Beam Models

Wenbin Yu*
Utah State University, Logan, Utah 84322-4130

April 13, 2012

1 Introduction

If a structure has one of its dimensions much larger than the other two, such as slender wings, rotor blades, level arms, shafts, channels, bridges and etc., we can simplify the analysis of such structures using beam models. The axis of the beam is defined along that longer dimension, and a cross-section normal to this axis is assumed to smoothly vary along the span of the beam. Although we could use the finite element method to routinely analyze complex structures, simple beam models are often used in the preliminary design stage because they can provide valuable insight into the behavior of the structures with much less effort. There are different beam models with different accuracy. The simplest one is the so-called classical model which can deal with extension, torsion, and bending in two transverse directions. There are at least three ways to derive this beam model: Newtonian method based on free body diagrams, variational method as an application of the Kantorovich method, and variational asymptotic method. Both the Newtonian method and the variational method are based on various ad hoc assumptions including kinematic assumptions such as the Euler-Bernoulli assumptions associated with extension and bending and the Saint Venant assumptions associated with torsion and kinetic assumptions for the 3D stress field within the structure. For this reason, we also term both the Newtonian method and the variational method as ad hoc approaches. Although the classical beam model is also commonly called Euler-Bernoulli beam model, it is misleading as the original Euler-Bernoulli beam model can only deal with extension and bending in two directions. We are usually taught the Newtonian method in our undergraduate study as it is intuitive for understanding. However, it is tedious and error-prone for development of new models and analysis of real structures. On the contrary, the variational method is systematic and easy to handle real structures. Mainly for this reason, the variational method is commonly employed in the literature to derive new models. The variational asymptotic method is a recent addition to the beam literature and it has the merits of the variational method without using ad hoc assumptions. We will present the details of these three methods for

*Associate Professor, Department of Mechanical and Aerospace Engineering.

constructing the classical model for isotropic, homogeneous beams (beam-like structure made of a single isotropic material) to appreciate the advantages and disadvantages of different methods.

Fundamentally speaking, a beam model, no matter how rudimentary or how sophisticated it is, is a one-dimensional (1D) model. It seeks to replace the governing equations of the original three-dimensional (3D) structure into a set of equations in terms of one fundamental variable, the beam axis. In other words, we need to replace the original 3D kinematics, kinetics, and energetics in terms of their 1D counterparts. Such a connection is usually not clearly pointed out in our study of beam theories.

As beam models can be considered as an approximation to the 3D elasticity theory, it is appropriate for us to review the basics of that theory. For simplicity, we restrict ourselves to material and geometric linear problems only. The theory of linear elasticity contains three parts including kinematics, kinetics and energetics. The kinematics deal with a continuous displacement field (u_i) and a continuous strain field (ε_{ij}) satisfying the following strain-displacement relations at any material point in the body:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

The kinetics deals with a continuous stress field (σ_{ij}) satisfying the following equilibrium equations at any material point in the body:

$$\sigma_{ji,j} + f_i = 0 \tag{2}$$

The energetics deals with the constitutive behavior of the material. For isotropic elastic material, it deals with the following constitutive relations satisfied at any material point in the body:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{3}$$

It is commonly called the generalized Hooke's law for isotropic materials. The 6×6 matrix is the compliance matrix with E as the Young's modulus and ν as the Possion's ratio. The constitutive relations in Eq. (3) can be simply inverted to obtain a 6×6 stiffness matrix.

The 15 equations in Eqs. (1), (2), (3) form the complete system to solve the 15 unknowns ($u_i, \varepsilon_{ij}, \sigma_{ij}$, note the symmetry of ε_{ij} and σ_{ij}). Clearly boundary is also part of the body, which implies that the above equations should also hold for points on the boundary. However, along the boundaries, we also know some information which can be considered as given to the solid in question. For example some boundary points are fixed. Hence along the boundary, we have some additional equations to satisfy. If the displacement of some boundary surfaces is prescribed to be u_i^*, then we require the displacement field to satisfy

$$u_i = u_i^* \tag{4}$$

on such boundary surfaces. If the traction of some boundary surfaces is prescribed to be t_i, then we require the stress field to satisfy

$$\sigma_{ij} n_i = t_j \tag{5}$$

2 Ad Hoc Approaches

The starting point of ad hoc approaches is the introduction of a set of kinematics assumptions which enables us to express the 3D displacements in terms of the 1D beam displacements, the 3D strain field in terms of 1D beam strains. Assumptions of the stress field are also used to relate the 3D stress field with the 3D strain field. Although these assumptions are commonly used in our textbooks, they are not emphatically pointed as one set of many possible assumptions. Students might mistakenly think these are the assumptions must be made for beam theory or, even worse, they might think that these assumptions represent a universal truth for beam-like structures. The reality is that these assumptions are usually reasonably justified for isotropic homogeneous beams featuring simple cross-sections and become questionable for beams of complex geometry made of general anisotropic, heterogeneous materials such as composite rotor blades. These assumptions are not absolutely needed if one uses the variational asymptotic method to construct the beam model, as we will show later.

2.1 Kinematics

As we have pointed out that the derivation of the classical beam model using the Newtonian method and the variational method starts from two types of ad hoc assumptions for kinematics: Euler-Bernoulli assumptions associated with extension and bending and Saint Venant assumptions associated with torsion.

2.1.1 The displacement field based on Euler-Bernoulli assumptions

The Euler-Bernoulli assumptions are

1. The cross-section is infinitely rigid in its own plane. Any material point in the plane of the cross-section solely consists of two rigid body translations.

2. The cross-section of a beam remain plane after deformation.

3. The cross-section remains normal to the deformed axis of the beam.

Experimental observations show that these assumptions are reasonable for slender structures made of isotropic materials with solid cross sections subjected to extension or bending deformations. When one or more of these conditions are not met, the classical beam model derived based on these assumptions may be inaccurate. Now, let us discuss the mathematical implication of the Euler-Bernoulli assumptions.

Consider a set of unit vectors $\hat{\boldsymbol{e}}_i$ with coordinates x_i (Here and throughout this chapter, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated

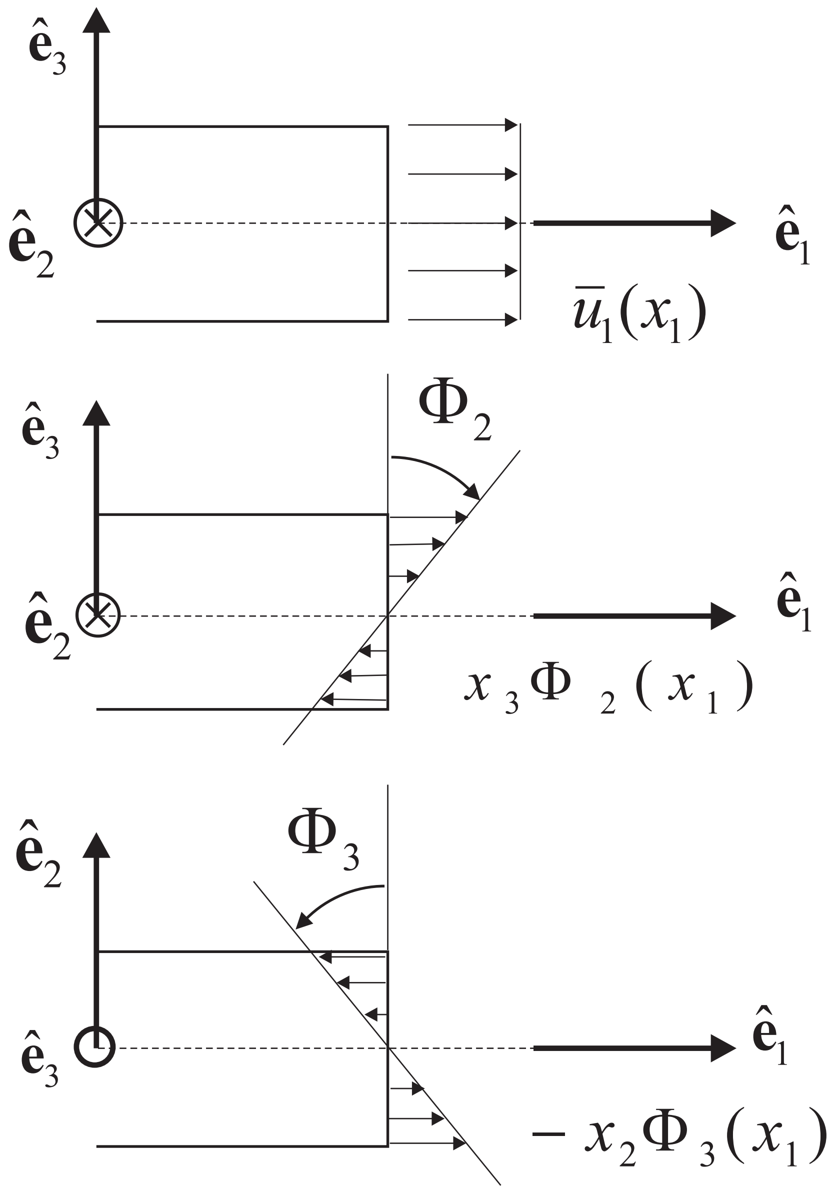

Figure 1: Decomposition of the axial displacement field.

indices are summed over their range except where explicitly indicated). This set of axes is attached at a point of the beam cross-section, $\hat{\boldsymbol{e}}_1$ is along the axis of the beam, and $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ define the plane of the cross-section. Let $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$, and $u_3(x_1, x_2, x_3)$ be the displacement of an arbitrary point of the beam in the $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ directions, respectively.

The first Euler-Bernoulli assumption states that the cross-section is infinitely rigid in its own plane, which implies that the displacement of any material point in the cross-sectional plane solely consists of two rigid body translations $\bar{u}_2(x_1)$ and $\bar{u}_3(x_1)$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1); \qquad u_3(x_1, x_2, x_3) = \bar{u}_3(x_1); \tag{6}$$

The second Euler-Bernoulli assumption states that the cross-section remains plane after deformation. This implies an axial displacement field consisting of a rigid body translation $\bar{u}_1(x_1)$, and two rigid body rotations $\Phi_2(x_1)$ and $\Phi_3(x_1)$, as depicted in Figure 1.

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) + x_3\Phi_2(x_1) - x_2\Phi_3(x_1); \tag{7}$$

Although the center of rotation is not necessarily at the origin of x_α, rotation around any other point can still be expressed using Eq. (7) as any axial displacements introduced by the shifting of the rotation center can be incorporated into the unknown function $\bar{u}_1(x_1)$. Note the sign convention: the rigid body translations of the cross-section $\bar{u}_i(x_1)$ are positive in the direction of the axes $\hat{\boldsymbol{e}}_i$; the rigid body rotations of the cross-section $\Phi_i(x_1)$ are positive if they rotate about the axes $\hat{\boldsymbol{e}}_i$, respectively. Only Φ_2 and Φ_3 are involved in the Euler-Bernoulli assumptions and Φ_1 will be introduced later in the displacement expressions according to the Saint Venant assumptions. Figure 2 depicts these various sign conventions. The reason there is a negative sign in the last term of Eq. (7) is because a positive Φ_3 will create a negative axial displacement for a positive x_2.

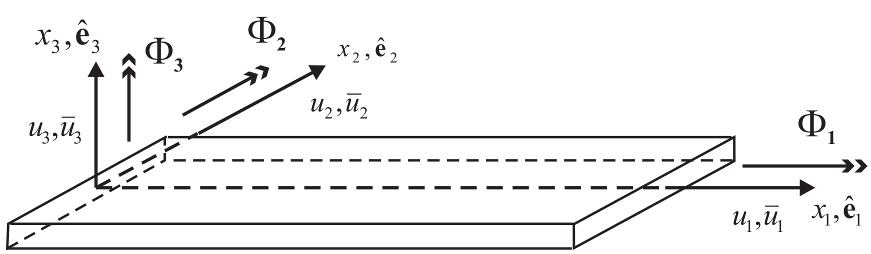

Figure 2: Sign convention for the displacements and rotations of a beam.

