INPUT-OUTPUT ECONOMICS

Big data meet computers. The 1951 article of Wassily Leontief on input-output economics described the excitement of a historical moment, when big data met computers. A sentence from the article: "...the method has had to await the modern high-speed computing machine as well as the present propensity of government and private agencies to accumulate mountains of data."

The historical moment has continued to this day. We still live in the age of BIG DATA.

A tribe, a region, a nation, or the entire world forms an economy. The economy produces and consumes commodities. As the economy grows, the number of commodities increases, and economic activities generate big data. What do we do with these data?

This problem has been known for a long time. A breakthrough took place in the twentieth century, when computers became available. Leontief, then professor of economics at Harvard University, in collaboration with the Bureau of Labor Statistics, developed a method to analyze the flow of commodities. This method of analysis is called the *input-output economics*. Leontief won a Nobel Prize in Economic Sciences in 1973.

Products described in terms of the algebra of scalars. The mathematical ideas of products (i.e., commodities) were known in antiquity. We express these ideas in terms of the *algebra of scalars*:

- Various amounts of a product form a scalar set, S.
- Making a product requires the input of many products.
- The input from a product T to make product S corresponds to a linear map from S to T.
- The price of a product S is a linear map from the scalar set of the product S to the scalar set of money M.

Total Output and Final Demand

Products. We divide an economy into n products, labeled as product 1, 2, ..., n. Various amounts of each product form a scalar set. Denote the scalar sets of the n products by $S_1, S_2, ..., S_n$. For example, various amounts of steel form a scalar set, and various number of cars form another scalar set.

Let $x_1, x_2, ..., x_n$ be the total outputs of the n products, and $d_1, d_2, ..., d_n$ be the final demands for the n products. They are measured on an annual basis. Both the total output x_i and the final demand d_i are elements in the scalar set S_i. The model aims to predict the total outputs $x_1, x_2, ..., x_n$, once the final demands $d_1, d_2, ..., d_n$ are known.

In a trivial economy, each product is made from scratch, and the total output of the product matches the final demand for the product:

$$x_1 = d_1$$
$$x_2 = d_2$$
$$\ldots\ldots$$
$$x_n = d_n$$

Making a product needs other products. In a nontrivial economy, each product is not made from scratch, but requires many products. For example, making cars requires steel, and even some cars. Let z_{ij} be the amount of product *i* needed to make the total output x_j of product *j*. Thus, z_{ij} is an element in the scalar set S_i

Input-output table in physical units. All the quantities z_{ij} form a $n \times n$ table, called the *input-output table* of the economy. We arrange the table such that entries in every row are elements in one scalar set, $z_{ij} \in S_i$. Each column of the table lists the input of products needed for making a product. Entries in each column do not add. The table also lists two other columns: the final demands and the total outputs.

Input-output table of a two-product economy

	product 1	product 2	final demand	total output
product 1	z_{11}	z_{12}	d_1	x_1
product 2	z_{21}	z_{22}	d_2	x_2

Here is an example adapted from Leontief's book, *Input-Output Economics*, 2nd edition, 1986. A simplified economy makes two and only two products: wheat and cloth.

Input-output table of a wheat-cloth economy in physical units

	wheat	cloth	final demand	total output
wheat	25 bushels	20 bushels	55 bushels	100 bushels
cloth	14 yards	6 yards	30 yards	50 yards

Various amounts of each product form a scalar set. For each product, we arbitrarily select a unit. The units chosen here are one bushel of wheat and one yard of cloth. Once a unit is selected for a product, an amount of the product is given by its magnitude, a real number.

