PDEs and Complex Analysis

On 26 October 2007, Albert Tsou sent in the following Q & A.

Q: Why did the mathematician name his dog "Cauchy"?
A: Because the dog left a residue at every pole.

As you know, certain sections in Saff and Snider upset me, and I have deviated from the book. We all agree that no text should be held as an authority. But now, both you and I are paying for the deviation.

The notes here follow closely my lectures. Several students have mentioned that their own notes may reproduce what was written on the blackboards, but not what I said in class, so that they found it hard to reproduce the ideas.

The notes, as well as the corresponding lectures, attempt to teach a particular application of complex analysis: using an analytic function to solve a partial differential equation. The primary example used here is electrostatics. I have several objects in mind:

- Link math to a physical problem. To make this link, you need to review the basics of electrostatics, to the extent that you can make physical sense of the mathematical results.
- Link PDE to analytic function. I use an approach to show that this link can be made for PDEs other than the Laplace equation. I learned the approach from technical papers when I was a graduate student here in late 80s, and used it in my papers on interfacial fracture mechanics. I have never seen this approach in any standard textbooks.
- Show techniques of using complex analysis to solve a PDE.

Thus, the "applied" content of these notes is somewhat more than the corresponding sections in Saff and Snider. One way to learn this material is to work through these notes, adding your own notes and inventing your own exercises along the way. For example, I don't have time to draw figures using WORD. Please sketch your own figures from the notes you took in class. Fill the 3 inch margin on the right side.

Laws of electrostatics. This review reminds you of the basics of electrostatics. If you feel uncomfortable with this review, please look at your physics textbook.

As an idealization, in this review we consider an ideal world composed of only two kinds of things: conductors and insulators. For example, metals are conductors, and air is an insulator. When a quantity of electric charge is injected into this world, the charge can flow in the conductors, but not in the insulators. In equilibrium, when all the electric charge stops flowing, the charge is trapped either in the interior of the insulators, or on the interfaces between the insulators and conductors; no electric charge remains in the interior of the conductors.

The world is three dimensional. We set up a system of coordinates, and write the coordinates of each point in the world as x, y, z.

Electric potential. The electric potential is a scalar, whose value may vary from one point in the world to another point. That is, the electric potential is a scalar field, written as a function $\phi(x, y, z)$. In equilibrium, the electric potential is uniform in a conductor, but is non-uniform in an insulator.

Electric field. The gradient of the electric potential in an insulator gives the electric field:

$$E_x = -\frac{\partial \phi(x, y, z)}{\partial x}, \quad E_y = -\frac{\partial \phi(x, y, z)}{\partial y}, \quad E_z = -\frac{\partial \phi(x, y, z)}{\partial z}.$$

The electric field in the insulator is a vector field. In equilibrium, the electric field in a conductor vanishes.

Electric displacement. Consider an arbitrary part of the world. This part has some shape and volume, and may contain both conductors and insulators. Let the net charge in this part of the world be Q. The electric displacement $\mathbf{D}(x, y, z)$ is a vector that satisfies Gauss's law:

$$\int \mathbf{D} \cdot \mathbf{n} dA = Q ,$$

where the integral extends over the surface that encloses the part, and $\mathbf{n}$ is the unit vector normal to the surface.

Material law. In an insulator, the electric displacement is linear in the electric field, namely,

$$\mathbf{D} = \varepsilon \mathbf{E} ,$$

where ε is the permittivity, a constant specific to a given insulator.

Action items for exercise. The above basic laws lead to the following items, which are essential for the understanding of electrostatics. Even if you do not wish to go through the derivations, you should try to understand the conclusions.

(a) The insulator can trap electric charge. Let $q(x, y, z)$ be the net electric charge per unit volume in the insulator. Use the divergence theorem to show that Gauss's law implies that

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = q .$$

(b) Consider two insulators meeting at an interface. Label one insulator by + and the other by -, and let the unit vector normal to the interface point to the insulator +. The electric displacement is discontinuous across the interface. Denote the component of the electric displacement normal to the interface in the two insulators by D_n^+ and D_n^-. Let ω be the electric charge per unit area of the interface. Show that Gauss's law implies that

$$D_n^+ - D_n^- = \omega .$$

(c) As an important special case of (b), consider an interface between an insulator and a conductor. In equilibrium, the electric displacement vanishes in the conductor, but not in the insulator.

