

On a given wire element, condition for transition between the bending stress initial distribution, given by Eq. (53b), and the modified distribution, is achieved when the maximum value of the stress difference on the limiting sections is reached. For a quarter-loop between \(\alpha = \pi/2\) and \(\alpha = \pi\), (Fig. 4, point 2’ to point 2), this condition yields:

\[\sigma_{b}\bigg|_{\alpha = \pi} - \sigma_{b}\bigg|_{\alpha = \pi/2} = \sigma_z \left(e^{-\tan \omega \mu (\pi/2)} - 1\right)\]  

(56)

Hence, using Eq. (43), combined with Eq. (47), it yields:

\[-\frac{V}{2} \sqrt{\frac{E}{J\bar{S}}} \frac{d_l}{2} \phi_1' = \sigma_z \left(e^{-\tan \omega \mu (\pi/2)} - 1\right)\]  

(56a)

Similarly, in the \([0, \pi/2]\) quarter-loop (Fig. 4, point 3 to point 3’):

\[-\frac{V}{2} \sqrt{\frac{E}{J\bar{S}}} \frac{d_l}{2} \phi_1'' = \sigma_z \left(1 - e^{\tan \omega \mu (\pi/2)}\right)\]  

(57)
This allows the calculation of \( V \) at which slip occurs at points 2 and 3 between that quarter-loop with respect to the rest of the strand cross-section (Fig. 4).

At point 2:

\[
V_{\alpha=\pi} = \frac{4\sigma_x \left(1 - e^{-\tan \alpha \mu (\pi/2)}\right)}{d_1 \sqrt{\frac{E}{J_1} \phi'_1}}
\]  \hspace{1cm} (58)

At point 3:

\[
V_{\alpha=0} = \frac{4\sigma_x \left(e^{\tan \alpha \mu (\pi/2)} - 1\right)}{d_1 \sqrt{\frac{E}{JS} \phi''_1}}
\]  \hspace{1cm} (59)

Respectively, \( J'_1 \) and \( J''_1 \) are the section moment of inertia for each of these two values of \( V \), while \( \phi'_1 \) and \( \phi''_1 \) are the corresponding correction factors. It should be noted that section 4 will slip before section 3 (Fig. 4), section 5 before section 4 etc. Indeed, on element 4-4', ratio of tangential force to limit of friction is larger than that ratio on element 3-3', and on 5-5', that ratio is larger than on 4-4' etc.

For an arbitrary angle \( \alpha \), it is given by Eqs (43) and (53b):

\[
\psi(\alpha) = \frac{\phi \frac{V}{2} \sqrt{\frac{E}{JS} d_1 \cos \alpha}}{\sigma_x \left(e^{\tan \alpha \mu (\pi/2 - \alpha)} - 1\right)}
\]  \hspace{1cm} (60)

It is easily seen that for given values of \( \Phi \) and \( J \), \( \psi(\alpha) \) is a maximum for \( \alpha \approx \pi/2 \) and decreases when going towards \( \alpha = 0 \) as well as towards \( \alpha = \pi \).

This means that on the outer layer wire sections located near neutral axis start slipping one after the other, up to section 2 (Fig. 4). In Fig. 6, the hatched and cross-hatched regions indicate the solid part of the cable cross-section before wire section 2 starts slipping. The cross-hatched region indicates the solid part before section 3 starts slipping. Thus, corresponding inertia and correction factor must be used in Eqs (58) and (59).

Same approach is to be used for inner layers. As already suggested, approximate values should be acceptable. Thus, after bending stresses have been determined in a given layer, it is possible to find in what proportion they are with the axial load stress. Consider the second layer, static condition

Friction forces = difference between forces on element limiting cross-sections
yields:
\[-(3\sigma_z + f(\alpha)) \tan \phi \mu d\alpha = d(f(\alpha))\]  
(61)

Constants of integration are obtained as in the previous calculation. Thus:
\[f(\alpha) = 3\sigma_z \left( e^{-\tan \mu \phi (\pi/2 \alpha)} - 1 \right)\]  
(61a)

and the value of transverse force \(V\) leading to impending slip at point 3 (Fig. 3), in the second layer corresponding section, that is, that value of \(V\) at which slip is complete in that layer, is given by:
\[V_{2,\alpha=0} = \frac{4 \times 3\sigma_z \left( e^{-\tan \mu \phi (\pi/2 \alpha)} - 1 \right)}{d_3 \sqrt{\frac{E}{J_2^S}} \phi^n_2}\]  
(59a)