The first row lists various amounts of wheat. Of the total output of 100 bushels of wheat, 25 bushels go to making wheat, 20 bushels go to making cloth, and 55 bushels go to the final demand. The scalars in the same scalar set are additive:

$$25 + 20 + 55 = 100 \text{ bushels of wheat}.$$

The second row lists various amounts of cloth. Of the total output of 50 yards of cloth, 14 yards go to making wheat, 6 yards go to making cloth, and 33 yards go to the final demand. The scalars in the same scalar set are additive:

$$14+6+30 = 50 \text{ yards of cloth}$$

The first column lists the amounts of products needed to make 100 bushels of wheat wheat: 25 bushels of wheat, and 14 yards of cloth.

The second column lists the amounts of products needed to make 50 yards of cloth: 20 bushels of wheat, and 6 yards of cloth.

The third column lists the final demands: 55 bushels of wheat, and 30 yards of cloth.

The last column lists the total outputs: 100 bushels of wheat, and 50 yards of cloth.

Input coefficients. Making one product requires the input of many products. Let C_{ij} be the amount of product *i* needed to make a unit of product *j*. Consequently, the amount of product *i* needed to make x_j amount of product *j* is

$$z_{ij} = C_{ij} x_j .$$

That is, C_{ij} maps the total output x_j to the amount z_{ij} of product *i*. The coefficient C_{ij} is a linear map from scalar set S_j to scalar set S_i,

$$C_{ij} : S_j \to S_i .$$

The coefficient, C_{ij}, is called an input coefficient. The input coefficients C_{ij} are determined from the measured data $x_1, x_2, ..., x_n$ and z_{ij} each year. The amount of product *i* needed to produce an amount of product *j* is proportional. The input coefficients are determined by the technology that makes each product. To produce a car, we need a certain amounts of metals, plastics, etc. Consequently, the coefficients C_{ij} are nearly constant year by year. In this analysis, we will assume that these coefficients are constant. By contrast, the total output of each product, x_j, varies year by year.

For the two-product economy, the input coefficients form a two-by-two table. Each entry of the table has a distinct unit.

Table of input coefficients of the wheat-cloth economy

	wheat	cloth
wheat	(25 bushels)/(100 bushels)	(20 bushels)/(50 yards)
cloth	(14 yards) /(100 bushels)	(6 yards)/ (50 yards)

Relate total output to final demand. For each product i, the total output x_i meets the demands of making all products plus the final demand:

$$x_1 = C_{11}x_1 + C_{12}x_2 + ... + C_{1n}x_n + d_1$$
$$x_2 = C_{21}x_1 + C_{22}x_2 + ... + C_{1n}x_n + d_2$$
$$......$$
$$x_n = C_{n1}x_1 + C_{n2}x_2 + ... + C_{nn}x_n = d_n$$

Rearrange each equation to put $x_1,...,x_n$ on the left side and put $d_1,...,d_n$ on the right side. We obtain that

$$\left(1-C_{11}\right)x_1 - C_{12}x_2 - ... - C_{1n}x_n = d_1$$
$$-C_{21}x_1 + \left(1-C_{22}\right)x_2 - ... - C_{1n}x_n = d_2$$
$$......$$
$$-C_{n1}x_1 - C_{n2}x_2 - ... + \left(1-C_{nn}\right)x_n = d_n$$

Given the output of every product, the above equations calculate the final demand for every product.

We can also express the final demands in terms of the total outputs. In solving the system of linear equations, we choose a unit for each product, so that the quantity of every product is represented by a real number, and every input coefficient is also a real number. This procedure allows us to add numbers in a column. Write the solution in the form

$$x_1 = A_{11}d_1 + A_{12}d_2 + ... + A_{1n}d_n$$
$$x_2 = A_{21}d_1 + A_{22}d_2 + ... + A_{2n}d_n$$
$$......$$
$$x_n = A_{n1}d_1 + A_{n2}d_2 + ... + A_{nn}d_n$$

We have assumed that the coefficients are constant from one year to another. The above equations relate the total outputs linearly to the final demands. If we wish to change the final demands, the above equations tell us the quantities of the products to be produced.

