

## SCALAR

**Various usages of the word scalar.** In linear algebra, the word *scalar* is used in two ways. First, the word scalar is used as a synonym to the word *number*, an element in the field  $F$ . Second, the word scalar is used in defining a linear form, an object to be studied in a later section. In this second usage, the scalars behave in a way different from numbers. Adding two scalars results in a scalar, and multiplying a scalar with a number resulting in a scalar; however, multiplying two scalars does not result in a scalar in the same set. The situation is the same in defining bilinear form, quadratic form, and inner product.

In physics, the word scalar is used to indicate properties like mass, volume and energy. The usage is consistent with the second usage in linear algebra, but is inconsistent with the first one in several ways:

- A physical property like mass is more than just a number; it has a unit.
- The multiplication defined on a field makes no sense to a physical quantity like mass: the multiplication of two elements in  $F$  gives yet another element in  $F$ , but the multiplication of two masses does not give another mass.
- If we regard both mass and volume as elements in the field  $F$ , then we need to assign a meaning to the addition of mass and volume. What does that even mean?

We will call each element in the field  $F$  a number, and will reserve the word scalar for another object in linear algebra, defined as follows.

### Scalar Set

**Definition.** We call a collection  $S$  of objects a scalar set if it is a one-dimensional vector space over a number field  $F$ . We call each element in  $S$  a scalar, and each element in  $F$  a number.

**Remark.** This definition relies on the definitions of number field and vector space. The definition invokes two sets  $F$  and  $S$ , as well as four binary maps. The set  $F$  is closed under two binary maps: adding two elements in  $F$  gives an element in  $F$ ,  $(F, F) \rightarrow F$ ; multiplying two elements in  $F$  gives an element in  $F$ ,  $(F, F) \rightarrow F$ . The set  $S$  is closed under two other binary maps: adding two elements in  $S$  gives an element in  $S$ ,  $(S, S) \rightarrow S$ ; multiplying an element in  $F$  and an element in  $S$  gives an element in  $S$ ,  $(F, S) \rightarrow S$ .

The four binary maps follow the familiar rules, as listed in the definitions of number field and vector space.

**Example.** The set of all numbers of form  $q\sqrt{2}$ , where  $q$  is a rational number, is a scalar set over the field of rational numbers.

**Example.** The set of all numbers of form  $bi$ , where  $b$  is a real number and  $i = \sqrt{-1}$ , is a scalar set over the field of real numbers.

**Example.** The set of all different amounts of money is a scalar field.

**Example.** The set of points on a straight line is not a scalar set, because it is unclear how we define the addition of two points, or the multiplication of a point and a number. However, we can form a scalar set by the following procedure. Mark a particular point on the line as the origin. The position of any point on the line relative to the origin defines a directed segment. The set of all directed segments is a scalar set over the field of real numbers. The addition of two directed segments  $x$  and  $y$  is a directed segment, formed by placing the tail of  $x$  at the origin, and placing the tail of  $y$  at the tip of  $x$ . The multiplication of a real number  $r$  and a segment  $x$  is a segment of length  $r$  times that of  $x$ .

**Example.** Similarly, the set of all times is not a scalar set, because it is unclear how we define the addition of two times, or the multiplication of a time and a number. However, we can form a scalar set by following a similar procedure. Mark a particular time as the reference. The difference of any other time relative to this reference defines a directed interval. The set of all directed intervals is a scalar set over the field of real numbers. The addition of two directed intervals  $x$  and  $y$  is a directed interval, formed by placing the tail of  $x$  at the reference, and placing the tail of  $y$  at the tip of  $x$ . The multiplication of a real number  $r$  and a n interval  $x$  is a segment of length  $r$  times that of  $x$ . We can use the same procedure to construct a scalar set by using differences in energy with respect to a reference point.

**A set of gold.** Consider a set of all different amounts of gold. We can define the addition of the pieces, but we do not have a sensible definition of the multiplication of the pieces. Thus, this set is not a number field. This set, however, is a one-dimensional vector space over the field of real numbers. That is, the set of pieces of all different amounts of gold is a scalar set. We stipulate the two binary maps in a natural way. Adding two pieces of gold corresponds to a piece of gold. Multiplying a real number  $r$  and a piece of gold corresponds to a piece of gold  $r$  times the amount.