The third Euler-Bernoulli assumption states that the cross-section remains normal to the deformed axis of the beam. This implies the equality of the slope of the beam and of the rotation of the section, as depicted in Figure 3

$$\Phi_3 = \bar{u}_2'; \quad \Phi_2 = -\bar{u}_3' \tag{8}$$

where the superscript prime expresses a derivative with respect to x_1. The minus sign in the second equation is a consequence of the sign convention on sectional displacements

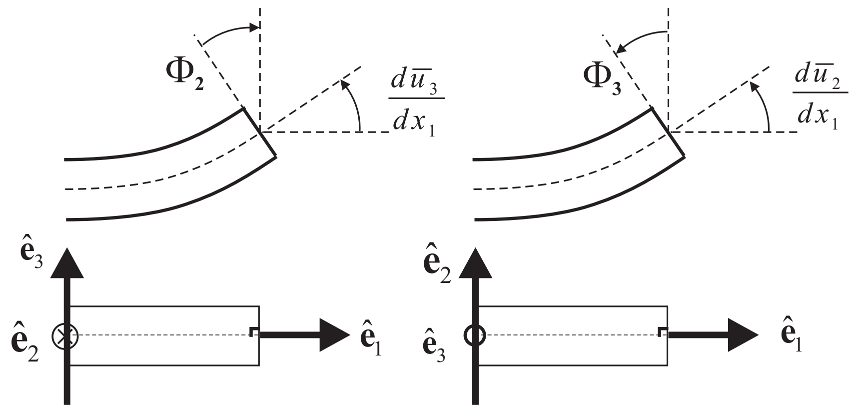

Figure 3: Beam slope and cross-sectional rotation.

and rotations. Substituting Eqs. (8) into Eq. (7), we can eliminate the sectional rotation from the axial displacement field. The complete 3D displacement field for a beam-like structure implied by the Euler-Bernoulli assumptions writes

$$\begin{aligned} u_1(x_1,x_2,x_3) &= \bar{u}_1(x_1) - x_3\bar{u}_3'(x_1) - x_2\bar{u}_2'(x_1); \\ u_2(x_1,x_2,x_3) &= \bar{u}_2(x_1); \\ u_3(x_1,x_2,x_3) &= \bar{u}_3(x_1). \end{aligned} \tag{9}$$

Here, we have to emphasize that the fact that in reality that the 3D displacements $u_i(x_1, x_2, x_3)$ are generally 3D unknown functions of x_1, x_2, x_3 as determined by physics. We have assumed a specific functional form for them in virtue of Euler-Bernoulli assumptions so that u_1 must be a linear combination of x_2, x_3 and some unknown 1D functions $\bar{u}_i$ which must be functions of x_1 only. The Euler-Bernoulli assumptions can be equivalently considered as constraining the structure in such a way that it must behave according to these assumptions, although we might not be able to apply such constraints physically. Because of these constraints, the overall system is more stiffer than the original structure. In other words, for a structure under the same load, displacements u_i obtained using the classical beam model based on the Euler-Bernoulli assumptions will be smaller than those obtained using a theory (for example 3D elasticity) without such assumptions. One or all of the three Euler-Bernoulli assumptions can be removed or replaced by other assumptions. For example, one can remove the third Euler-Bernoulli assumptions, which implies the cross-section remains as a plane during deformation but it not necessarily remains as normal to the beam axis. This is actually the starting point of the derivation of the Timoshenko beam model. As the Timoshenko beam model has one less assumption, it is expected that the displacements obtained by Timoshenko beam model will be larger than those obtained using the classical beam model based on the Euler-Bernoulli assumptions.

2.1.2 The displacement field based on Saint Venant assumptions

The displacement field based on Euler-Bernoulli assumptions, Eqs. (9), have been proven to be adequate for isotropic homogeneous beams featuring extension and bending. However, it provides a poor representation of beams under torsion, which is often present in structures, and in fact many important structural components are designed to carry torsional loads primarily. If the beam is twisted, the cross section will be warped and cannot remain plane in general. For this reason, Saint Venant relaxed the second Euler-Bernoulli assumption and introduced the following assumptions:

1. The shape and size of the cross section in its own plane are preserved, which implies each cross section rotates like a rigid body.
2. The cross section does not remain plane after deformation but warp proportionally according to the rate of twist.
3. The rate of twist is uniform along the beam, which implies the twist angle is a linear function of the beam axis.

According to the first assumption of Saint Venant, the in-plane displacement due to the twist angle $\Phi_1(x_1)$ (see Figure 2) can be described as

$$u_2(x_1, x_2, x_3) = -x_3\Phi_1(x_1); \qquad u_3(x_1, x_2, x_3) = x_2\Phi_1(x_1). \tag{10}$$

According to the second assumption of Saint Venant, the axial displacement field is proportional to the twist rate $\kappa_1 = \Phi_1'$ and has an arbitrary variation over the cross-section described by the unknown warping function $\Psi(x_2, x_3)$ such that

$$u_1(x_1, x_2, x_3) = \Psi(x_2, x_3)\ \kappa_1. \tag{11}$$

It is noted that the twist rate is constant according to the third assumption. $\Psi(x_2, x_3)$ is usually called Saint Venant warping function and solved separately over the cross-sectional domain according to the elasticity theory. It is governed by the following equation

$$\frac{\partial^2\Psi}{\partial x_2^2} + \frac{\partial^2\Psi}{\partial x_3^2} = 0. \tag{12}$$

at all points of the cross-section A along with stress free boundary conditions along the boundary curve of the cross-section. The warping function vanishes for a isotropic, homogenous, circular cross-section.

It is common that the beam could subjected to extension, bending, and torsion. We need to develop a model which could simultaneously handle all these deformation modes. This can be achieved by combining the displacement expressions in Eqs. (9), Eqs. (10), and Eq. (11) such that

$$\begin{aligned} u_1(x_1, x_2, x_3) &= \bar{u}_1(x_1) - x_3\bar{u}_3'(x_1) - x_2\bar{u}_2'(x_1) + \Psi(x_2, x_3)\ \kappa_1; \\ u_2(x_1, x_2, x_3) &= \bar{u}_2(x_1) - x_3\Phi_1(x_1); \\ u_3(x_1, x_2, x_3) &= \bar{u}_3(x_1) + x_2\Phi_1(x_1). \end{aligned} \tag{13}$$

Clearly, the complete 3D displacement field of the beam can be expressed in terms of three sectional displacements $\bar{u}_1(x_1)$, $\bar{u}_2(x_1)$, $\bar{u}_3(x_1)$ and one sectional rotation $\Phi(x_1)$. This important simplification resulting from the Euler-Bernoulli assumptions and Saint Venant assumptions allows the development of the classical beam model in terms of $\bar{u}_i$ and Φ, which are unknowns functions of the beam axis x_1 only, a 1D formulation. In other words, through these two types of assumptions, we relate the 3D displacements, $u_i(x_1, x_2, x_3)$, in terms of 1D beam displacements, $\bar{u}_i(x_1), \Phi_1(x_1)$.

2.1.3 The strain field

To deal with geometrical linear problem, we use the infinitesimal strain field defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{14}$$

Substituting the displacement field in Eqs. (13), we obtain the following 3D strain field as

$$\varepsilon_{11}(x_1, x_2, x_3) = \bar{u}_1'(x_1) - x_3\bar{u}_3''(x_1) - x_2\bar{u}_2''(x_1). \tag{15}$$

$$\varepsilon_{22} = \varepsilon_{33} = 2\varepsilon_{23} = 0 \tag{16}$$

$$2\varepsilon_{12} = \left(\frac{\partial \Psi}{\partial x_2} - x_3\right) \kappa_1; \quad 2\varepsilon_{13} = \left(\frac{\partial \Psi}{\partial x_3} + x_2\right) \kappa_1; \tag{17}$$

At this point it is convenient to introduce the following notation for the 1D beam strains

$$\epsilon_1(x_1) = \bar{u}_1'(x_1); \quad \kappa_1(x_1) = \Phi_1'(x_1) \quad \kappa_2(x_1) = -\bar{u}_3''(x_1); \quad \kappa_3(x_1) = \bar{u}_2''(x_1). \tag{18}$$

where ϵ_1 is the axial strain, κ_1 the twist rate, and κ_2 and κ_3 the curvature about the axes $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$, respectively. Eq. (18) can be considered as the 1D beam strain-displacement relations. It is pointed out here that expressing the twist rate κ_1 as a function of x_1 is a direct violation of the third Saint Venant assumption of uniform torsion which is used to obtain the expression for ε_{11} in Eq. (15). Nevertheless, it is a common practice that the classical beam model derived based on this assumption is frequently used to analyze beams with twist rates varying along the beam axis. This type of inconsistency frequently happens for models derived using ad hoc assumptions such as the Euler-Bernoulli assumptions and Saint Venant assumptions.

Using the definition in Eqs. (18), we can express the axial strain distribution ε_{11} in Eq. (15) as

$$\varepsilon_{11}(x_1, x_2, x_3) = \epsilon_1(x_1) + x_3\kappa_2(x_1) - x_2\kappa_3(x_1); \tag{19}$$

The vanishing of the in-plane strain field as implied by Eq. (16) is a direct consequence of assuming the cross-section to be infinitely rigid in its own plane. The strain measures ϵ_1, κ_i are usually collectively terms as classical beam strain measures. The original 3D strain field is expressed in terms of the classical beam strain measures, which are 1D functions of x_1 and we have now completed the expressions for 3D kinematics including the displacement field $u_i(x_1, x_2, x_3)$ and the strain field $\varepsilon_{ij}(x_1, x_2, x_3)$ in terms of 1D kinematics including beam displacement variables $\bar{u}_i(x_1), \Phi_1(x_1)$ and the classical beam strain measures $\epsilon_1(x_1), \kappa_i(x_1)$.

2.2 Kinetics

Having known the strain field, we can obtain the stress field in the beam using the generalized 3D Hooke's law if the material is linear elastic. For example, for an isotropic material, we have

$$\sigma_{11} = (\lambda{+}2G)\varepsilon_{11} \qquad \sigma_{22} = \sigma_{33} = \lambda\varepsilon_{11} \qquad \sigma_{12} = 2G\varepsilon_{12} \qquad \sigma_{13} = 2G\varepsilon_{13} \qquad \sigma_{23} = 0 \tag{20}$$

where $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $G = \frac{E}{2(1+\nu)}$ is the shear modulus. Although this stress field naturally flow from the generalized Hooke's law, it does not agree with the experimental measurements very well. We have to introduce additional assumptions regarding the stress field to provide more accurate approximation of the reality. Because the dimension of the cross-section is much smaller comparing to the length of the beam axis, we can assume that $\sigma_{22} \approx 0$ and $\sigma_{33} \approx 0$ in comparison to σ_{11}. This assumption clearly conflicts with the stress field in Eq. (20) obtained from the strain field which is obtained from the displacement field based on the Euler-Bernoulli assumptions and the Saint Venant assumptions. The reason is that the first assumption of Euler-Bernoulli and Saint Venant, cross section remains rigid in its own plane, clearly violates the reality. We all know that when beam is deformed, the cross section will deform in its own plane due to Poisson's effect. For this very reason, we overrule the previous assumptions for obtaining kinematics and introduce the following assumptions for the stress field:

$$\sigma_{22} = \sigma_{33} = \sigma_{23} = 0 \tag{21}$$

With this additional assumption, we end up with the following stress field:

$$\sigma_{11} = E\varepsilon_{11} \qquad \sigma_{12} = 2G\varepsilon_{12} \qquad \sigma_{13} = 2G\varepsilon_{13} \qquad \sigma_{22} = \sigma_{33} = \sigma_{23} = 0 \tag{22}$$

Clearly the above stress field is not the same as those in Eq. (20), which implies that the stress field in Eq. (22) conflicts with our starting Euler-Bernoulli and Saint Venant assumptions. In fact, it is obtained from the strain-stress relations for isotropic materials with the assumption that $\sigma_{22} = \sigma_{33} = 0$, which implies in fact $\varepsilon_{22} = \varepsilon_{33} = -\nu/E\varepsilon_{11}$ as a direct consequence of the Hooke's law. This implication contradicts with the strain field in Eq. (16) obtained using the Euler-Bernoulli assumptions and the Saint Venant assumptions except when $\nu = 0$ which in general is not true. The kind of contradictions are common in structural models derived based on ad hoc assumptions. Nevertheless, such inconsistencies are used in the derivation of the classical beam model and commonly taught in textbooks. These contractions can be partially justified by the fact that we need to rely on the Euler-Bernoulli assumptions and Saint Venant assumptions to obtain a simple expression of the 3D kinematics in terms of 1D kinematics and we also use the stress assumptions in Eq. (21) so that the results can better agree with reality. A sad fact is that such inconsistencies are seldom clearly pointed out and criticized. As a summary, to derive the classical beam model based on ad hoc assumptions, we have to first use Euler-Bernoulli assumptions and Saint Venant assumptions to related 3D kinematics with 1D kinematics, and then use the stress assumption in Eq. (21) to obtain the 3D stress field. In other words, in our further derivations, we use the 3D strains as expressed in Eqs. (19), (16), and (17) and the 3D stresses as expressed in Eq. (22), despite of the fact that they are obtained through a set of conflicting assumptions.