Let D_n be the component of the electric displacement in the insulator normal to the interface, taking D_n to be positive when it points toward the conductor. Let ω be the electric charge per unit area of the interface. Show that Gauss's law implies that

$$D_n = \omega .$$

(d) An insulator has a dielectric constant ε, and a distribution of electric charge $q(x, y, z)$. Show that the electric potential $\phi(x, y, z)$ satisfies Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{q}{\varepsilon} .$$

Parallel-plate capacitor. A parallel-plate capacitor is a layer of an insulator sandwiched between two conductors. The two conductors, known as the electrodes, are connected to a battery through an external circuit. The battery supplies a voltage V between the two electrodes, and pumps electric charge from one electrode to the other. The electric charge resides at the interfaces between the insulator and the two electrodes. In equilibrium, the electric charge is $+Q$ at one interface is, and is $-Q$ at the other interface.

When the thickness of the insulator is small compared to its length and width, the electric field is uniform in the insulator, except in the small regions near the edges of the insulator. Let us ignore these edges and focus on the interior of the insulator. Let A be the area of each electrode, and L be the thickness of the insulator. Let y be the coordinate in the direction through the thickness of the insulator. The electric potential in the insulator is linear in y, given by

$$\phi = Vy / L .$$

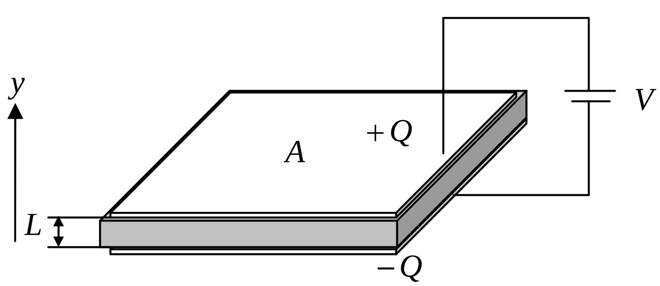
The electric field in the insulator is

$$E = V / L .$$

The electric displacement is

$$D = Q / A .$$

Both the electric field and the electric displacement are uniform fields in the insulator, pointing in the direction of $-y$.



Action items for exercise. The following items ask you to think physically about electrostatics.

(a) The capacitance *C* of a capacitor is defined by the magnitude of the electric charge on one electrode divided by the voltage applied by the battery, namely, $C = Q/V$. Show that the capacitance of the parallel-plate capacitor is

$$C = \varepsilon A / L,$$

(b) Describe an experimental procedure to measure permittivity of an insulator.

(c) Find values for the permittivity of water and glass in the literature. Why is the permittivity of water so much larger than glass? Give a molecular description.

Electrostatics in two dimensions. Consider a situation that can be modeled by a two-dimensional field of electric potential $\phi(x, y)$. When the insulator does not have any net electric charge in its interior, show that the field of electric potential is governed by the Laplace equation:

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0, \quad \text{for } (x,y) \text{ inside insulator.}$$

We will assume that the surface of the insulator is covered with conductors, so that the electric potential on the surface of the insulator is held at prescribed values:

$\phi(x, y) = \text{prescribed value}$,for (x, y) on the surface of the insulator.

The Laplace equation and the boundary condition together constitute a boundary value problem, whose solution gives the electric potential field $\phi(x, y)$ in the insulator.

Once the electric potential is solved, the electric field is give by the gradient:

$$E_x = -\frac{\partial \phi(x,y)}{\partial x}, \quad E_y = -\frac{\partial \phi(x,y)}{\partial y}.$$

The electric displacement is given by

$$D_x = \varepsilon E_x, \quad D_y = \varepsilon E_y.$$

The electric charge density on the surface of the insulator is

$$\omega = D_x n_x + D_y n_y,$$

where n_x and n_y are the components of the unit vector normal to the surface.

Represent electrostatics using a complex potential. This problem guides you through the steps that link electrostatics to functions of a complex variable.