## Unit and Magnitude

**Definition.** Of all vector spaces, one-dimensional vector spaces have a unique attribute: elements in a one-dimensional vector space *scale* with one another by numbers in  $F$ . Let  $u$  be a nonzero element in  $S$ . An element  $s$  in  $S$  scales with  $u$ , namely,

$$s = s_M u,$$

where  $s_M$  is a number in  $F$ . We call  $u$  a unit of the scalar set  $S$ , and  $s_M$  the magnitude of the scalar  $s$  relative to the unit  $u$ .

**Remark.** A scalar set is, by definition, a one-dimensional vector space. Nonetheless it is helpful to replace generic terms for a vector space with specific terms for a scalar set. We replace the phrase “one-dimensional vector space” with the phrase “scalar set”. We call an element in the vector space a vector, and call an element in the scalar set a scalar. A basis of an  $n$ -dimensional vector space consists of  $n$  linearly independent elements in the vector space, whereas a unit of a scalar set is a nonzero element in the scalar set. We replace the phrase “components of a vector with respect to a basis” with the phrase “magnitude of a scalar relative to a unit”.

**Do not confuse a scalar with its magnitude.** For a three-dimensional vector space  $V$  over a number field  $F$ , given a basis  $e_1, e_2, e_3$ , a vector  $x$  in  $V$  is a linear combination,  $x = x^1 e_1 + x^2 e_2 + x^3 e_3$ , where the components  $x^1, x^2, x^3$  are numbers in  $F$ . We should not confuse a vector with its components. For the set of directed segments in a plane, a vector is an object, a directed segment, and its components are real numbers.

Similarly, for a scalar set  $S$  over a number field  $F$ , given a unit  $u$ , a scalar  $s$  in  $S$  scales with the unit,  $s = s_M u$ , where the magnitude  $s_M$  is a number in  $F$ . For the set of gold, the scalar is an object, a piece of gold, and the magnitude is a real number. We do not confuse a piece of gold with a number.

**Extensive property.** A piece of a substance, such as gold, has many physical properties, including volume, shape, color, temperature, mass, energy. A physical property is extensive if it is proportional to the amount of the substance. Volume, mass and energy are extensive properties. Shape, color, temperature are not extensive properties.

We can use a one-dimensional vector space  $S$  to model an extensive physical property such as mass. In this model,  $F$  is the field of real numbers. We define the addition of two masses by lumping them together. We define the multiplication of a mass and a real number  $r$  by another mass  $r$  times the amount.

We use a particular element in  $S$  as the unit for this quantity. For example, for the scalar set  $S$  of all masses, the unit mass, kilogram, is the mass of a block metal, called the International Prototype Kilogram (IPK), preserved in a vault located in Sevres, France. Any other mass equals this unit times a real number. For example, 1.7 kg means a mass, which is 1.7 times the mass of the IPK. The mass of the IPK is an element in the scalar set of masses, and by an international convention we agree to call it a unit of mass. The mass 1.7 kg is another element in the set.

Like mass, each of extensive physical properties, including energy, entropy, and electric charge, forms a scalar set.

**Counter example.** The set of all temperatures is not a scalar set.

### Change of Unit

Let  $u$  and  $\tilde{u}$  be two non-zero scalars in a scalar set  $S$  over a number field  $F$ . The two scalars are proportional to each other:

$$\tilde{u} = pu,$$

where  $p$  is a number in  $F$ , and is the magnitude of the scalar  $\tilde{u}$  relative to the scalar  $u$ .

A scalar  $s$  in  $S$  scales with either unit:

$$s = s_M u = \tilde{s}_M \tilde{u},$$

where the number  $s_M$  in  $F$  and is the magnitude of the scalar  $s$  relative to the unit  $u$ , and the number  $\tilde{s}_M$  is the magnitude of the scalar  $s$  relative to the unit  $\tilde{u}$ .