We also need to note that the transverse shear stresses in Eq. (22) here are only caused by twist in view of Eq. (17) and the transverse shear stresses due to flexure, denoted as $\sigma^*_{1\alpha}$ for distinction, cannot be obtained this way. This is due to the third Euler-Bernoulli assumption. When we assume cross-section remains normal to the beam axis during deformation, we effectively assume that the beam is infinitely rigid in transverse shear in flexure. Hence, the transverse shear stresses due to flexure, although exist in

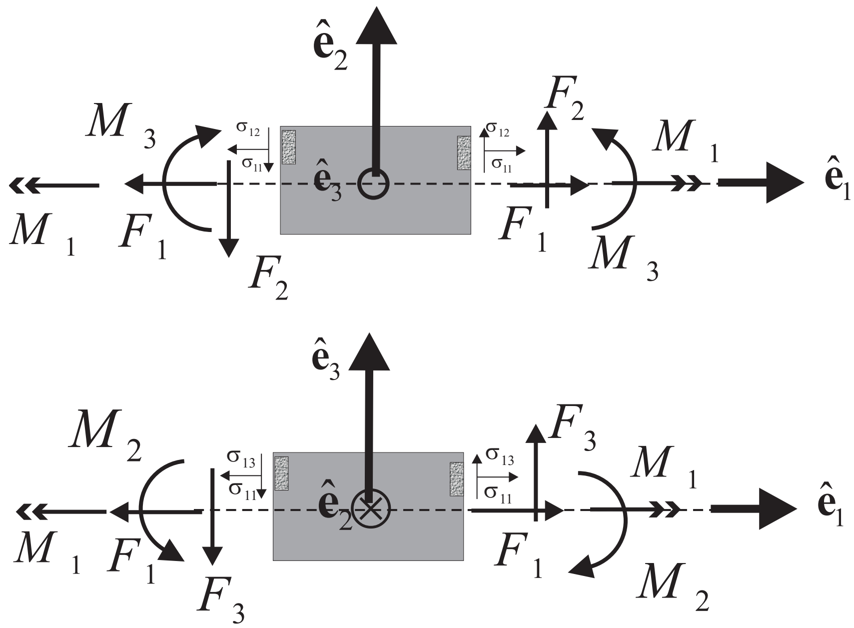

Figure 4: Sign convention for sectional stress resultants

general, cannot be obtained based on constitutive relations but must be determined from equilibrium considerations as will be shown later.

To complete the 1D beam model, we also need to introduce a set of 1D kinetic variables called sectional stress resultants to relate with it 3D counterparts, the 3D stress field. The sectional stress resultants are defined as follows:

$$
\begin{aligned}
&\int \sigma_{11} \mathrm{d}A = F_1 \\
&\int (\sigma_{13} x_2 - \sigma_{12} x_3) \mathrm{d}A = M_1 \\
&\int \sigma_{11} x_3 \mathrm{d}A = M_2 \\
&\int \sigma_{11} x_2 \mathrm{d}A = -M_3
\end{aligned} \tag{23}
$$

where A denotes the cross-sectional domain. The sign convention is determined by the definition as depicted in Figure 4 for a differential beam segment. F_α are not defined similarly as the first equation in Eq. (23) as we have pointed out that $\sigma_{1\alpha}$ in Eq. (22) are only twist-induced transverse shear stresses which are statically equivalent to the twisting moment M_1 only. Thus, the corresponding transverse stress resultants due to twist-induced transverse shear stresses will vanish. For this reason, we define the transverse shear resultants in terms of the flexure-induced transverse shear stresses instead, such that

$$
\int \sigma^*_{1\alpha} \mathrm{d}A = F_\alpha \tag{24}
$$

As will be shown later, F_α are not kinetic variables in the 1D classical beam model and are only used for deriving the equilibrium equations using the Newtonian approach.

It is timely noted that to complete the kinetics part, we need to establish governing equations among the 1D kinetic variables F_1, M_i which will be furnished by either Newtonian method or the variational method later.

2.3 Energetics

Substituting the 3D strain field in Eqs. (19), (16), and (17) into the 3D stresses in Eq. (22), then into Eq. (23), we have

$$\begin{aligned} F_1 &= \int E\left(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3\right) \mathrm{d}A = S_{11}\epsilon_1 + S_{13}\kappa_2 + S_{14}\kappa_3 \\ M_1 &= \int G\left(x_2(\Psi_{,3} + x_2) - x_3(\Psi_{,2} - x_3)\right)\kappa_1 \mathrm{d}A = S_{22}\kappa_1 \\ M_2 &= \int x_3 E\left(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3\right) \mathrm{d}A = S_{13}\epsilon_1 + S_{33}\kappa_2 + S_{34}\kappa_3 \\ M_3 &= -\int x_2 E\left(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3\right) \mathrm{d}A = S_{14}\epsilon_1 + S_{34}\kappa_2 + S_{44}\kappa_3 \end{aligned} \tag{25}$$

with

$$\begin{aligned} S_{11} &= \int E \mathrm{d}A \qquad S_{13} = \int E x_3 \mathrm{d}A \qquad S_{14} = -\int E x_2 \mathrm{d}A \\ S_{22} &= \int G\left(x_2^2 + x_3^2 + x_2\Psi_{,3} - x_3\Psi_{,2}\right) \mathrm{d}A \\ S_{33} &= \int E x_3^2 \mathrm{d}A \qquad S_{34} = -\int E x_2 x_3 \mathrm{d}A \qquad S_{44} = \int E x_2^2 \mathrm{d}A \end{aligned} \tag{26}$$

These are commonly called beam stiffness. As the beam is made of a single isotropic material, then the constants E and G can be factored out.

Eq. (25) can be rewritten in the following matrix form

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & 0 & S_{13} & S_{14} \\ 0 & S_{22} & 0 & 0 \\ S_{13} & 0 & S_{33} & S_{34} \\ S_{14} & 0 & S_{34} & S_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \tag{27}$$

Here S_{11} is the extension stiffness, S_{13} and S_{14} are the extension-bending coupling stiffness, S_{22} is the torsional stiffness, S_{33} and S_{44} are the bending stiffness, S_{34} is the cross bending stiffness. Eq. (27) can be considered as the constitutive relations for the classical beam model, the 1D counterpart of the 3D generalized Hooke's law. The 4×4 symmetric matrix is commonly called classical beam stiffness matrix. Because of the assumptions we have used and the restriction that our beam is made of a single isotropic material, the torsional behavior is automatically decoupled from extension and bending, implied by the fact that the entries on the second row and second column are zero except the diagonal term, the torsional stiffness. That is also the reason that why in our undergraduate study, torsion

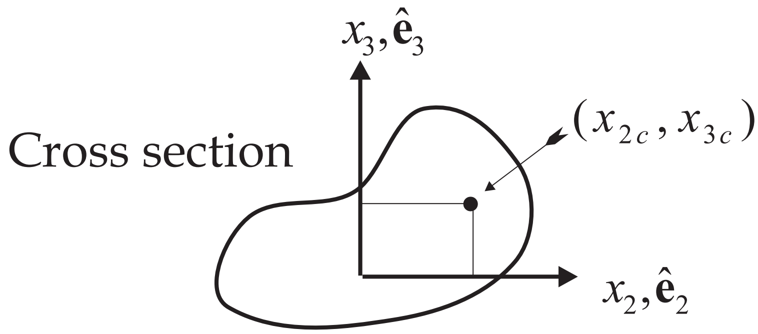

Figure 5: Sketch of extension center location.

is taught separately from extension and bending. For a composite beam structure, the stiffness matrix could be fully populated such that

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \qquad \begin{Bmatrix} \epsilon_1 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{bmatrix} \begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{28}$$

The 4×4 matrix in the right equation is commonly called compliance matrix which is the inverse of the stiffness matrix. The stiffness matrix (thus the compliance matrix) for composite beams cannot be simply evaluated using the integrals in Eq. (26) but a numerical approach is usually needed.

2.3.1 Extension center

It is also possible to decouple extension and bending by choosing the origin of the coordinates x_α in such a way that extension-bending coupling stiffness computed with respect to this newly chosen origin vanish. Such a point is normally called the centroid but we prefer the name of extension center for the reason that when an extension force is applied at this point, no bending deformation will be caused. Suppose we choose the origin at the extension center with location at (x_{2c}, x_{3c}) (see Figure 5) with respect to the original x_i coordinate system, then we have

$$\begin{aligned} S_{13}^* &= \int E(x_3 - x_{3c})\mathrm{d}A = 0 \Longrightarrow x_{3c} = \frac{S_{13}}{S_{11}} \\ S_{14}^* &= -\int E(x_2 - x_{2c})\mathrm{d}A = 0 \Longrightarrow x_{2c} = \frac{-S_{14}}{S_{11}} \end{aligned} \tag{29}$$

In other words, if we know the classical stiffness matrix in an arbitrary coordinate, we can compute the extension center according to the above formulas. A coordinate system x_i^* with the origin of x_α^* located at (x_{2c}, x_{3c}) and $x_1^* = x_1$ is called the centroidal coordinate system of the beam. Although as long as the origin is at the centroid, the coordinate system is called centroidal coordinate system. We usually also choose x_α^* to be parallel to our original coordinate system x_α. All our previous formulations remain exactly the same in this new coordinate system (we need to replace x_i with x_i^*, of course), except that the constitutive relations will become

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2^* \\ M_3^* \end{Bmatrix} = \begin{bmatrix} S_{11} & 0 & 0 & 0 \\ 0 & S_{22} & 0 & 0 \\ 0 & 0 & S_{33}^* & S_{34}^* \\ 0 & 0 & S_{34}^* & S_{44}^* \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \kappa_1 \\ \kappa_2^* \\ \kappa_3^* \end{Bmatrix} \tag{30}$$

Here superscript star is used to denoted that these quantities are computed with respect to the centroidal coordinate system x_i^* and $S_{33}^*, S_{34}^*, S_{44}^*$ are different from these values in the original arbitrary coordinate system x_i.

2.3.2 Principal bending axes

Because of the special choice of centroidal coordinate system, extension is now decoupled from the bending in the classical beam stiffness matrix. However, the bending deformations in two directions are still coupled if the cross bending stiffness S_{34}^* is not zero. What we can is to rotate the coordinates x_α^* in such a way that cross bending stiffness computed with the rotated axes vanish. Denote the rotated axes as $\bar{x}_\alpha$, we call the coordinate system $\bar{x}_i$ with $\bar{x}_1 = x_1^*$ as the principal centroidal axes of bending. According to what we have learned in our undergraduate solid mechanics, the rotated angle can be computed from the following expressions

$$\sin 2\alpha = -\frac{S_{34}^*}{H} \qquad \cos 2\alpha = \frac{S_{44}^* - S_{33}^*}{2H} \tag{31}$$

with $H = \sqrt{\frac{(S_{44}^* - S_{33}^*)^2}{4} + (S_{34}^*)^2}$. In the principal centroidal coordinate system of bending, we have

$$\bar{S}_{34} = 0 \qquad \bar{S}_{33} = \frac{S_{44}^* + S_{33}^*}{2} - H \qquad \bar{S}_{44} = \frac{S_{44}^* + S_{33}^*}{2} + H \tag{32}$$

$\bar{S}_{33}$ and $\bar{S}_{44}$ are called principle bending stiffnesses. In other words in the principal centroidal coordinate system, the stifffness matrix becomes a diagonal matrix and we can completely decouple all the four fundamental deformation modes (extension, torsion, and bending in two directions) and we can study each deformation separately and that is what exactly we have learned in our undergraduate studies. To achieve such decoupling, we need to first locate the centroid of the cross-section, then we need to identify the principal bending directions of the cross-section, and finally write our equations in the principal centroidal coordinate system. More specifically, we need to follow the following steps:

- Compute the stiffness matrix Eq. (27) according to Eqs. (26) in any arbitrary user chosen coordinate system.
- Locate the centroid according to the formulas in Eq. (29). Compute $S_{33}^*, S_{34}^*, S_{44}^*$ in the centroidal coordinate system x_i^*.
- Locate the direction of principal bending axes by computing the rotating angles according to Eq. (31), and compute the principal bending stiffness according to Eq. (32).

The principal centroidal coordinate system $\bar{x}_i$ has the origin located at the centroid and $\bar{x}_\alpha$ align with the principal bending axes. The formulations we have thus far remain the same for the principal centroidal coordinate system except the classical stiffness matrix becomes a diagonal matrix with $S_{11}, S_{22}, \bar{S}_{33}, \bar{S}_{44}$ on the diagonal. Note, each cross-section only has one unique principal centroidal coordinate system. For simple cross-sections it is easy to identify such coordinate system. For realistic structures, it is not trivial to do so. Particularly, for a general composite beam, the 4×4 classical beam stiffness matrix could be fully populated which means all the four deformation modes can be fully coupled and such a decoupling may only cause confusion and not of much use any longer.