(a) Let

$$\xi = x + py,$$

where p is a number to be determined. Write the field of electric potential as a function of a single variable,

$$\phi(x, y) = f(\xi).$$

Show that, provided $p = i$ or $p = -i$, any differentiable function $f(\xi)$ satisfies the Laplace equation.

(b) The Laplace equation is a linear, homogenous PDE. Consequently, the general solution is a linear superposition of the two solutions:

$$\phi(x, y) = f(z) + g(\bar{z}),$$

where f is a differentiable function of $z = x + iy$, and g is a differentiable function of $\bar{z} = x - iy$.

(c) The electric potential is real-valued. Show that the electric potential is represented as

$$\phi(x, y) = \operatorname{Re}[G(z)],$$

where $G(z)$ is an analytic function of z. The function $G(z)$ is called the complex potential.

Proof. Let us introduce some notation using an example, $f(z) = (z - a)^2$. Then, $f(\bar{z}) = (\bar{z} - a)^2$, $\bar{f}(\bar{z}) = (\bar{z} - \bar{a})^2$, and $\bar{f}(z) = (z - \bar{a})^2$) Because the electric potential is real-valued, $\phi = \overline{\phi}$, and therefore

$$f(z) + g(\bar{z}) = \bar{f}(\bar{z}) + \bar{g}(z)$$

Using this relation, we rewrite $\phi(x, y) = f(z) + g(\bar{z})$ as

$$\phi(x, y) = \frac{f(z) + g(\bar{z})}{2} + \frac{\bar{f}(\bar{z}) + \bar{g}(z)}{2}$$

Introduce a new function

$$G(z) = f(z) + \bar{g}(z).$$

Note that this new function $G(z)$ is an analytic function of *z*. Thus,

$$\phi(x, y) = \frac{G(z) + \overline{G}(\bar{z})}{2} = \operatorname{Re}[G(z)].$$

(d) Show that

$$-E_x + iE_y = \frac{dG(z)}{dz}.$$

Proof. Note that

$$E_x = -\frac{\partial \phi(x, y)}{\partial x} = -\frac{dG(z)}{2dz} - \frac{d\overline{G}(\bar{z})}{2d\bar{z}}$$

$$E_y = -\frac{\partial \phi(x, y)}{\partial y} = -i\frac{dG(z)}{2dz} + i\frac{d\overline{G}(\bar{z})}{2d\bar{z}}$$

A combination of the above two equations give the desired proof.

(e) For a given boundary value problem, what determines the function $G(z)$?

Answer: The boundary condition:

$\phi(x, y) = \operatorname{Re}[G(z)] = \text{prescribed value}$, for (x, y) on the surface of the insulator.

The above approach to express the solution of a PDE in terms of an analytic function of a complex variable is applicable to any PDE with the following attributes:

- The PDE is homogenous.
- The PDE is linear.
- Each term in the PDE contains the same order of differentiation.

Two examples follow.

Exercise: use an analytic function to represent the solution of a partial differential equation. A real-valued function, $V(x, y)$, satisfies a partial differential equation (PDE):

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2} = 0 .$$

Let $\xi = x + py$, where p is a complex number to be determined. Show that the solution to this PDE can be represented as

$$V(x, y) = \mathrm{Re}[G(\xi)],$$

where $G(\xi)$ is an analytic function of ξ. Determine the value of p.

Exercise: use an analytic function to represent the solution of a partial differential equation. A real-valued function, $V(x, y)$, satisfies a partial differential equation (PDE):

$$6\frac{\partial^4 V}{\partial x^4} + 5\frac{\partial^4 V}{\partial x^2 \partial y^2} + \frac{\partial^4 V}{\partial y^4} = 0 .$$

Show that the solution to this PDE can be represented as

$$V(x, y) = \mathrm{Re}[G(\xi) + H(\eta)],$$

where $G(\xi)$ is any differentiable function of the variable $\xi = x + py$, and $H(\eta)$ is any differentiable function of the variable $\eta = x + qy$. Determine the values of p and q.

Read Section 2.5 of Saff and Snider, Harmonic Functions.