Complete slip for the \(n^{\text{th}}\) layer (counting from the outer layer) is achieved for a transverse force value given by:
\[V_{n,\alpha=0} = \frac{4 \times (2n-1)\sigma_z \left( e^{-\tan \mu \phi (\pi/2 \alpha)} - 1 \right)}{d_n \sqrt{\frac{E}{J_n^S}} \phi^n_n}\]  
(59b)

Value of \(J^n_n\) is determined as in the first layer case. Parameter \(d_n\) is the corresponding layer diameter, while \(\phi^n_n\) is the correction factor. It is assumed that slip starts in one layer only after the adjacent external layer has reached the complete slip state. Thus, a proof is necessary to show that, for all sections, and at a given state of a section, \(V_{(n+1)\text{min}}\) is greater than \(V_{n,\alpha=0}\).

\(V_{n,\alpha=0}\) and \(V_{(n+1)\text{min}}\) are obtained from Eqs (59b) and (52c) respectively, with appropriate index modification:
\[V_{(n+1)\text{min}} = \frac{(2n+1) \times 4\mu \tan \phi \sigma_z}{\phi^n_n d_{(n+1)} \sqrt{\frac{E}{S J^n_n}}}\]

In this equation, the \(\Phi\) and \(J\) indices correspond to the state of the given section before complete slip in the \(n^{\text{th}}\) layer. For this section state, if \(V_{(n+1)\text{min}}\) is greater than \(V_{n,\alpha=0}\), slip in the \(n^{\text{th}}\) will be complete before slip starts in the \((n+1)^{\text{th}}\) layer. Hypothesis will be validated if:
\[\frac{V_{(n+1)\text{min}}}{V_{n,\alpha=0}} > 1\]

Using the above expressions:
\[
\frac{V_{(n+1)\min}}{V_{n,n=0}} = \frac{2n+1}{2n-1} \frac{\mu \tan \omega}{e^{\mu \tan \omega \left(\frac{\pi}{2}\right)} - 1} d_n
\]

With a \( p \)-layer cable:

\[
d_n = (2p - 2n + 2) \delta \quad d_{(n+1)} = (2p - 2n) \delta
\]

leading to:

\[
\frac{V_{(n+1)\min}}{V_{n,n=0}} = \frac{2n+1}{2n-1} \frac{p - n + 1}{p - n} \frac{\mu \tan \omega}{e^{\mu \tan \omega \left(\frac{\pi}{2}\right)} - 1}
\]

One sees that this ratio decreases when \( \mu \) and \( p \) increase. For \( \mu \) and \( p \) values which will be used subsequently (up to 6 layers for \( p \)), it will be verified that for all layers and all values of \( \mu \):

\[
\frac{V_{(n+1)\min}}{V_{n,n=0}} > 1
\]

and the assumption of layer by layer unlocking of the strand is verified.

Taking Eq. (59b), corresponding to complete slip in a given layer, and Eq. (52c), corresponding to impending slip, the ratio of corresponding transverse forces is:

\[
\frac{V_{n,\min}}{V_{n,n=0}} = \frac{\mu \tan \omega}{e^{\mu \tan \omega \left(\frac{\pi}{2}\right)} - 1} \sqrt{\frac{J_n \phi''}{J_n \phi'}}
\]  \hspace{2cm} (62)

Similarly, for a given layer, correction factors which correspond to beginning and complete slip have a ratio \( \phi''/\phi_n \) which may be approximated in the following way. In Eq. (59b), neglecting the transition from \( \phi_n \) to \( \phi''_n \) in the section evolution process, the calculated transverse force \( V_{n1} \) is smaller than the “true” value. On the other hand, neglecting the variation of section inertia from \( J_n \) to \( J''_n \) leads to a transverse force \( V_{n2} \) which is larger than the “true” value. That “true” value \( V_n \) lies between \( V_{n1} \) and \( V_{n2} \) and an acceptable approximation is given by:

\[
V_n = \frac{V_{n1} + V_{n2}}{2}
\]