2.3.3 Extension center of composite beams

To obtain the extension center and principal bending axes for a general composite beam is more involved than the simple formulas we obtained previously for isotropic homogenous beams. Nevertheless, they can be obtained by slightly modifying the definitions of extension center and principal bending axes. As all the deformation modes of the classical beam model could be fully coupled, the extension and principal bending axes can only be rigorously defined at the cross-sectional level. In other words, we modify our definition of extension center as the point on the cross-section when only axial force resultant F_1 is applied at this point, no bending curvatures will be caused. Imagining F_1 is applied at the centroid in Figure 5, pointing toward the reader, then with respect to the coordinate system x_i, F_1 will also generate bending moments $M_2 = F_1 x_{3c}$ and $M_3 = -F_1 x_{2c}$. Using the second equation in Eq. (28), we have

$$\begin{Bmatrix} \epsilon_1 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{bmatrix} \begin{Bmatrix} F_1 \\ 0 \\ F_1 x_{3c} \\ -F_1 x_{2c} \end{Bmatrix} \tag{33}$$

The definition of extension center requires $\kappa_2 = \kappa_3 = 0$, which implies the following two equations

$$(c_{13} + c_{33} x_{3c} - c_{34} x_{2c}) = 0 \qquad (c_{14} + c_{34} x_{3c} - c_{44} x_{2c}) = 0 \tag{34}$$

which can be used to locate the position of extension center as

$$x_{2c} = \frac{c_{14} c_{33} - c_{13} c_{34}}{c_{33} c_{44} - c_{34}^2} \qquad x_{3c} = \frac{c_{14} c_{34} - c_{13} c_{44}}{c_{33} c_{44} - c_{34}^2} \tag{35}$$

It can be easily verified that the above formula will be reduced to be the same as those in Eq. (5) if the flexibility constants are obtained by inverting the stiffness matrix in Eq (27) for an isotropic homogenous beam.

Relocating the origin of the coordinate system to the extension center, we can obtain the centroidal coordinate system and the compliance matrix in this coordinate system will have the following form.

$$\begin{Bmatrix} \epsilon_1 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} c_{11}^* & c_{12}^* & 0 & 0 \\ c_{12}^* & c_{22}^* & c_{23}^* & c_{24}^* \\ 0 & c_{23^*} & c_{33}^* & c_{34}^* \\ 0 & c_{24}^* & c_{34}^* & c_{44}^* \end{bmatrix} \begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{36}$$

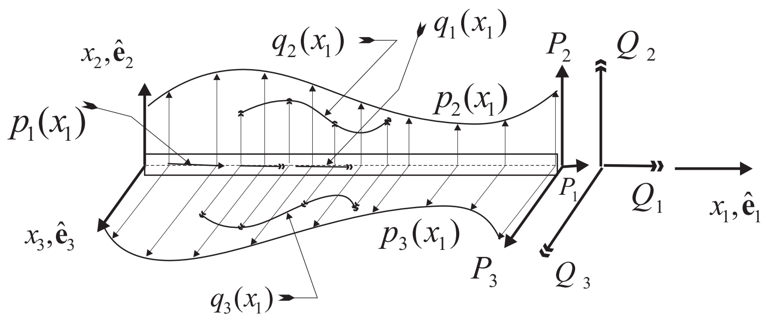

Figure 6: Beam with arbitrary three-dimensional loading.

Note all the beam strains and sectional resultants are defined in the centrodial coordinate system x_i^*. Note here, even if we vanished the two extension-bending coupling entries in the compliance matrix, the extension is still coupled with the torsion which itself could be coupled with the bending. That is to say, we can not completely decouple extension from bending for general composite beams. This is the reason why I believe extension center and principal axes of bending are not very meaningful quantities for composite beams.

2.4 Equilibrium equations

In our classical beam problem, we are solving for the unknown beam displacements $(\bar{u}_i, \Phi_1)$, bean strains (ϵ_1, κ_i), and stress resultants (F_1, M_i), a total of 12 unknowns. Thus far, we have obtained four equations for the 1D strain-displacement relations in Eq. (18), and four equations for the 1D constitutive relations in Eq. (27), a total of eight equations. We are lacking of four equations to form a complete system. These four equations can be derived using either Newtonian method or variational method.

2.4.1 Newtonian method

To use Newtonian method to derive the equilibrium equations of the classical beam model, we need to consider the equilibrium of a differential beam element using some free body diagrams, which is the focus of this section.

Consider a beam of arbitrary cross-sectional shape subjected to a complex three-dimensional loading as sketched in Figure 6. This loading consists of distributed and concentrated axial and transverse loads, as well as distributed and concentrated moments. The axial and transverse distributed loads $p_1(x_1)$, $p_2(x_1)$, and $p_3(x_1)$ act in the direction $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$, respectively. The same convention is used for the concentrated loads P_1, P_2, and P_3. The distributed moments $q_1(x_1), q_2(x_1)$ and $q_3(x_1)$ act about the axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$, respectively. The concentrated moments Q_1, Q_2, and Q_3 act about the same axes. Figure 6 depicts concentrated forces and moments acting at the tip of the beam, but in practical situations, such concentrated loads could be applied at any span-wise location. Note here, we consider the distributed loads in terms of distributed forces p_i and distributed moments q_i acting at the origin of x_α and they are functions of x_1 only. In other words the distribution is only along the beam axis and not distributed along the

Loading Type	Notation	Units
Distributed loads	$p_1(x_1), p_2(x_1), p_3(x_1)$	N/m
Concentrated loads	P_1, P_2, P_3	N
Distributed moments	$q_1(x_1), q_2(x_1), q_3(x_1)$	N·m/m
Concentrated moments	Q_1, Q_2, Q_3	N·m

Table 1: Loading components acting on the beam.

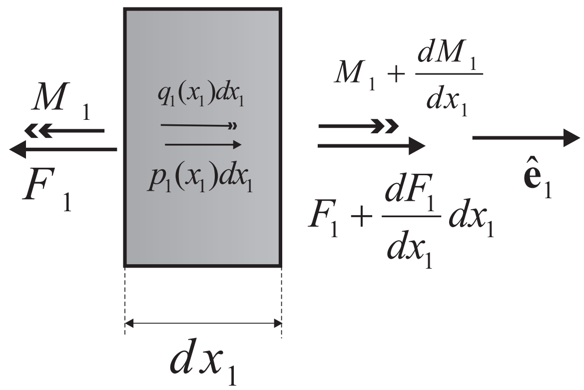

Figure 7: Free body diagram for the axial forces.

cross-section. In reality, in the original 3D structure, within the framework of 3D elasticity, there are distributed body forces as functions of x_i and distributed surfaces tractions along the boundary surfaces. The 1D loads should relate with the 3D loads in such a way that they are statically equivalent: summation of forces and summation of moments in three directions of the 3D loads should be equal to those of the 1D loads. How to achieve it systematically will be given in the next section when we derive the classical beam model using the variational method. The notation used here for the various loads is summarized in Table 1.

The equilibrium equations can be derived considering free body diagrams of a differential beam element. Let us focus on the equilibrium along the beam axis direction first. Consider an infinitesimal slice of the beam of length $\mathrm{d}x_1$ as depicted in Figure 7. Summing all the forces in the axial direction yields the following equation

$$\frac{\mathrm{d}F_1}{\mathrm{d}x_1} = -p_1(x_1) \tag{37}$$

Summing all the moments in the axial direction yields the following equation

$$\frac{\mathrm{d}M_1}{\mathrm{d}x_1} = -q_1(x_1) \tag{38}$$

The first sketch in Figure 8 depicts the transverse loads and bending moments acting on an infinitesimal slice of the beam, focusing on the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$ plane. A summation of the

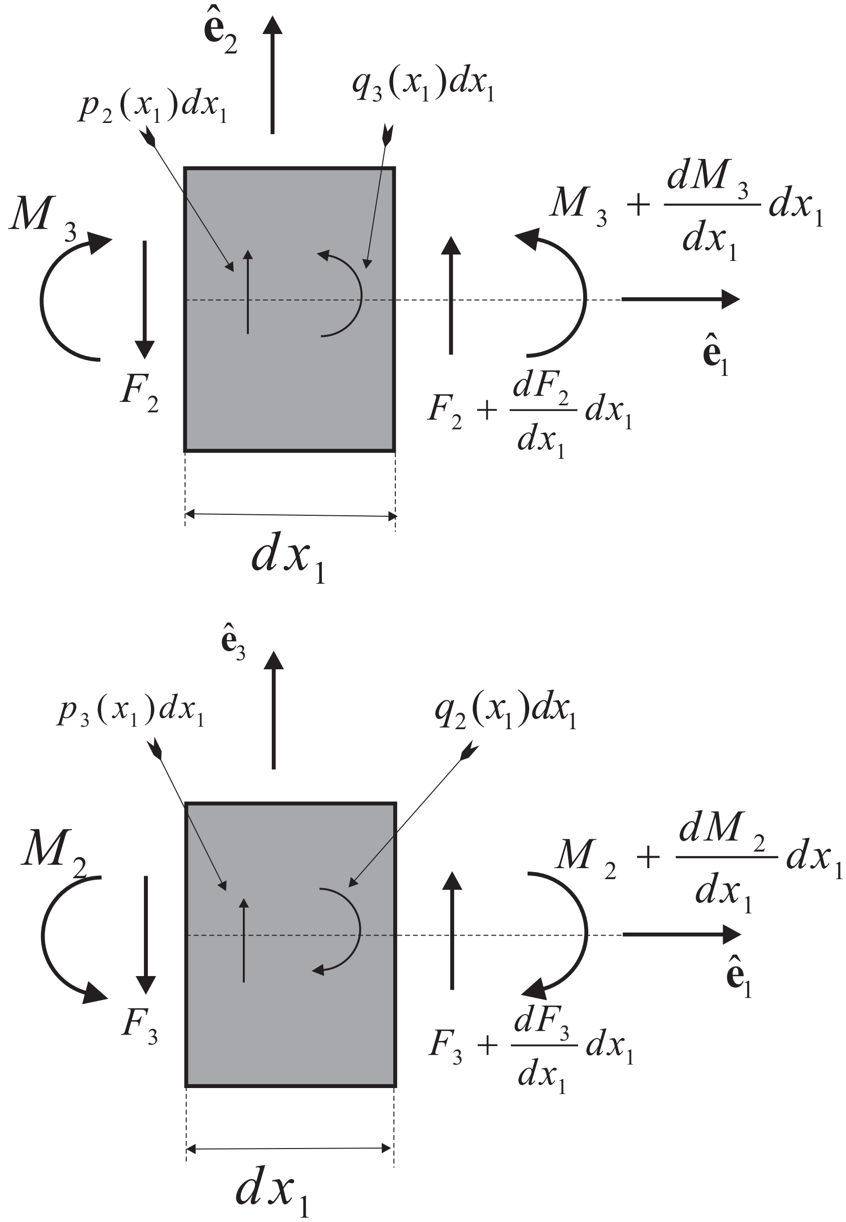

Figure 8: Free body diagram for the transverse shear forces and bending moments.

forces along axis $\hat{\boldsymbol{e}}_2$ gives the transverse force equilibrium equation in this direction

$$\frac{\mathrm{d}F_2}{\mathrm{d}x_1} = -p_2(x_1). \tag{39}$$

A summation of the moments taken about the origin about an axis parallel to $\hat{\boldsymbol{e}}_3$ yields

$$\frac{dM_3}{\mathrm{d}x_1} + F_2 = -q_3(x_1). \tag{40}$$

Similarly, the second sketch of Figure 8 also depicts the transverse loads and bending moments acting on an infinitesimal slice of the beam, focusing on the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_3)$ plane. Summing the forces along axis $\hat{\boldsymbol{e}}_3$ gives the second transverse force equilibrium equation

$$\frac{\mathrm{d}F_3}{\mathrm{d}x_1} = -p_3(x_1), \tag{41}$$

and the summing of the moments taken about the origin about an axis parallel to $\hat{\boldsymbol{e}}_2$ leads to

$$\frac{dM_2}{\mathrm{d}x_1} - F_3 = -q_2(x_1). \tag{42}$$

The shear forces F_2 and F_3 can be eliminated from the equilibrium equations by taking a derivative of Eqs. (42) and (40), then introducing Eqs. (41) and (39), respectively, to yield the bending moment equilibrium equations

$$\frac{d^2M_2}{\mathrm{d}x_1^2} = -p_3(x_1) - \frac{\mathrm{d}q_2}{\mathrm{d}x_1}; \tag{43}$$

$$\frac{d^2M_3}{\mathrm{d}x_1^2} = p_2(x_1) - \frac{\mathrm{d}q_3}{\mathrm{d}x_1}. \tag{44}$$

The four equations in Eqs. (37), (38), (43), (44) are the last four equations we need to complete the classical beam theory.