A uniform electric field. Consider a uniform electric field E pointing in the *x*-direction. Using $-E_x + iE_y = \frac{dG(z)}{dz}$, we obtain that

$$\frac{dG(z)}{dz} = -E .$$

Integrating, we obtain that

$$G(z) = -Ez + \text{complex constant}$$

The electric potential is

$$\phi(x, y) = \mathrm{Re}[G(z)] = -Ex + \text{real constant} .$$

Exercise. A uniform electric field points in the direction at an angle γ from the *x*-axis. Determine the complex potential and the electric potential.

Section 3.4 of Saff and Snider: Washers, Wedges and Walls. This section uses the function $\log z$ to solve a few types of boundary value problems. Recall that $\log z$ is a multi-valued function, with a branch point at $z = 0$. Once you have selected a branch cut, you can write

$$\log z = \log|z| + i\arg(z) = \log r + i\theta .$$

The function $\log z$ has the real part $\log r$ and the imaginary part θ.

Washers. An insulator takes the shape of a cylindrical shell. The electric potentials are prescribed at constant levels in the inner and outer surfaces of the shell. On a perimeter of a circle, r is a constant, namely, $\mathrm{Re}[\log z]$ is constant. Consequently, the complex potential for the cylindrical shell is

$$G(z) = A\log z + B ,$$

where A and B are real-valued constants. The electric potential is

$$\phi(x, y) = \mathrm{Re}[G(z)] = A\log r + B$$

The constants A and B are selected to give the prescribed electric potentials on the inner and outer surfaces of the cylindrical shell.

Wedges. An insulator takes the shape of a wedge. The electric potential is prescribed at a constant level at each edge of the wedge. At a fixed θ, $\mathrm{Im}[\log z]$ is constant. Consequently, the electric potential for the wedge is

$$G(z) = -iA\log z + B ,$$

where A and B are real-valued constants. The electric potential is

$$\phi(x, y) = \mathrm{Re}[G(z)] = A\theta + B$$

The constants A and B are selected to give the prescribed electric potentials on the two edges of the wedge.

Walls. An insulator occupies a half space, $y \geq 0$. The surface of the insulator, $y = 0$, is divided into segments by a set of points $x_1, x_2, x_3 \ldots$. Each segment is prescribed with a constant level of electric potential. What is the field of electric potential inside the insulator. The complex potential takes the form

$$G(z) = -iA_1 \log(z - x_1) - iA_2 \log(z - x_2) - iA_3 \log(z - x_3) - \ldots + B ,$$

where $A_1, A_2, A_3, \ldots, B$ are real-valued constants selected to give the electric potentials prescribed on all the segments. The electric potential is

$$\phi(x, y) = \mathrm{Re}[G(z)] = A_1\theta_1 + A_2\theta_2 + A_3\theta_3 + \ldots + B ,$$

where $\theta_1, \theta_2, \theta_3$,… are polar angles centered at the points $x_1, x_2, x_3...$ The electric field in the insulator is given by

$$-E_x + iE_y = \frac{dG(z)}{dz} = -\frac{iA_1}{z - x_1} - \frac{iA_2}{z - x_2} - \frac{iA_3}{z - x_3} - ...$$

A uniform electric field perturbed by a cylindrical conductor of circular cross-section. We have outlined the solution to this problem in class.

(a) Use words, pictures and equations to state the problem clearly.

(b) Trace the steps that lead to the solution

$$G(z) = -E_\infty \left(z - \frac{a^2}{z} \right).$$

Solution. At a point far from the cylinder, the electric field is uniform, so that

$$G(z) \to -E_\infty z \quad as\ |z| \to \infty .$$

We need to find an function analytic outside the circle $|z| = a$, such that $\mathrm{Re}[G(z)] = 0$ for *z* on the circle (i.e., $z = ae^{i\theta}$). By inspection, the function is

$$G(z) = -E_\infty \left(z - \frac{a^2}{z} \right).$$

(c) Sketch the contours of constant potential.

(d) Sketch the lines of electric field.

(e) Let θ be the polar angle. Determine the electric field in the insulator, around the surface of the conductor, as a function of θ.

(f) How is the electric charge distributed on the surface of the cylindrical conductor?