For the two-product economy, the equations become that

$$(1-0.25)x_1 - 0.40x_2 = d_1 \quad \text{bushels of wheat}$$
$$-0.14x_1 + (1-0.12)x_2 = d_2 \quad \text{yards of cloth}$$

Given the output of every product, the above equations calculate the final demand for every product.

Conversely, given the final demand for every product, we can solve these equations to find the output of every product:

$$x_1 = 1.457d_1 + 0.6623d_2$$
$$x_2 = 0.2318d_1 + 1.241d_2$$

For example, set $d_1 = 55$ and $d_2 = 30$ on the right sides of the two equations, and they will give $x_1 = 100$ and $x_2 = 50$ on the left.

Level of aggregation. In specifying a product, we have much flexibility to choose levels of aggregation. For example, we may regard the automobiles as a single product. In this case, we cannot use individual automobiles as a unit of the scalar set, because automobiles have many makes. We may use some dollar amount as a unit. Of course, we can also model each make of automobiles as a distinct product.

A primitive economy may be modeled with very few products, such as a two-product economy: wheat and cloth. A modern economy is routinely modeled with hundreds or more products. On the website of the Bureau of Economic Analysis, you can find data of production and consumption of products.

Price and Value Added

Input-output table in terms of money. Entries in a column of the input-output table in physical units are quantities of different products; they do not add. We now introduce another scalar set: money. Various amount of money (e.g., dollars) form a scalar set, *M*. The monetary values of different products are additive.

We next give a price to each product. A price *p* of a product is a linear map from the scalar set of the product *S* to the scalar set of money *M*,

$$p : S \to M .$$

In a *n*-product economy, let x_i be the total output of product *i*, and p_i be a price of the product. The monetary value of the total output the product is $p_i x_i$. The payment for the product *j* to make the total output x_i of product *i* is $p_j z_{ji}$

Let v_i be the value added per unit output of product *i*. The value added is proportional to the total output, $v_i x_i$.

For the two-product economy, assume that the two products have the prices

$$p_{\text{wheat}} = 2\ \$/(\text{bushel of wheat})$$
$$p_{\text{cloth}} = 5\ \$/(\text{yard of cloth})$$

We can then convert the input-output table from the physical units to monetary unit. We also add two rows, one for value added by the product, and the other for the total value produced by the product. Every entry in the table is now in the unit of $.

Input-output table of the wheat-cloth economy in $

	wheat	cloth	final demand	total output
wheat	50	40	110	200
cloth	70	30	150	250
value added	80	180		
total value	200	250		

Relate price to value added. Making the amount x_j of product j needs the inputs of $z_{1j}, z_{2j}, ..., z_{nj}$ amounts of the n products, for which payment is $p_1 z_{1j} + p_2 z_{2j} + ... + p_n z_{nj}$:

$$p_j x_j = p_1 z_{1j} + p_2 z_{2j} + ... + p_n z_{nj} + v_j x_j .$$

Divide this equation by x_j, and we obtain that

$$p_j = p_1 C_{1j} + p_2 C_{2j} + ... + p_n C_{nj} + v_j .$$

This equation holds for every product, giving n equations.

$$p_1 = p_1 C_{11} + p_2 C_{21} ... + p_n C_{n1} + v_1$$
$$p_2 = p_1 C_{12} + p_2 C_{22} ... + p_n C_{n2} + v_2$$
$$.......$$
$$p_n = p_1 C_{1n} + p_2 C_{2n} ... + p_n C_{nn} + v_n$$

Note that the input coefficient of a row now becomes a column.