Substituting the 1D strain-displacement relations in Eq. (18) into the 1D constitutive relations in Eq. (27), then into the four 1D equilibrium equations, we obtain the following displacement formulation of the classical beam theory:

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(S_{11}\bar{u}_1' - S_{13}\bar{u}_3'' + S_{14}\bar{u}_2''\right) = -p_1 \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(S_{22}\Phi_1'\right) = -q_1 \tag{46}$$

$$\frac{d^2}{\mathrm{d}x_1^2}\left(S_{13}\bar{u}_1' - S_{33}\bar{u}_3'' + S_{34}\bar{u}_2''\right) = -p_3(x_1) - \frac{\mathrm{d}q_2}{\mathrm{d}x_1} \tag{47}$$

$$\frac{d^2}{\mathrm{d}x_1^2}\left(S_{14}\bar{u}_1' - S_{34}\bar{u}_3'' + S_{44}\bar{u}_2''\right) = p_2(x_1) - \frac{\mathrm{d}q_3}{\mathrm{d}x_1} \tag{48}$$

Note if we strictly abide our Saint Venant assumptions that $\kappa_1 = \Phi_1'$ is a constant, the second equation will be true only if $q_1(x_1)$ vanish. However, in general for a beam structure, we will have such distributed moment exist and we usually choose to violate the Saint Venant assumption. In other words, κ_1 is determined by the second equation and consider Saint Venant assumption was used as an approximation for obtaining the displacement field and the strain field.

A total of 12 boundary conditions are needed for us to determine the constants associated with solving the system of differential equations. Supposing the beam is clamped at its root $x_1 = 0$, we will have the following six boundary conditions at this point:

$$\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0; \quad \Phi_1 = 0; \quad \bar{u}_2' = \bar{u}_3' = 0 \tag{49}$$

which corresponds to vanishing displacements and rotations at the root of the beam in view of Eq. (8) due to the third Euler-Bernoulli assumption. If the beam is also subjected to concentrated loads (P_i) and moments (Q_i) at its tip ($x_1 = L$), then we have the following another six boundary conditions at this point

$$F_1 = P_1; \quad F_2 = P_2; \quad F_3 = P_3; \quad M_1 = Q_1; \quad M_2 = Q_2; \quad M_3 = Q_3 \tag{50}$$

Introducing the sectional constitutive laws, Eq. (27) and using the definition of the sectional strains Eq. (18) and the equilibrium equations in Eqs. (40) and (42) yields the boundary conditions in Eq. (50) expressed in terms of displacements as

$$\begin{aligned}
S_{11}\bar{u}_1' - S_{13}\bar{u}_3'' + S_{14}\bar{u}_2'' &= P_1 \\
-\frac{\mathrm{d}}{\mathrm{d}x_1}\left(S_{14}\bar{u}_1' - S_{34}\bar{u}_3'' + S_{44}\bar{u}_2''\right) &= P_2 + q_3(L) \\
\frac{\mathrm{d}}{\mathrm{d}x_1}\left(S_{13}\bar{u}_1' - S_{33}\bar{u}_3'' + S_{34}\bar{u}_2''\right) &= P_3 - q_2(L) \\
S_{22}\Phi_1' &= Q_1 \\
S_{13}\bar{u}_1' - S_{33}\bar{u}_3'' + S_{34}\bar{u}_2'' &= Q_2 \\
S_{14}\bar{u}_1' - S_{34}\bar{u}_3'' + S_{44}\bar{u}_2'' &= Q_3
\end{aligned} \tag{51}$$

The governing equations of the problem are in the form of the four coupled differential equations (45), (46), (47), and (48), for the four beam displacements $\bar{u}_i$ and Φ_1. The equations are second order in the axial displacement $\bar{u}_1$ and Φ_1, and fourth order in the transverse displacements $\bar{u}_2$, and $\bar{u}_3$. There are 12 associated boundary conditions, six at each end of the beam Eqs. (49) and (51). Boundary conditions corresponding to various end configurations can be derived based on equilibrium considerations using free body diagrams at the boundary point.

If we choose the principal centroidal axes $\bar{x}_i$ of bending as the coordinate system, the four classical deformation modes (extension, twist, and bending directions) will be completely decoupled and the corresponding governing different equations are simplified as:

$$(S_{11}\bar{u}_1')' = -p_1 \quad (S_{22}\Phi_1')' = -q_1 \quad (S_{33}\bar{u}_3'')'' = p_3(x_1) + q_2' \quad (S_{44}\bar{u}_2'')'' = p_2(x_1) - q_3' \tag{52}$$

The boundary conditions in Eq. (51) will be simplified as

$$\begin{aligned}
&S_{11}\bar{u}_1' = P_1 \quad -(S_{44}\bar{u}_2'')' = P_2 + q_3(L) \quad -(S_{33}\bar{u}_3'')' = P_3 - q_2(L) \\
&S_{22}\Phi_1' = Q_1 \quad S_{33}\bar{u}_3'' = -Q_2 \quad S_{44}\bar{u}_2'' = Q_3
\end{aligned} \tag{53}$$

How to solve these equations has been extensively covered in our undergraduate solid mechanics course.

2.5 Variational method

The equilibrium equations of the classical beam model can be derived in a more systematic fashion using the variational method based on the Kantorovich method. From the view point of Kantorovich method, our objective is to reduce to original 3D problem to a 1D problem so we try to approximate the original 3D fields in terms of 1D unknown functions of the beam axis x_1 and some known functions of the cross-sectional coordinates x_α. To this end, we consider the displacement field based on the Euler-Bernoulli assumptions and Saint Venant assumptions, Eq. (13), as an approximate trial function for the 3D displacement field, and the stress field in Eq. (22) as an approximate trial function for the 3D stress field. For the original 3D structure, the load can be applied either as distributed body force f_i and/or surface tractions t_i. The principle of virtual work of the beam structure can be stated as

$$\frac{1}{2}\int_0^L \delta U_{1D} \mathrm{d}x_1 = \delta W \tag{54}$$

with U_{1D} understood as the 1D strain energy density along the beam axis, defined as

$$U_{1D} = \frac{1}{2}\left\langle \sigma_{ij}\varepsilon_{ij} \right\rangle \tag{55}$$

where the angle bracket denotes the integration over the cross-section. The virtual work δW due to applied loads can be expressed as

$$\delta W = \int_0^L \left(\langle f_i \delta u_i \rangle + \oint_{\partial\Omega} t_i \delta u_i \mathrm{d}s \right) \mathrm{d}x_1 + \langle t_i \delta u_i \rangle |_{x_1=0} + \langle t_i \delta u_i \rangle |_{x_1=L} \tag{56}$$

Here $\partial\Omega$ denotes the lateral surface of the beam structure, and the last two terms denote the integration evaluated by the root surface ($x_1 = 0$) and the tip surface ($x_1 = L$), respectively. Substituting the 3D displacement field expressed in Eq. (13) into Eq. (56), we have

$$\delta W = \int_0^L (p_i \delta \bar{u}_i + q_i \delta \Phi_i)\, \mathrm{d}x_1 + (P_i \delta \bar{u}_i + Q_i \delta \Phi_i)\, |_{x_1=0} + (P_i \delta \bar{u}_i + Q_i \delta \Phi_i)\, |_{x_1=L} \tag{57}$$

where $\Phi_2 = -\bar{u}_3'$ and $\Phi_3 = \bar{u}_2'$ due to the third assumption of Euler-Bernoulli and

$$
\begin{aligned}
p_i(x_1) &= \langle f_i \rangle + \oint_{\partial\Omega} t_i \mathrm{d}s \\
q_1(x_1) &= \langle x_2 f_3 - x_3 f_2 \rangle + \oint_{\partial\Omega} (x_2 t_3 - x_3 t_2) \mathrm{d}s \\
q_2(x_1) &= \langle x_3 f_1 \rangle + \oint_{\partial\Omega} x_3 t_1 \mathrm{d}s \\
q_3(x_1) &= -\langle x_2 f_1 \rangle - \oint_{\partial\Omega} x_2 t_1 \mathrm{d}s \\
P_i &= \langle t_i \rangle \\
Q_1 &= \langle x_2 t_3 - x_3 t_2 \rangle \\
Q_2 &= \langle x_3 t_1 \rangle \\
Q_3 &= -\langle x_2 t_1 \rangle
\end{aligned}
\tag{58}
$$

Here we actually provided a systematic way to obtain the distributed forces $p_i(x_1)$ and moments $q_i(x_1)$ along the beam axis, and the concentrated forces P_i and moments Q_i we used in the Newtonian method based on the original applied body forces f_i and surface tractions t_i in the 3D structure. The concentrated forces P_i and Q_i should be evaluated on the end surfaces at either $x_1 = 0$ or $x_1 = L$. Note in deriving Eq. (57), we have to realize that the warping function is already known and the twist rate κ_1 is also assumed to be a constant according to the Saint Venant assumptions.

Substituting the 3D stress field expressed in Eq. (22) into Eq. (55), we have

$$
U_{1D} = \frac{1}{2} \left\langle E\varepsilon_{11}^2 + 4G(\varepsilon_{12}^2 + \varepsilon_{13}^2) \right\rangle \tag{59}
$$

Substituting the 3D strain field expressed in Eqs. (19), (16), (17) into the above equation, we have

$$
U_{1D} = \frac{1}{2} \left(S_{11}\epsilon_1^2 + 2S_{13}\epsilon_1\kappa_2 + 2S_{14}\epsilon_1\kappa_3 + S_{33}\kappa_2^2 + 2S_{34}\kappa_2\kappa_3 + S_{44}\kappa_3^2 + S_{22}\kappa_1^2 \right) \tag{60}
$$

Here the stiffness constants $S_{11}, S_{13}, \ldots$ are the same as those defined in Eq. (26). Note here

$$
S_{22} = \left\langle G \left[(\Psi_{,2} - x_3)^2 + (\Psi_{,3} + x_2)^2 \right] \right\rangle \tag{61}
$$

which can be shown to be the same as that defined in Eq. (26) because

$$
\begin{aligned}
&\langle G \left[(\Psi_{,2} - x_3)\Psi_{,2} + (\Psi_{,3} + x_2)\Psi_{,3} \right] \rangle = \langle G \left\{ [(\Psi_{,2} - x_3)\Psi]_{,2} + [(\Psi_{,3} + x_2)\Psi]_{,3} \right\} \rangle \\
&= \oint_{\partial A} G\Psi((\Psi_{,2} - x_3)n_2 + (\Psi_{,3} + x_2)n_3) \mathrm{d}s
\end{aligned}
\tag{62}
$$

vanishes due to the fact that Ψ must satisfy the governing equation in Eq. (12) in the cross-sectional domain and the stress free boundary conditions along the boundary curve of the cross-section, denoted using ∂A.

Carrying out the partial derivatives of U_{1D} in Eq. (60) and in view of Eq. (25), we obtain

$$\begin{aligned}
\frac{\partial U_{1D}}{\partial \epsilon_1} &= (S_{11}\epsilon_1 + S_{13}\kappa_2 + S_{14}\kappa_3) = F_1 \\
\frac{\partial U_{1D}}{\partial \kappa_1} &= S_{22}\kappa_1 = M_1 \\
\frac{\partial U_{1D}}{\partial \kappa_2} &= (S_{13}\epsilon_1 + S_{33}\kappa_2 + S_{34}\kappa_3) = M_2 \\
\frac{\partial U_{1D}}{\partial \kappa_3} &= (S_{14}\epsilon_1 + S_{34}\kappa_2 + S_{44}\kappa_3) = M_3
\end{aligned} \tag{63}$$

This gives another way to define the sectional stress resultants as conjugates to the 1D beam strains in terms of the 1D strain energy density, *i.e.*, the stress resultant can be defined as the partial derivative of the 1D strain energy density with respect to the corresponding 1D beam strain measures and these equations can also be written in the same matrix form as Eq. (27). In other words the variational method provides another way to derive the same energetics as we have presented previous in Section 2.3.