A uniform electric field perturbed by a strip of conductor. A strip of conductor lies in a vacuum; see a figure below. The width of the strip is 2*a*. The length of the strip is much larger than the width, and the thickness of the strip is negligible compared to the width. Remote from the strip, the electric field is uniform, of magnitude E_∞, directed parallel to the width of the strip. Let *x* axis be along the width of the strip, and the origin of the axis coincide with the center of the strip.

(a) Confirm that

$$G(z) = -E_\infty (z^2 - a^2)^{1/2}$$

satisfies all the boundary conditions. Choose the strip as the branch cut of the function $G(z)$.

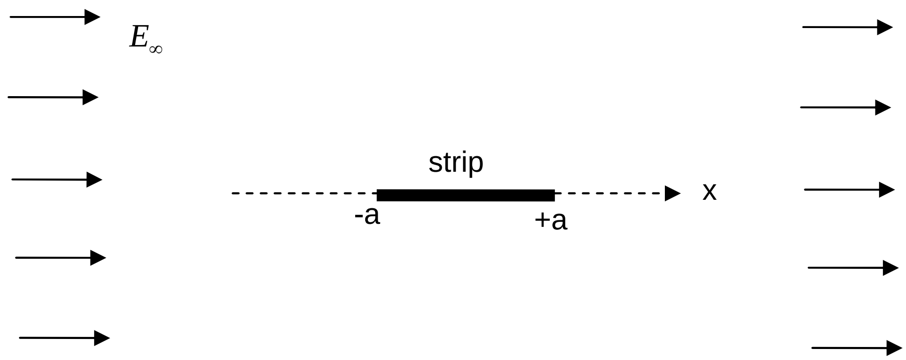
(b) Sketch the contours of constant potential.

(c) Sketch the lines of electric field.

(d) Determine the electric field on the top and bottom faces of the strip.

(e) Determine the distribution of electric charge on the strip.

(f) What would the electric field in the vacuum be if the remote electric field were normal to the faces of the strip?



Conformal mapping. A function $w = f(z)$ maps one complex number to another complex number. The function is said to be conformal at a point z_0 if $f'(z_0)$ exists and $f'(z_0) \neq 0$.

Here is a geometric interpretation of a conformal mapping. Write $w_0 = f(z_0)$. Let z be another point on the z-plane, and $w = f(z)$ be the corresponding point on the w-plane. Because the function is analytic, we can use the Taylor series:

$$w - w_0 = f'(z_0)(z - z_0),$$

where high order terms have been neglected. Observe that $z - z_0$ is a vector in the z-plane, and $w - w_0$ is the corresponding vector in the w-plane. The above relation linearly maps the vector $z - z_0$ to the vector $w - w_0$.

We now fix the point z_0. As z varies, the vector $z - z_0$ varies in length and direction. The linear relation, $w - w_0 = f'(z_0)(z - z_0)$, maps every vector $z - z_0$ by multiplying the same complex number $f'(z_0)$, independent of z. Consequently, the linear relation magnifies the length of every vector $z - z_0$ by the same factor $|f'(z_0)|$, and rotates the direction of every vector $z - z_0$ by the same angle $\arg[f'(z_0)]$. That is, when a function is analytic and $f'(z_0) \neq 0$, the function preserves the image locally.

In general, $f'(z_0)$ varies when z_0 varies. Consequently, the small vectors in different neighborhood magnify by different factors and rotate by different angle. That is, the conformal mapping usually distort image globally. The only exception is the linear map $w = az + b$, whose derivative is the constant *a.* The linear map translates, magnifies and rotates; it preserves images globally. It is boring.

When a function is conformal, the function is a one-to-one map, and its inverse function exists. We will write the function and its inverse as

$$w = f(z), \quad z = \Gamma(w).$$

Non-conformal mapping. To appreciate the conformal mapping, we should talk about non-conformal mapping. A function can be non-conformal in several ways. For example, the following transformation

$$u = u(x, y), \quad v = v(x, y)$$

maps a point on the (x, y) plane to a point on the (u, v) plane. Even if the two functions are differentiable, the transformation will usually be non-conformal, because the two functions may not be conjugate to each other.

As a second example, a function $f(z)$ fails to be a conformal map at a point z_0 because z_0 is a pole or a branch point. Examples are $(z - z_0)^{-1}$ and $\log(z - z_0)$.