Using Eq. (59b), it yields:

\[
\frac{\phi''_n}{\phi_n} \approx \frac{2 \sqrt{J''_n}}{\sqrt{J''_n} + \sqrt{J_n}}
\]  \hspace{2cm} (62a)

and Eq. (62) is transformed to:
\[ \frac{V_{n,\text{min}}}{V_{n,=0}} = \frac{\mu \tan \theta}{e^{\mu \tan \theta (\pi/2)} - 1} \left( \frac{2\sqrt{J_n}}{\sqrt{J_n^2 + J_n}} \right) \] (62b)

In Fig. 7, this equation has been drawn with respect to layer number for three different cables. Curve 1 is for a 91 wires, 44 mm diameter spiral cable, with 4 mm diameter wires. Curve 2 is for a 61 wires, 32 mm diameter spiral cable, with 3.5 mm diameter wires. Finally, curve 3 is for a 37 wires, 28 mm diameter spiral cable, with 4 mm diameter wires. Curves have been calculated taking a friction factor of \( \mu = 0.35 \). Although this value does not have any impact as the expression

\[ \frac{\mu \tan \theta}{e^{\mu \tan \theta (\pi/2)} - 1} \]

has been kept constant in the ensuing calculations. One can see that ratio \( \frac{V_{n,\text{min}}}{V_{n,=0}} \) has a value of about 0.7 (Fig. 7). Such value, much smaller than 1, shows that complete “unlocking” of a layer does not occur all at once, that is, it is a continuous process. However, it is generally sufficient to assume that complete slip occurs instantaneously. The corresponding transverse force is then taken as the average of the \( V_{n,=0} \) and \( V_{n,\text{min}} \) values.

There remains to determine which value should be given to friction coefficient \( \mu \). In the case of a spiral cable, Isaachsen\(^{19} \) takes \( \mu = 0.2 \). The same value is taken by Stephan\(^{20} \) in his calculation of tensile stresses in a spiral cable. According to Benoit\(^{21} \), research data by Hirschland\(^{22} \) on crane cable stiffness tend to show that \( \mu \) is probably greater than 0.4. According to his own tests, Rubin\(^{23} \) finds \( \mu = 0.277 \) for a crane cable, without taking into account inter-strand friction. Performing tensile tests on spliced strands, Overlach\(^{24} \) gets \( \mu = 0.12 \). However, Findeis\(^{25} \) takes \( \mu = 0.35 \) as inter-layer friction factor. Besides, he considers that same layer wires are contacting with a friction factor of \( \mu = 0.2 \), explained

\(^{19}\) Z.V.D.I. 1907, p. 656
\(^{20}\) Dinglers polytechnisches Journal 1909, p. 755
\(^{21}\) Benoit and Woernle, The cable problem, p. 22 et 23
\(^{22}\) Hirschland, On cable deformations, Thesis, Hannover, 1906
\(^{23}\) Rubin, Research on crane and elevator cable strength, Thesis, Karlsruhe, 1920
\(^{24}\) Overlach, On axial splices, Graduation Thesis, Karlsruhe, 1931
\(^{25}\) Findeis, Ropeway design and formulas, p. 69
by an “initial pressure” resulting from the stranding process. Based on tests by Kroen\textsuperscript{26}, which are themselves based on tests by Meyer\textsuperscript{27} and Roch\textsuperscript{28}, he also considers that in a helical strand, broken wires recover their full carrying capacity after between 2 and 2.5 lay lengths.

In a spiral cable, it is well known that at points of contact between adjacent layers, because wires cross each other, pressure may be very high, leading to plastic deformations on top of the elastic ones. As shown by most research data, average friction factor is probably quite high. For small contact areas, as in the present situation, steel/steel tests performed by Rennie\textsuperscript{29} have yielded $\mu = 0.4$. Besides, it should be noted that wire deformation at contact points provide some kind of grip preventing relative motion. Henceforth, it seems realistic to assume a value of $\mu$ between 0.3 and 0.4.

This concludes the study of a cable section evolution under imposed bending. Results obtained by Findeis\textsuperscript{30} on this problem should be mentioned. His results differ so much from the above that the discrepancy ought to be explained. For example, he finds that outer layer impending slip occurs when transverse force is between 1/30 and 1/25 of axial load (corresponding to a stress of 3000 to 4000 kgf/cm\textsuperscript{2}, \textit{i.e.} 300 to 400 MPa). Results obtained through Eq. (52c) are but a fraction of these values.