Substituting Eqs. (57), into Eq. (54), we can rewrite the principal of virtual work energy in a 1D form as

$$\int_0^L \delta U_{1D} \mathrm{d}x_1 = \int_0^L (p_i\delta\bar{u}_i + q_i\delta\Phi_i)\,\mathrm{d}x_1 + (P_i\delta\bar{u}_i + Q_i\delta\Phi_i)\,|_{x_1=0} + (P_i\delta\bar{u}_i + Q_i\delta\Phi_i)\,|_{x_1=L} \tag{64}$$

which implies the following

$$0 = \int_0^L (\delta U_{1D} - p_i\delta\bar{u}_i - q_i\delta\Phi_i)\,\mathrm{d}x_1 - (P_i\delta\bar{u}_i + Q_i\delta\Phi_i)\,|_{x_1=0} - (P_i\delta\bar{u}_i + Q_i\delta\Phi_i)\,|_{x_1=L} \tag{65}$$

The variation of 1D strain energy density U_{1D} can be evaluated based on Eq. (63) as

$$\delta U_{1D} = F_1\delta\epsilon_1 + M_i\delta\kappa_i = F_1\delta\bar{u}_1' + M_1\delta\Phi_1' - M_2\delta\bar{u}_3'' + M_3\delta\bar{u}_2'' \tag{66}$$

Realizing $\Phi_2 = -\bar{u}_3'$ and $\Phi_3 = \bar{u}_2'$ and carrying out integration by parts for the integral term in Eq. (65), we can rewrite Eq. (65) as

$$\begin{aligned}
0 = &\int_0^L ((M_3'' + q_3' - p_2)\delta\bar{u}_2 - (M_2'' + q_2' + p_3)\delta\bar{u}_3 - (F_1' + p_1)\delta\bar{u}_1 - (M_1' + q_1)\delta\Phi_1)\,\mathrm{d}x_1 \\
&- [(P_1 + F_1)\delta\bar{u}_1 + (Q_1 + M_1)\delta\Phi_1 + (P_2 - M_3' - q_3)\delta\bar{u}_2 + (P_3 + M_2' + q_2)\delta\bar{u}_3 + \\
&(Q_3 + M_3)\delta\bar{u}_2' - (Q_2 + M_2)\delta\bar{u}_3']_{x_1=0} \\
&- [(P_1 - F_1)\delta\bar{u}_1 + (Q_1 - M_1)\delta\Phi_1 + (P_2 + M_3' + q_3)\delta\bar{u}_2 + (P_3 - M_2' - q_2)\delta\bar{u}_3 + \\
&(Q_3 - M_3)\delta\bar{u}_2' - (Q_2 - M_2)\delta\bar{u}_3']_{x_1=L}
\end{aligned} \tag{67}$$

As $\bar{u}_i$ and Φ_1 are the four unknown functions of the classical beam model, they can vary independently. The corresponding Euler-Lagrange equations are

$$F_1' + p_1 = 0 \quad M_1' + q_1 = 0 \quad M_2'' + q_2' + p_3 = 0 \quad M_3'' + q_3' - p_2 = 0 \tag{68}$$

which are the same as the equilibrium equations we obtained in Eqs. (37), (38), (43), (44) using Newtonian method. The boundary conditions can be deduced from the last four lines of Eq. (67). If we know certain displacement variables ($\bar{u}_i$, Φ_1, $\bar{u}'_\alpha$) are prescribed, then their variations must be zero. For example if the root is clamped, we have $\bar{u}_i = 0, \Phi_1 = 0, \bar{u}'_\alpha = 0$, which implies $\delta\bar{u}_i = 0, \delta\Phi_1 = 0, \delta\bar{u}'_\alpha = 0$, which will automatically vanish the boundary terms evaluated at $x_1 = 0$ in Eq. (67). If the tip displacements and rotations are free to vary, then the coefficients in front of the variation of these variables must be zero, that is we have

$$F_1 = P_1 \quad M_1 = Q_1 \quad P_2 = -M'_3 - q_3 \quad P_3 = M'_2 + q_2 \quad Q_2 = M_2 \quad Q_3 = M_3 \tag{69}$$

This boundary condition is the same as those in Eq. (50) in view of Eqs. (40) and (42).

Using the 1D constitutive relations in Eq. (27) and the sectional strain definitions in Eq. (18), we can formulate the governing equations and the boundary conditions in terms of $\bar{u}_i, \Phi_1$ exactly the same as those we derived using the Newtonian method.

Although both the Newtonian method and the variational method based on the same set of ad hoc assumptions necessary to obtain the displacement field in Eq. (13), the strain field in Eqs. (19), (16), (17), and the stress field in Eq. (22), there are some difference between these two methods.

- We does not have to introduce the transverse shear stress resultants for the derivation using the variational approach.
- The variational method can establish a rational connection between the applied loads in the original 3D structure and the final 1D beam model.
- Although lack of being intuitive, the variational approach is more systematic. As far as one is careful about the derivation, it is not easy to make a sign error like commonly happen in Newtonian approach particularly for deriving the boundary conditions.
- As the variational approach is based on the Kantrovich method, it is easy to extend this derivation for higher-order models by using a different set of assumptions for the 3D displacement field in terms of 1D unknown functions, while such extensions using Newtonian approach is much more difficult.

However, because both methods are based on a host of ad hoc assumptions, they feature the same set of contradictions as we discussed carefully in previous sections. In the next section, we will use the variational asymptotic method to construct the classical beam model without invoking any ad hoc assumptions thus avoiding the awkward self-contractions.

3 Variational Asymptotic Method

The whole purpose of beam model is to approximate the original 3D model with a 1D beam model formulated in terms of unknown functions of the beam axis. Our motivation comes from the fact that the cross-sectional domain is much smaller than the span of the

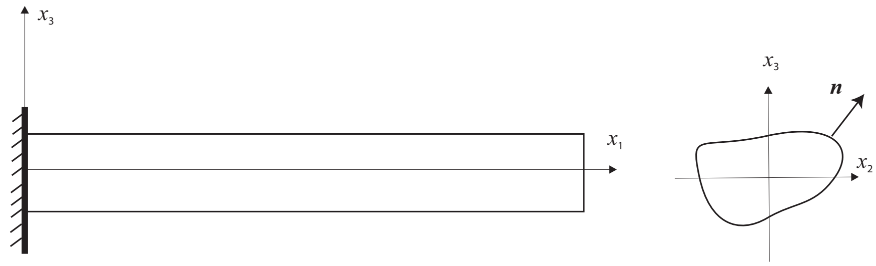

Figure 9: Sketch of a beam

beam structure. This fact of smallness of the cross-section compared to the beam span can be exploited using the variational asymptotic method to derive the classical beam model. For illustrative purpose, we consider a prismatic beam, with L as the length and h as the dimension of the cross-section (see Figure 9). Then we know that $\delta = h/L$ as a small parameter. Suppose the 3D displacements are $u_i(x_1, x_2, x_3)$, then the 3D strains as defined in linear elasticity are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{70}$$

To proceed using the variational asymptotic method, we need to have some very basic knowledge of order analysis. For a continuous differentiable function, $f(x)$ for $x \in [a, b]$. If we denote the order of $f(x)$ as $\bar{f}$, then $\frac{df}{dx}$ is of the order of $\frac{\bar{f}}{b-a}$, denoting as $\frac{df}{dx} \sim \frac{\bar{f}}{b-a}$. Then it is obvious that $u_{i,1} \sim \bar{u}_i/L$ and $u_{i,\alpha} \sim \bar{u}_i/h$, and $u_{i,1} \ll u_{i,\alpha}$ because $\delta = h/L \ll 1$.

The 3D strain field can be written explicitly as

$$\begin{aligned} \varepsilon_{11} &= u_{1,1} \\ 2\varepsilon_{12} &= u_{1,2} + u_{2,1} \\ 2\varepsilon_{13} &= u_{1,3} + u_{3,1} \\ \varepsilon_{22} &= u_{2,2} \\ 2\varepsilon_{23} &= u_{2,3} + u_{3,2} \\ \varepsilon_{33} &= u_{3,3} \end{aligned} \tag{71}$$

The total potential energy of the original 3D structure is given as follows

$$\Pi = \frac{1}{2}\int_0^L U_{1D} \mathrm{d}x_1 - W \tag{72}$$

with twice of the 1D strain energy density expressed in a different form as

$$\begin{aligned} 2U_{1D} =& \left\langle E\varepsilon_{11}^2 \right\rangle + \left\langle G(2\varepsilon_{12})^2 + G(2\varepsilon_{13})^2 + G(2\varepsilon_{23})^2 \right\rangle \\ &+ \left\langle \frac{E}{(1+\nu)(1-2\nu)} \begin{Bmatrix} \nu\varepsilon_{11} + \varepsilon_{22} \\ \nu\varepsilon_{11} + \varepsilon_{33} \end{Bmatrix}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \nu\varepsilon_{11} + \varepsilon_{22} \\ \nu\varepsilon_{11} + \varepsilon_{33} \end{Bmatrix} \right\rangle \end{aligned} \tag{73}$$

Note although this form is different from that in Eq. (55), they are identical to each other after some algebraic manipulations.

In view of Eq. (56), the work done by applied loads in the original 3D structure can be obtained as

$$W = \int_0^L \left(\langle f_i u_i \rangle + \oint_{\partial\Omega} t_i u_i \mathrm{d}s \right) \mathrm{d}x_1 + \langle t_i u_i \rangle |_{x_1=0} + \langle t_i u_i \rangle |_{x_1=L} \tag{74}$$

We have assumed that the 3D strain field is small as we are working within the framework of linearity elasticity, *i.e.*, $\hat{\epsilon} = O(\varepsilon_{ij}) \ll 1$ with $\hat{\epsilon}$ denoting the characteristic magnitude of the 3D strain field. From Eqs. (71), we can conclude that

$$u_i = O(L\hat{\epsilon}) \tag{75}$$

The 1D strain energy density will be of the order of $\bar{\mu}h^2\hat{\epsilon}^2$ with $\bar{\mu}$ denoting the order of the elastic constants. The condition of the boundedness of deformations for $h/L \to 0$ puts some constraints on the order of the external forces. It is clear that the work done must be of the same order as the strain energy, *i.e.*, $f_i u_i h^2 \sim t_i u_i h \sim \bar{\mu}h^2\hat{\epsilon}^2$. In view of Eq. (75), we have

$$f_i h \sim t_i \sim \bar{\mu}\frac{h}{L}\hat{\epsilon} \tag{76}$$

Substituting the strain field in Eq. (71) into the total potential energy of the original structure in Eq. (72) and dropping smaller terms, we obtain:

$$\begin{aligned} 2\Pi = &\left\langle G u_{1,2}^2 + G u_{1,3}^2 + G(u_{2,3} + u_{3,2})^2 \right\rangle \\ &+ \left\langle \frac{E}{(1+\nu)(1-2\nu)} \begin{Bmatrix} u_{2,2} \\ u_{3,3} \end{Bmatrix}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{2,2} \\ u_{3,3} \end{Bmatrix} \right\rangle \end{aligned} \tag{77}$$

Note these kept terms, in the order of $\bar{\mu}L^2\hat{\epsilon}^2$ are much larger than those neglected in the strain energy and in the work done which are in the order of $\bar{\mu}h^2\hat{\epsilon}^2$. The behavior of the structure is governed by the principle of minimum total potential energy. The quadratic form in Eq. (77) will reach its absolute minimum zero if the following conditions can be satisfied:

$$u_{1,2} = u_{1,3} = u_{2,2} = u_{3,3} = u_{2,3} + u_{3,2} = 0 \tag{78}$$

which has the following solution

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) \tag{79}$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1) - x_3\Phi_1(x_1) \tag{80}$$

$$u_3(x_1, x_2, x_3) = \bar{u}_3(x_1) + x_2\Phi_1(x_1) \tag{81}$$

where $\bar{u}_i$ and Φ_1 are arbitrary unknown 1D functions of x_1. Although we have found an expression for the 3D displacement field in terms of 1D functions of x_1, we are not sure whether we have included all the terms corresponding to the classical beam model yet. We need to continue our variational asymptotic procedure by perturbing the displacement field such that

$$\begin{aligned} u_1(x_1, x_2, x_3) &= \bar{u}_1(x_1) + v_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) &= \bar{u}_2(x_1) - x_3\Phi_1(x_1) + v_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) &= \bar{u}_3(x_1) + x_2\Phi_1(x_1) + v_3(x_1, x_2, x_3) \end{aligned} \tag{82}$$

with v_1 asymptotically smaller than $\bar{u}_1$, v_2 asymptotically smaller than $\bar{u}_2 - x_3\Phi_1$, and v_3 asymptotically smaller than $\bar{u}_3 + x_2\Phi_1$. Because $\bar{u}_i, \Phi_1$ are four arbitrary functions, for definiteness of the expression in Eq. (82), we need to introduce four constraints for newly introduced 3D functions v_i. The choice of four constraints is directly related with how we define the four 1D functions ($\bar{u}_i(x_1)$, $\Phi_1(x_1)$) in terms of the 3D displacement field $u_i(x_1, x_2, x_3)$. If we choose the constraints as

$$\langle v_i \rangle = 0 \qquad \langle v_{3,2} - v_{2,3} \rangle = 0 \tag{83}$$

It implies the following definitions of $\bar{u}_i(x_1)$, $\Phi_1(x_1)$) in terms of 3D displacements as

$$\begin{aligned}
A\bar{u}_1(x_1) &= \langle u_1(x_1, x_2, x_3) \rangle \\
A\Phi_1(x_1) &= \frac{1}{2} \langle u_{3,2} - u_{2,3} \rangle \\
A\bar{u}_2(x_1) &= \langle u_2(x_1, x_2, x_3) \rangle + \langle x_3 \rangle \Phi_1(x_1) \\
A\bar{u}_3(x_1) &= \langle u_3(x_1, x_2, x_3) \rangle - \langle x_2 \rangle \Phi_1(x_1)
\end{aligned}$$

with A denoting the cross-sectional area. If the origin of x_α is at the geometric center of the cross-section (*i.e.*, $\langle x_\alpha \rangle = 0$), $\bar{u}_i$ is defined as the average of corresponding 3D displacement u_i over the cross-section and Φ_1 is defined as the average of corresponding 3D axial rotations $\frac{1}{2}(u_{3,2} - u_{2,3})$ over the cross-section. For simplicity of the following derivations, we restrict x_α to originate from the geometric center.