As a third example, even when a function is analytic, the function may still fail to be a conformal map because $f'(z) = 0$. If this equality holds in a domain, the function is a constant. Such a function maps every point on the z-plane to a constant on the w plane: this trivial function distorts everywhere.

If $f'(z) = 0$ only holds for an isolated point, the situation is more interesting. Say a function $f(z)$ is analytic at a point z_0, and $f'(z_0) = 0$. We assume $f''(z_0) \neq 0$. An example of such a function is $w = f(z) = z^2$ at $z_0 = 0$. We know this function is a two-to-one mapping, and has a branch point on the w-plane. It turns out that this behavior is generic. Once again use the Taylor series:

$$w - w_0 = \frac{f''(z_0)}{2}(z - z_0)^2.$$

Because $f'(z_0) = 0$, the leading order term is quadratic. The quadratic transformation doubles the angle.

Bilinear transformation (or Mobius transformation):

$$w = \frac{az + b}{cz + d}.$$

When $ad = bc$, this function reduces to a constant, and is therefore non-conformal everywhere. Except this trivial case, the bilinear transformation maps the entire plane to another entire plane. This transformation has a pole at $z = -d / c$. The bilinear transformation maps a circle to another circle. If the circle on the z-plane passes the point $z = -d / c$, the corresponding image on the w-plane is a straight line, which is taken to be a circle of infinite radius.

A circle is determined by three points. A bilinear transformation that maps the three points z_1, z_2, z_3 to the three points w_1, w_2, w_3 takes the form:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Mapping the exterior of a unit circle to the exterior of some other curve. Consider a unit circle on the w-plane. An analytic function outside this circle can be represented by the Laurent series. Consider a particular Laurent series:

$$z = a_1 w + a_0 + \frac{a_{-1}}{w} + \frac{a_{-2}}{w^2} + ...$$

Here we have dropped high order positive powers. This way, when $|w|$>>1, the function is a linear map. The coefficients can be selected to fit the curve on the z-plane. Consider a very special case:

$$z = a_1 w + \frac{a_{-1}}{w},$$

where a_1 and a_{-1} are real-valued. When w is on a unit circle, $w = e^{i\gamma}$, the corresponding curve on the z plane is give by

$$x + iy = a_1 e^{i\gamma} + a_{-1}e^{-i\gamma} = (a_1 + a_{-1})\cos\gamma + i(a_1 - a_{-1})\sin\gamma$$

Comparing the real and imaginary parts, we obtain that

$$x = (a_1 + a_{-1})\cos\gamma, \quad y = (a_1 - a_{-1})\sin\gamma$$

That is, the corresponding curve on the z-plane is an ellipse, provided we identify $a_1 + a_{-1} = a$ and $a_1 - a_{-1} = b$, with *a* and *b* being the semi-axes of the ellipse.

Show that the transformation

$$z = \Gamma(w) = \frac{a+b}{2}w + \frac{a-b}{2w}$$

is conformal for w outside the unit circle. What happens at the limiting case $b = 0$?

Mapping a boundary value problem. Recall the boundary value problem on the z-plane. An insulator occupies a domain D_z in the *z*-plane. The electric potential is given by

$$\phi(x,y) = \mathrm{Re}[G(z)],$$

where $G(z)$ is an analytic function in the domain, and is determined by the boundary condition

$$\phi(x,y) = \mathrm{Re}[G(z)] = \text{prescribed value}, \quad \text{for } (x,y) \text{ on the}$$

surface of the insulator.

Now consider a conformal transformation

$$z = \Gamma(w),$$

Which maps domain D_w in the w-plane to the domain D_z in the *z*-plane. The electric potential is given by

$$\phi(x,y) = \mathrm{Re}[G(\Gamma(w))].$$

The composite function $H(w) = G(\Gamma(w))$ is an analytic function in the domain D_w, and is determined by the boundary condition

$$\phi(x,y) = \mathrm{Re}[H(w)] = \text{prescribed value}\ , \qquad \text{for } w \text{ on the}$$

boundary of the domain D_w.