Rewrite the above equations as

$$p_1(1 - C_{11}) - p_2 C_{21} ... - p_n C_{n1} = v_1$$
$$-p_1 C_{12} + p_2(1 - C_{22}) ... - p_n C_{n2} = v_2$$
$$.......$$
$$-p_1 C_{1n} - p_2 C_{2n} ... + p_n(1 - C_{nn}) = v_n$$

These equations express the values added per unit outputs in terms of the prices. We can also invert the relation and write

$$
\begin{aligned}
p_1 &= A_{11}v_1 + A_{21}v_2 + ... + A_{n1}v_n \\
p_2 &= A_{12}v_1 + A_{22}v_2 + ... + A_{n2}v_n \\
&...... \\
p_n &= A_{1n}v_1 + A_{2n}v_2 + ... + A_{nn}v_n
\end{aligned}
$$

Compare the relation between price and value added to the relation between demand and output, and we note that the coefficients of a row now become coefficients of a column.

Gross Domestic Product

Definition. The gross domestic product (GDP) is defined in two ways. First, the GDP is defined as the sum of the monetary values of the final demands:

$$\text{GDP} = p_1 d_1 + p_2 d_2 + ... + p_n d_n .$$

Second, the GDP is defined as the sum of value added of the total outputs:

$$\text{GDP} = v_1 x_1 + v_2 x_2 + ... + v_n x_n .$$

Theorem. The two definitions of GDP give the same value, namely,

$$p_1 d_1 + p_2 d_2 ... + p_n d_n = v_1 x_1 + v_2 x_2 + ... + v_n x_n$$

Proof. To prove this identity, recall the relation between the total outputs and final demands,

$$
\begin{aligned}
x_1 &= C_{11}x_1 + C_{12}x_2 + ... + C_{1n}x_n + d_1 \\
x_2 &= C_{21}x_1 + C_{22}x_2 + ... + C_{1n}x_n + d_2 \\
&...... \\
x_n &= C_{n1}x_1 + C_{n2}x_2 + ... + C_{nn}x_n = d_n
\end{aligned}
$$

and the relations between the prices and values added,

$$
\begin{aligned}
p_1 &= p_1 C_{11} + p_2 C_{21} ... + p_n C_{n1} + v_1 \\
p_1 &= p_1 C_{12} + p_2 C_{22} ... + p_n C_{n2} + v_2 \\
&....... \\
p_n &= p_1 C_{1n} + p_2 C_{2n} ... + p_n C_{nn} + v_n
\end{aligned}
$$

Multiplying equations in the first set by $p_1, p_2, ..., p_n$ and adding the n-equations, we obtain that

$$
\begin{aligned}
p_1 x_1 + p_2 x_2 + ... + ... p_n x_n = \; & p_1 C_{11} x_1 + p_1 C_{12} x_2 + ... + p_1 C_{1n} x_n + p_1 d_1 \\
& + p_1 C_{21} x_1 + p_1 C_{22} x_2 + ... + p_1 C_{1n} x_n + p_1 d_2 \\
& ...... \\
& + p_1 C_{n1} x_1 + p_1 C_{n2} x_2 + ... + p_1 C_{nn} x_n + p_1 d_n
\end{aligned}
$$

Similarly, multiplying equations in the second set by $x_1, x_2, \ldots, x_n$ and adding the n-equations, we obtain that

$$\begin{aligned} p_1 x_1 + p_2 x_2 + \ldots + \ldots p_n x_n = & \; x_1 C_{11} p_1 + x_1 C_{21} p_2 + \ldots + x_1 C_{n1} p_n + x_1 v_1 \\ & + x_2 C_{12} p_1 + x_2 C_{22} p_2 + \ldots + x_3 C_{1n} p_n + x_1 v_2 \\ & \ldots\ldots \\ & + x_n C_{1n} p_1 + x_n C_{2n} p_2 + \ldots + x_n C_{nn} p_n + x_n v_n \end{aligned}$$

A comparison of the two equations gives that

$$p_1 d_1 + p_2 d_2 + \ldots + p_n d_n = v_1 x_1 + v_2 x_2 + \ldots + v_n x_n .$$

References

W.W. Leontief, Input-output economics. Scientific American 185, 15-21, October 1952.

W. Leontief, Input-Output Economics, Second edition, Oxford University Press, 1986.