Substituting this displacement field in Eq. (82) into Eq. (71), we can obtain the following 3D strain field as

$$\begin{aligned}
\varepsilon_{11} &= \epsilon_1 + v_{1,1} \\
2\varepsilon_{12} &= v_{1,2} + \bar{u}_2' - x_3\kappa_1 + v_{2,1} \\
2\varepsilon_{13} &= v_{1,3} + \bar{u}_3' + x_2\kappa_1 + v_{3,1} \\
\varepsilon_{22} &= v_{2,2} \\
2\varepsilon_{23} &= v_{2,3} + v_{3,2} \\
\varepsilon_{33} &= v_{3,3}
\end{aligned} \tag{84}$$

Here we let $\bar{u}_1' = \epsilon_1$ and $\Phi_1' = \kappa_1$ as we defined previously. However, we do not have to assume that κ_1 is constant as what we did previously in the Saint Venant assumptions.

Substituting the displacement field in Eqs. (82) and the 3D strain field in Eqs. (84) into the total potential energy of the original 3D structures in Eq. (72) and dropping smaller terms, we have

$$\begin{aligned}
2\Pi = {} & \langle E \rangle \epsilon_1^2 + \left\langle G(v_{1,2} + \bar{u}_2' - x_3\kappa_1)^2 + G(v_{1,3} + \bar{u}_3' + x_2\kappa_1)^2 + G(v_{2,3} + v_{3,2})^2 \right\rangle \\
& + \left\langle \frac{E}{(1+\nu)(1-2\nu)} \begin{Bmatrix} \nu\epsilon_1 + v_{2,2} \\ \nu\epsilon_1 + v_{3,3} \end{Bmatrix}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \nu\epsilon_1 + v_{2,2} \\ \nu\epsilon_1 + v_{3,3} \end{Bmatrix} \right\rangle \\
& - \int_0^L \left(\langle f_i\bar{u}_i + (x_2 f_3 - x_3 f_2)\Phi_1 \rangle + \oint_{\partial\Omega} t_i\bar{u}_i + (x_2 t_3 - x_3 t_2)\Phi_1 \mathrm{d}s \right) \mathrm{d}x_1 \\
& - \langle t_i\bar{u}_i + (x_2 t_3 - x_3 t_2)\Phi_1 \rangle |_{x_1=0} - \langle t_i\bar{u}_i + (x_2 t_3 - x_3 t_2)\Phi_1 \rangle |_{x_1=L}
\end{aligned} \tag{85}$$

of which the v_α related terms will reach the absolute minimum value zero if the following conditions are satisfied:

$$v_{2,3} + v_{3,2} = 0 \tag{86}$$

$$\nu\epsilon_1 + v_{2,2} = 0 \tag{87}$$

$$\nu\epsilon_1 + v_{3,3} = 0 \tag{88}$$

which has the following solution

$$v_\alpha = -x_\alpha \nu \epsilon_1 \tag{89}$$

with the unknown functions of x_1 can be absorbed into $\bar{u}_2(x_1)$ and $\Phi_1(x_1)$. The conditions for the minimum of the v_1 related terms in Eq. (85) has to be obtained using the usual steps of calculus of variations. The corresponding Euler-Lagrange equation is

$$v_{1,22} + v_{1,33} = 0 \tag{90}$$

over the cross-sectional domain and the boundary condition along the cross-sectional boundary curve is

$$(v_{1,2} + \bar{u}_2' - x_3\kappa_1)n_2 + (v_{1,3} + \bar{u}_3' + x_2\kappa_1)n_3 = 0 \tag{91}$$

with n_α denoting the components along x_α of outward normal vector $\boldsymbol{n}$ of the boundary curve (see Figure 9). The solution of v_1 can be written in the following form

$$v_1 = -x_\alpha u_\alpha' + \Psi(x_2, x_3)\kappa_1 \tag{92}$$

with $\Psi(x_2, x_3)$ as the Saint Venant warping function satisfying governing equations in Eq. (12) along with the stress free boundary conditions such that

$$(\Psi_{1,2} - x_3\kappa_1)n_2 + (\Psi_{1,3} + x_2\kappa_1)n_3 = 0 \tag{93}$$

along the boundary curve of the cross-section. The constraint in Eq. (83) for v_i implies we should constrain Ψ such that $\langle \Psi(x_2, x_3) \rangle = 0$ which helps solve the Saint Venant warping function uniquely.

Substituting the solutions for v_i in Eqs. (92) and (89) into Eq. (82), we can express the 3D displacement field as

$$\begin{aligned} u_1 &= \bar{u}_1(x_1) - x_\alpha u_\alpha' + \Psi(x_2, x_3)\kappa_1 \\ u_2 &= \bar{u}_2(x_1) - x_3\Phi_1(x_1) - x_2\nu\epsilon_1 \\ u_3 &= \bar{u}_3(x_1) + x_2\Phi_1(x_1) - x_3\nu\epsilon_1 \end{aligned} \tag{94}$$

Now, we know that the asymptotical expansion of the 3D displacement field will be spanned by $\bar{u}_i$ and Φ_1 as no new degrees of freedom will appear according to the variational asymptotic method. However, we are still not sure whether we have included all the orders needed for the classical beam model. For this purpose, we perturb the displacement field one more time such that

$$\begin{aligned} u_1 &= \bar{u}_1(x_1) - x_\alpha \bar{u}_\alpha' + \Psi(x_2, x_3)\kappa_1 + w_1(x_1, x_2, x_3) \\ u_2 &= \bar{u}_2(x_1) - x_3\Phi_1(x_1) - x_2\nu\epsilon_1 + w_2(x_1, x_2, x_3) \\ u_3 &= \bar{u}_3(x_1) + x_2\Phi_1(x_1) - x_3\nu\epsilon_1 + w_3(x_1, x_2, x_3) \end{aligned} \tag{95}$$

with the constraints on v_i passed onto w_i following the same reasoning we have used for obtaining Eq. (83). That is we have

$$\langle w_i \rangle = 0 \qquad \langle w_{3,2} - w_{2,3} \rangle = 0 \tag{96}$$

The 3D strain field corresponding to the displacement field in Eq. (95) is

$$\begin{aligned}
\varepsilon_{11} &= \epsilon_1 + x_3\kappa_2 - x_2\kappa_3 + \Psi(x_2, x_3)\kappa_1' + w_{1,1} \\
2\varepsilon_{12} &= (\Psi_{,2} - x_3)\kappa_1 + w_{1,2} + w_{2,1} - \nu x_2\epsilon_1' \\
2\varepsilon_{13} &= (\Psi_{,3} + x_2)\kappa_1 + w_{1,3} + w_{3,1} - \nu x_3\epsilon_1' \\
\varepsilon_{22} &= -\nu\epsilon_1 + w_{2,2} \\
2\varepsilon_{23} &= w_{2,3} + w_{3,2} \\
\varepsilon_{33} &= -\nu\epsilon_1 + w_{3,3}
\end{aligned} \tag{97}$$

Here we let $\bar{u}_2'' = \kappa_3$ and $-\bar{u}_3'' = \kappa_2$ as we defined previously. Clearly from these equations, we can estimate that $\epsilon_1 \sim h\kappa_i \sim \hat{\epsilon}$.

Substituting the displacement field in Eqs. (95) and the 3D strain field in Eqs. (97) into the total potential energy of the original 3D structures in Eq. (72) and dropping smaller terms, we have

$$\begin{aligned}
2\Pi = &\left\langle E(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3)^2 \right\rangle + \left\langle G[(\Psi_{,2} - x_3)\kappa_1 + w_{1,2}]^2 + G[(\Psi_{,3} + x_2)\kappa_1 + w_{1,3}]^2 \right\rangle \\
&+ \left\langle G(w_{2,3} + w_{3,2})^2 \right\rangle + \left\langle \frac{E}{(1+\nu)(1-2\nu)} \begin{Bmatrix} \nu(x_3\kappa_2 - x_2\kappa_3) + w_{2,2} \\ \nu(x_3\kappa_2 - x_2\kappa_3) + w_{3,3} \end{Bmatrix}^T \right. \\
&\left. \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \nu(x_3\kappa_2 - x_2\kappa_3) + w_{2,2} \\ \nu(x_3\kappa_2 - x_2\kappa_3) + w_{3,3} \end{Bmatrix} \right\rangle \\
&- \int_0^L (p_i\bar{u}_i + q_i\Phi_i)\ \mathrm{d}x_1 - (P_i\bar{u}_i + Q_i\Phi_i)|_{x_1=0} - (P_i\bar{u}_i + Q_i\Phi_i)|_{x_1=L}
\end{aligned} \tag{98}$$

with p_i, q_i, P_i, Q_i defined the same as those in Eqs. (58). The minimization of this functional in Eq. (108) will be reached by $w_1 = 0$ and the following conditions:

$$w_{2,3} + w_{3,2} = 0 \tag{99}$$

$$\nu(x_3\kappa_2 - x_2\kappa_3) + w_{2,2} = 0 \tag{100}$$

$$\nu(x_3\kappa_2 - x_2\kappa_3) + w_{3,3} = 0 \tag{101}$$

which can be solved along with the constraints in Eq. (96), yielding

$$\begin{aligned}
w_2 &= (\langle x_2x_3 \rangle - x_2x_3)\nu\kappa_2 + (x_2^2 - x_3^2 - \langle x_2^2 \rangle + \langle x_3^2 \rangle)\frac{\nu\kappa_3}{2} \\
w_3 &= (x_2x_3 - \langle x_2x_3 \rangle)\nu\kappa_3 + (x_2^2 - x_3^2 - \langle x_2^2 \rangle + \langle x_3^2 \rangle)\frac{\nu\kappa_2}{2}
\end{aligned} \tag{102}$$

Now we have obtained for all the contributions to the classical beam model and it can be easily verified that any further perturbation will not add any major terms to this beam model as far as the total potential energy of the structure is concerned.

The complete 3D displacement field corresponding to the classical beam model is

$$
\begin{aligned}
u_1 &= \bar{u}_1(x_1) - x_\alpha \bar{u}'_\alpha + \Psi(x_2, x_3)\kappa_1 \\
u_2 &= \bar{u}_2(x_1) - x_3\Phi_1(x_1) + \underline{(\langle x_2x_3\rangle - x_2x_3)\nu\kappa_2 + (x_2^2 - x_3^2 - \langle x_2^2\rangle + \langle x_3^2\rangle)\frac{\nu\kappa_3}{2}} \\
u_3 &= \bar{u}_3(x_1) + x_2\Phi_1(x_1) + \underline{(x_2x_3 - \langle x_2x_3\rangle)\nu\kappa_3 + (x_2^2 - x_3^2 - \langle x_2^2\rangle + \langle x_3^2\rangle)\frac{\nu\kappa_2}{2}}
\end{aligned}
\tag{103}
$$

Comparing to the displacement field based on Euler-Bernoulli assumptions and Saint Venant assumptions in Eqs. (13), the variational asymptotic method obtained additional terms which are underlined in Eq. (103).

Substituting the solutions for w_i into Eq. (97) and dropping the terms smaller than the order of $\hat{\epsilon}$, the complete 3D strain field corresponds to the classical beam model is

$$
\begin{aligned}
\varepsilon_{11} &= \epsilon_1 + x_3\kappa_2 - x_2\kappa_3 \\
2\varepsilon_{12} &= (\Psi_{,2} - x_3)\kappa_1 \\
2\varepsilon_{13} &= (\Psi_{,3} + x_2)\kappa_1 \\
\varepsilon_{22} &= -\nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) \\
2\varepsilon_{23} &= 0 \\
\varepsilon_{33} &= -\nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3)
\end{aligned}
\tag{104}
$$

Comparing to the strain field obtained based on Euler-Bernoulli assumptions and Saint Venant assumptions, ε_{22} and ε_{33} are different.