Once the two functions $\Gamma(w)$ and $H(w)$ are determined, the electric field is given by

$$-E_x + iE_y = \frac{dG(z)}{dz} = \frac{dH(w)}{dw}\frac{dw}{dz} = \frac{dH(w)}{dw}\left(\frac{d\Gamma}{dw}\right)^{-1}.$$

A uniform electric field perturbed by a cylindrical conductor of an elliptic cross section. We have outlined the solution to this problem in class.
(a) Use words, pictures and equations to state the problem clearly.
(b) Find a conformal transformation that maps a unit circle on the w-plane to an ellipse on the z-plane.

Answer:

$$z = \Gamma(w) = \frac{a+b}{2}w + \frac{a-b}{2w}$$

(c) Find the analytic function $H(w)$.

Solution. At a point far from the cylinder, the electric field is uniform, so that

$$G(z) \to -E_\infty z \qquad as\ |z| \to \infty .$$

Because $z \to \frac{a+b}{2}w$ as $|z| \to \infty$, we obtain that

$$H(w) \to -\frac{E_\infty(a+b)}{2}w \qquad as\ |w| \to \infty$$

We need to find an function analytic $H(w)$ outside the circle $|w| = 0$, such that $\mathrm{Re}[H(w)] = 0$ for *w* on the unit circle (i.e., $w = e^{i\gamma}$). By inspection, the function is

$$H(w) = -\frac{E_\infty(a+b)}{2}\left(w - \frac{1}{w}\right).$$

(d) Sketch the contours of constant potential on the z-plane.
(e) Sketch the lines of electric field on the z-plane.
(f) Determine the electric field at point $z = a$.

Solution. The electric field is given by

$$-E_x + iE_y = \frac{dH(w)/dw}{d\Gamma(w)/dw} = \frac{-\frac{E_\infty(a+b)}{2}\left(1 + \frac{1}{w^2}\right)}{\frac{a+b}{2} - \frac{a-b}{2w^2}}.$$

Using the mapping $z = \Gamma(w)$, the point $z = a$ corresponds to $w = 1$. Thus,

$$-E_x + iE_y = -E_\infty \left(1 + \frac{a}{b}\right).$$

Consequently, the electric field at the tip of an ellipse is

$$E = E_\infty \left(1 + \frac{a}{b}\right)$$

The electric field is in the direction normal to the conductor. When the ellipse is very long and thin, *a* >> *b*, the electric field at the tip of the ellipse can be many times the applied electric field. This effect is called field concentration.

Example 1 in Section 7.1 of the Saff and Snider. We have gone over this example in class. Our approach attempts to describe the generic steps.

(a) State the problem clearly. Sketch the constant potential lines and the electric field lines.

(b) Find a conformal transformation that maps the circle to the straight line in Fig. 7.4.

Solution. A bilinear map will do. We choose three points:

$$f(-1) = 0, \quad f(i) = i, \quad f(1) = \infty .$$

Using the general rule:

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

We find that

$$\frac{(w - 0)(i - \infty)}{(w - \infty)(i - 0)} = \frac{(z + 1)(i - 1)}{(z - 1)(i + 1)}.$$

Solving for *w*, we obtain that

$$w = f(z) = \frac{1 + z}{1 - z}.$$

The inverse function is

$$z = \Gamma(w) = \frac{w - 1}{w + 1}.$$

(c) Find the complex potential $H(w)$.

Solution. On the w-plane, this problem becomes a wall problem, with the boundary conditions $\phi = 1$ when y > 0, and $\phi = -1$ when y < 0. The solution is

$$H(w) = -i \frac{2}{\pi} \log w .$$

On the w-plane, we choose the branch cut along the negative *u* axis, such that the argument of *w* is restricted as $-\pi < \arg(w) < \pi$. Note that

$$\phi(x, y) = \operatorname{Re}[H(w)] = \frac{2}{\pi} \arg(w).$$

This function does satisfy the boundary condition for ϕ.

(d) Determine electric field at $z = 0$.

Solution: The electric field is given by

$$-E_x + iE_y = \frac{dH(w)/dw}{d\Gamma(w)/dw} = \frac{-i\dfrac{2}{\pi w}}{\dfrac{2}{(w+1)^2}}.$$

Using the mapping $z = \Gamma(w)$, the point $z = 0$ corresponds to $w = 1$. Thus,

$$-E_x + iE_y = -i\frac{4}{\pi}.$$

Consequently, the electric field at the center of the insulator is

$$E_x = 0, \quad E_y = -\frac{4}{\pi}.$$

(e) Interpret this result in physical terms.