Findeis’ approach is based on a half-loop only. He assumes, without proving it, that complete wire loop slip occurs when stress limit condition is reached at any point of that loop. As this entails a variation in the bending moment, and assuming further that it is a constant on a half-loop, he assumes a sine variation of the bending stress. Let $b$ be that bending stress (Findeis notation). He derives the limit stress which must act at both ends of the half-loop. According to his theory, it is given by:

$$X = 0.64 b$$

While being the average bending stress on the considered wire cross-sections, this does not represent the sliding force. Obviously, this limit force is the difference between the stresses at both ends, \textit{i.e.}

$$X = \sigma_b - [-\sigma_b] = 2\sigma_b = 2b$$

As this sliding force is too low, I guess that the resulting limit of friction is too high. As it overestimates the friction forces between same layer wires, as well as those arising from the stranding process, it may be concluded that Findeis’ determination for adjacent layer friction limit is incorrect. Let $z$ that wire stress component from cable axial load, and $u$ its component normal to cable axis, according to Findeis:

$$r_3 = u(e^{\mu z} - 1) = z \tan \phi (e^{\mu z} - 1)$$

For a half-loop, $\alpha = \pi$ with $\mu = 0.35$, he gets:

$$r_3 = 2.0 \times z \tan \phi$$

\textsuperscript{26} Österreichische Zeitschrift für Berg- u. Hüttenwesen, 1906, p. 109
\textsuperscript{27} Zeitschrift für Berg-, Hütten u. Salinenwesen, 1885
\textsuperscript{28} Jahrbuch für das Berg- u. Hüttenwesen im Königreich Sachsen, 1898
\textsuperscript{29} Dubbel, Machine Design Pocket Handbook (Summary), 4\textsuperscript{th} Ed., p. 308
\textsuperscript{30} Findeis, Ropeway design and formulas, p. 57 etc.
In this equation, one should replace \( u \), that is \( z \tan \omega \), by the stress minimum value (which takes place on the cable concave side). Besides stress \( z \), there is a bending component. In the equation, one should have \( (z - \sigma_b) \) instead of \( z \). While \( \sigma_b \) is often negligibly small compared to \( z \), this is not always the case. This can be seen in the “Vigiljochbahn” example, where Findeis has computed the compressive stress, in the cable concave side. He finds a value for \( \sigma_b \) which is larger than the axial load stress component \( z \), so that, in that region, wires are in a compression state, which might even lead to buckling!

Even assuming that \( z \) is the minimum stress in the bent wire loop, the preceding equation is slightly incorrect. An improved approach could be as follows (Fig. 5):

Normal contact force between a wire element and the adjacent inner layer: \( z \tan \omega \, d\alpha \)
Friction = stress increase: \( dz = -z \tan \omega \, \mu \, d\alpha \)
Whence: \( z = C e^{-\tan \omega \, \mu \, \alpha} \)
Letting \( \alpha = \pi \), \( z = z_1 \), it yields: \( C = z_1 e^{\tan \omega \, \mu \, \pi} \) and \( z = z_1 e^{\tan \omega \, \mu \, (\pi - \alpha)} \)

Maximum value of stress, which occurs at the loop end (\( \alpha = 0 \)) is given by:

\[ z_2 = z_1 e^{\tan \omega \, \mu \, \pi} \]

The corresponding friction force is the difference between stresses acting at both ends of the loop, that is: \( r_3 = z_2 - z_1 \), yielding:

\[ r_3 = z_1 \left( e^{\tan \omega \, \mu \, \pi} - 1 \right) \]  
(63a)

Instead of this equation, Findeis rather finds:

\[ r_3 = z_1 \tan \omega \left( e^{\mu \, \pi} - 1 \right) \]  
(63b)

For example, taking \( \tan \omega = 0.25 \) and \( \mu = 0.35 \), Eq. (63a) yields:

\[ r_3 = 0.32 \, z_1 \]

while Eq. (63b) yields:

\[ r_3 = 0.5 \, z_1 \]

Calculating ratio V/S, if one takes these corrections into account when determining \( r_3 \) and \( X \), one gets only 1/5 the Findeis value.

*The last three more technologically oriented parts of the paper have not been translated.*