The complete stress field using the Hooke's law will be

$$
\begin{aligned}
&\sigma_{11} = E(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) \\
&\sigma_{12} = G(\Psi_{,2} - x_3)\kappa_1 \quad \sigma_{13} = G(\Psi_{,3} + x_2)\kappa_1 \\
&\sigma_{22} = \sigma_{33} = \sigma_{23} = 0
\end{aligned}
\tag{105}
$$

which is the same as those we assumed before in Eq. (22) in the ad hoc approaches, although none of the assumptions has been used in obtaining this.

Substituting the solutions for w_i into Eq. (108), we will obtain the potential energy of the classical beam model and carry out the variation will result in the same variational statement as that in Eq. (64), which implies we will have the same 1D constitutive relations as those in Eq. (27), the same 1D governing different equations as those in Eqs. (37), (38), (43), (44), and the same boundary conditions as those in Eq. (50).

3.1 A shortcut for the variational asymptotic derivation

We have used three perturbations to derive the classical beam model. A shortcut is possible for us to derive the same model using one perturbation, which is what we adopted in the formulation of VABS, a world-known commercial software for modeling composite beams.

To construct 1D classical beam model, the 3D displacement field must be expressed in terms of the four unknown function $\bar{u}_i$ and Φ_1. Let us introduce the following change of variables

$$
\begin{aligned}
u_1 &= \underline{\bar{u}_1(x_1) - x_\alpha \bar{u}'_\alpha} + w_1(x_1, x_2, x_3) \\
u_2 &= \underline{\bar{u}_2(x_1) - x_3 \Phi_1(x_1)} + w_2(x_1, x_2, x_3) \\
u_3 &= \underline{\bar{u}_3(x_1) + x_2 \Phi_1(x_1)} + w_3(x_1, x_2, x_3)
\end{aligned}
\tag{106}
$$

The underline terms can be understood as the displacements introduced by the deformation of the beam axis in terms of $\bar{u}_i$ and Φ_1 if one assuming the cross-section is not deformable. The fact that the cross-section is deformable is captured by w_i which are called generalized warping functions as the cross-section can deform both in-plane and out-of-plane which are asymptotically smaller than those underlined terms. Although w_i are not the same as those used in Eqs. (95), the constraints in Eq. (96) can be used if we define $\bar{u}_i$ and Φ_1 according to the following definitions:

$$
\begin{aligned}
A\Phi_1(x_1) &= \frac{1}{2} \langle u_{3,2} - u_{2,3} \rangle \\
A\bar{u}_2(x_1) &= \langle u_2(x_1, x_2, x_3) \rangle + \langle x_3 \rangle \Phi_1(x_1) \\
A\bar{u}_3(x_1) &= \langle u_3(x_1, x_2, x_3) \rangle - \langle x_2 \rangle \Phi_1(x_1) \\
A\bar{u}_1(x_1) &= \langle u_1(x_1, x_2, x_3) \rangle + \langle x_\alpha \rangle \bar{u}'_\alpha
\end{aligned}
$$

The 3D strain field corresponding to the displacement field in Eq. (106) is

$$
\begin{aligned}
\varepsilon_{11} &= \epsilon_1 + x_3\kappa_2 - x_2\kappa_3 + w_{1,1} \\
2\varepsilon_{12} &= w_{1,2} - x_3\kappa_1 + w_{2,1} \\
2\varepsilon_{13} &= w_{1,3} + x_2\kappa_1 + w_{3,1} \\
\varepsilon_{22} &= w_{2,2} \\
2\varepsilon_{23} &= w_{2,3} + w_{3,2} \\
\varepsilon_{33} &= w_{3,3}
\end{aligned}
\tag{107}
$$

Substituting the displacement field in Eqs. (106) and the 3D strain field in Eqs. (107) into the total potential

$$
\begin{aligned}
2\Pi = {} & \left\langle E(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3)^2 \right\rangle + \left\langle G(w_{1,2} - x_3\kappa_1)^2 + G(w_{1,3} + x_2\kappa_1)^2 \right\rangle \\
& + \left\langle G(w_{2,3} + w_{3,2})^2 \right\rangle + \left\langle \frac{E}{(1+\nu)(1-2\nu)} \begin{Bmatrix} \nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) + w_{2,2} \\ \nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) + w_{3,3} \end{Bmatrix}^T \right. \\
& \left. \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) + w_{2,2} \\ \nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) + w_{3,3} \end{Bmatrix} \right\rangle \\
& - \int_0^L (p_i\bar{u}_i + q_i\Phi_i)\ \mathrm{d}x_1 - (P_i\bar{u}_i + Q_i\Phi_i)|_{x_1=0} - (P_i\bar{u}_i + Q_i\Phi_i)|_{x_1=L}
\end{aligned}
\tag{108}
$$

The warping functions that minimize the above energy functional are governed by the Euler-Lagrange equations of this energy functional, given by

$$
w_{1,22} + w_{1,33} = 0 \tag{109}
$$

$$
2(1-\nu)w_{2,22} + (1-2\nu)w_{2,33} + w_{3,23} - 2\nu\kappa_3 = 0 \tag{110}
$$

$$
2(1-\nu)w_{3,33} + (1-2\nu)w_{3,22} + w_{2,23} + 2\nu\kappa_2 = 0 \tag{111}
$$

and the associated boundary conditions

$$n_3(x_2\kappa_1 + w_{1,3}) + n_2(w_{1,2} - x_3\kappa_1) = 0 \tag{112}$$

$$n_3(w_{2,3} + w_{3,2}) + \frac{2n_2}{1-2\nu}\left[\nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) + \nu w_{3,3} + (1-\nu)w_{2,2}\right] = 0 \tag{113}$$

$$n_2(w_{2,3} + w_{3,2}) + \frac{2n_3}{1-2\nu}\left[\nu(\epsilon_1 + x_3\kappa_2 - x_2\kappa_3) + \nu w_{2,2} + (1-\nu)w_{3,3}\right] = 0 \tag{114}$$

where n_α is the direction cosine of outward normal with respect to x_α. Here, to maintain a simpler derivation, we do not use Lagrange multipliers to enforce the constraints of Eqs.(96). Instead, we keep these constraints in mind and check whether they can be satisfied by the solution. It can be observed that Eq. (109) and (112) are just the equations of Saint-Venant warping $\Psi(x_2, x_3)$ in elasticity textbooks such as , except

$$w_1(x_1, x_2, x_3) = \Psi(x_2, x_3)\kappa_1(x_1) \tag{115}$$

Hence the first approximation of the out-of-plane warping w_1 can be solved by the methods used to solve the Saint Venant torsion problem commonly found in elasticity textbooks. According to the theory of elasticity, Ψ can be determined up to a constant, and one can choose the constant so that the constraint $\langle w_1 \rangle = 0$ is satisfied. The following functions of w_α satisfy the other constraints and solve Eqs. (110), (111), (113) and (114):

$$\begin{aligned} w_2 &= -x_2\nu\epsilon_1 + (\langle x_2x_3\rangle - x_2x_3)\nu\kappa_2 + (x_2^2 - x_3^2 - \langle x_2^2\rangle + \langle x_3^2\rangle)\frac{\nu\kappa_3}{2} \\ w_3 &= -x_3\nu\epsilon_1 + (x_2x_3 - \langle x_2x_3\rangle)\nu\kappa_3 + (x_2^2 - x_3^2 - \langle x_2^2\rangle + \langle x_3^2\rangle)\frac{\nu\kappa_2}{2} \end{aligned} \tag{116}$$

Substituting the solutions for w_i into Eq. (106), we obtain the same displacement as Eq. (103). Substituting the solutions for w_i into Eq. (107), we obtain the same strain field as Eq. (104). Using the 3D Hooke's law, we will obtain the same stress field as in Eq. (105). In other words, we obtained the same solution for relating the original 3D elasticity to the classical beam model as we derived previously using three perturbations in the previous section in a much quicker way.

4 Problems

1. Verify that the formulas in Eq. (35) is the same as those in Eq. (29) for isotropic homogenous beams.

2. Following the procedure in this chapter for deriving the classical beam model, derive the Timoshenko beam model by removing the third Euler-Bernoulli assumptions using the Newtonian approach.

3. Consider the thin-walled, L shaped cross-section of a beam as shown in Figure 10. Let $b = 0.25\ m$, $h = 0.1\ m$, and $t = 2.5 \times 10^{-3}\ m$.

   (a) Find the location of the centroid of the section and verify that the formulas in Eq. (35) will give the same results as those in Eq. (5).

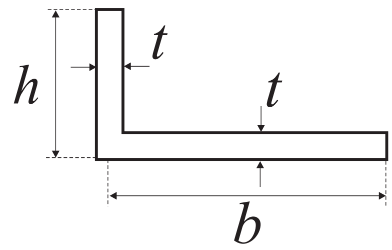

Figure 10: Thin-walled, L shaped cross-section.

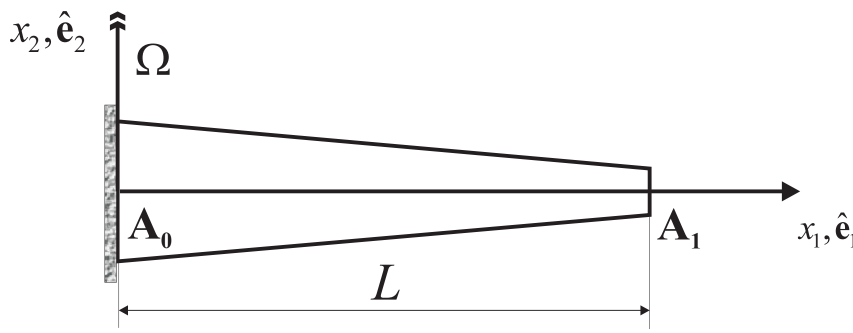

Figure 11: A helicopter blade rotating at an angular speed Ω.

(b) Find the orientation of the principal centroidal axes of bending.

4. A helicopter blade of length L is rotating at an angular velocity Ω about the $\hat{\mathbf{e}}_2$ axis, as depicted in Figure 11. The blade is homogeneous and its cross-section linearly tapers from an area A_0 at the root to $A_1 = A_0/2$ at the tip, *i.e.*

$$A(x_1) = A_0 + (A_1 - A_0)\,\frac{x_1}{L} = A_0\,(1 - \frac{x_1}{2L}).$$

(a) Solve the governing differential equations of this problem to find the axial displacement $\bar{u}_1(x_1)$ and the axial force distribution $F_1(x_1)$.

(b) Find an approximate solution of the problem. Assume the axial displacement field in the form of $\bar{u}_1(x_1) = ax_1 + bx_1^2$.

(c) Compare the solution obtained in parts (1) and (2). Plot the exact and approximate axial displacement fields $\bar{u}_1(x_1)$ on the same plot. Plot the exact and approximate axial force $F_1(x_1)$ on the same another plot. Comment on your results. How would you improve the approximate solution?

5. Consider the cantilevered beam shown in Figure 12. It is subjected to a uniform transverse loading and has a tip spring of stiffness.

(a) Solve the problem exactly to find the deflection of the beam and the force in the tip spring using the differential statement.

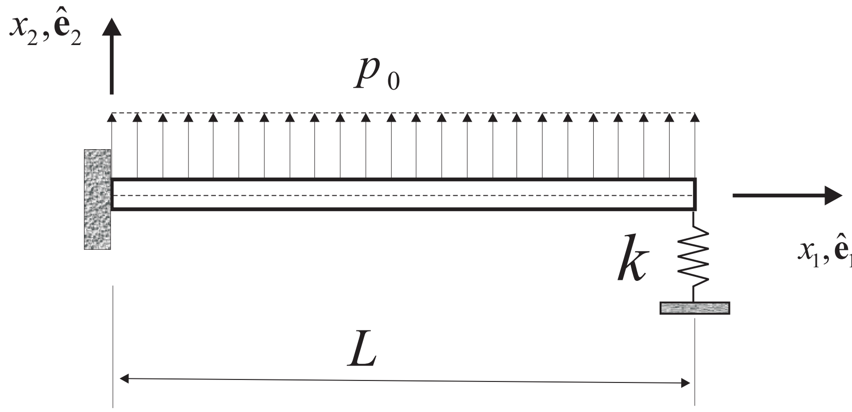

Figure 12: A cantilever with a tip spring.

(b) Use the energy approach based on the Principle of Minimum Total Potential Energy to solve for deflection of the beam by choosing $\bar{u}_2(x_1) = c_1 x_1^2 + c_2 x_1^3$, where c_1 and c_2 are constants to be determined by the Ritz method. Quantify your errors if there is a difference between exact solution and approximate solution for the maximum deflection and the force in the spring. What will the results if the trial function is chosen to be $\bar{u}_2(x_1) = c_1 x_1^2 + c_2 x_1^3 + c_3 x_1^4$?