Earlier notes on Thermal Conduction

1. Thermal conduction. This problem guides you through the elements of thermal conduction. (a) Explain in physical terms the basic modes of heat transfer: Conduction, convection and radiation.

(b) Let (x, y, z) be a system of coordinates, and *t* be time. The temperature in the conductor is a function of position and time, $T(x, y, z, t)$. That is, the temperature in the conductor is a scalar field. What is a scalar? Give three more examples of scalars among commonly used physical quantities.

(c) What is heat capacity? Outline an experimental method to measure heat capacity. Find the value of heat capacity of a window glass in the literature. Cite the reference. How much energy is needed to increase the glass window of your room by 10 K? Give rough dimensions of your window.

(d) Use words, figures and symbols to define the quantity heat flux. Why is heat flux a vector?

(e) Fourier's law states that the heat flux is proportional to the temperature gradient. Paraphrase this law into equations. When was Fourier born and when did he die?

(f) Outline a procedure to experimentally determine thermal conductivity. Find the value of heat conductivity of a glass in the literature. Cite the reference. How much energy goes through the glass of your window? Assume the temperature difference between inside and outside is 5 K.

(g) Apply the law of conservation of energy to a volume element for a time increment. Use words, draw figures, and derive the equation

$$\rho c\frac{\partial T}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0.$$

(h) Show that a combination of Fourier's law and the law of conservation of energy leads to

$$\frac{\partial T}{\partial t} = D\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

(i) What is the dimension of *D*? How is *D* related to density, heat capacity, and conductivity?

2. Evolving temperature field in a sphere of glass. A sphere of glass, radius *R*, is held at temperature T_0 for a while, so that this temperature is established uniformly inside the sphere. The sphere is then thrown into a pool of hot water, of temperature T_w. Let *r* be the distance from the center to a point in the sphere. The temperature field inside the sphere is a function of position and time, $T(r,t)$.

(a) Show that the temperature field inside the sphere evolves according to the form

$$T = T_0 + (T_w - T_0) f\left(\frac{r}{R}, \frac{t}{R^2 / D}\right),$$

where *f* is a function of two dimensionless variables as indicated.

(b) Sketch this evolving temperature field.

(c) Estimate the time needed for the sphere to equilibrate with the temperature of the water. Assume that the radius of the sphere is 1 cm.

3. Steady-state temperature field in two dimensions. This problem guides you through the steps that link the physical problem to complex analysis.

(a) What is an equilibrium temperature field? What is a steady-state temperature field?

(b) Show that a steady-state temperature field in two dimensions is governed by the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

(c) Let $\xi = x + py$, where p is a number to be determined. Write the temperature field as a function of a single variable, $T(x,y) = f(\xi)$. Show that the Laplace equation is satisfied by either $p = i$ or $p = -i$.

(d) The Laplace equation is a linear, homogenous PDE. Consequently, the general solution is a linear superposition of the two solutions:

$$T(x,y) = f(z) + g(\bar{z}),$$

where *f* and *g* are functions of the variables as indicated. Recall that the temperature is real-valued. Show that the steady-state field can be represented as

$$T(x,y) = \mathrm{Re}[F(z)],$$

where $F(z)$ is an analytic function of z.

(e) For a given boundary value problem, what determines the function $F(z)$?

4. A boundary value problem. This problem confirms our everyday experience that spatial variation of temperature in a small region does not affect the temperature remote from this region. Imagine that a thermal conductor occupies a half space $y > 0$. The temperature on the surface of the conductor is prescribed as

$$T(x,0) = T_{av} + A\cos\left(\frac{2\pi x}{L}\right)$$

where T_{av} is the average temperature on the surface, *A* is the amplitude of the variation, and *L* is the period of variation. In the conductor, very remote from the surface, the temperature is also T_{av}.

(a) Sketch the steady-state isotherms inside the conductor.

(b) Solve for the steady-state temperature field inside the conductor using the complex variable method.