A combination of the above expressions gives that

$$s_M = p\tilde{s}_M.$$

This expression relates three numbers in  $F$ . The magnitude of a scalar converts in a way opposite to the way in which the unit of the scalar set converts. Thus, the scalar set is contravariant.

**Example.** Kilogram and pound are two units of mass. They convert to each other by 1 kilogram = 2.20462 pounds. Thus, a 10-pound turkey is 4.5 kilograms.

**Example.** Money in the United States comes in many units: cent, nickel, dime, quarter, and dollar. These units are represented by distinct physical objects.

### Linear Function

Let  $S$  and  $T$  be two scalar sets over a number field  $F$ . Write a map  $g: S \rightarrow T$  as

$$t = g(s),$$

where  $s$  is an element in  $S$ , and  $t$  is an element in  $T$ . The map is a linear function if

- 1)  $g(x+y) = g(x) + g(y)$  for any  $x$  and  $y$  in  $S$
- 2)  $g(\alpha x) = \alpha g(x)$  for any  $\alpha$  in  $F$  and any  $x$  in  $S$

We write the linear map  $g(s)$  as  $gs$  if it does not cause any confusion.

**Remark.** We have just written conditions 1 and 2 out of habit. Because  $S$  is a scalar set, (2) implies (1). Let  $u$  be a non-zero element in  $S$ . The two scalars  $x$

and  $y$  in  $S$  scale as  $x = \tilde{x}u$  and  $y = \tilde{y}u$ , where  $\tilde{x}$  and  $\tilde{y}$  are numbers in  $F$ . Consequently,

$$g(x+y) = g((\tilde{x} + \tilde{y})u) = (\tilde{x} + \tilde{y})g(u) = \tilde{x}g(u) + \tilde{y}g(u) = g(x) + g(y).$$

**Example.** Let  $U$  be the set of various amounts of US dollars, and  $E$  be the set of various amounts of Euros. A linear map  $r: U \rightarrow E$  is a rate of exchange.

**Example.** Let  $G$  be the set of various amounts of gold, and  $C$  be the set of various amounts of money. A linear map  $p: G \rightarrow C$  is a price of gold.

**Example.** Let  $M$  be the set of different masses of gold, and  $V$  be the set of different volumes. The linear map  $\rho: V \rightarrow M$  is the density of gold.

**Example.** Let  $T$  be the set of directed intervals of time, and  $D$  be the set of directed segments on a geometric line. The linear map  $c: T \rightarrow D$  is the speed.

**Example.** Let  $T$  be the set of directed intervals of time, and  $E$  be the set of differences in energy. The linear map  $P: T \rightarrow E$  is the power.

**Example.** Let  $T$  be the set of directed intervals of time, and  $C$  be the set of various amounts of electric charge. The linear map  $I: T \rightarrow C$  is the electric current.

**Example.** Let  $C$  be the set of various amounts of electric charge, and  $E$  be the set of various amounts of energy. The linear map  $\phi: C \rightarrow E$  is the electric potential.

**Example.** Let  $N$  be the set of various amounts of a species of molecules, and  $E$  be the set of various amounts of energy. The linear map  $\mu: N \rightarrow E$  is the chemical potential.

**Magnitude of a linear map.** The magnitude of a linear map is relative to the units of the two scalar sets. Let  $S$  and  $T$  be two scalar sets over a number field  $F$ , and  $g: S \rightarrow T$  be a linear map. The linear map associates an element  $s$  in  $S$  to an element  $t$  in  $T$ :

$$t = gs.$$

Let  $u$  be a unit of  $S$ . An element  $s$  in  $S$  scales with  $u$ , namely,

$$s = s_M u.$$

This expression defines the number  $s_M$  in  $F$  as the magnitude of the scalar  $s$  relative to the unit  $u$ . Similarly, let  $v$  be a unit of  $T$ . An element  $t$  in  $T$  scales with  $v$ , namely,

$$t = t_M v.$$

This expression defines the number  $t_M$  in  $F$  as the magnitude of the scalar  $t$  relative to the unit  $v$ .

Because  $gs$  is a linear map, we write

$$gs = g(s_M u) = s_M gu .$$

Note that  $u$  is an element in  $S$ , so that  $gu$  is an element in  $T$ , and must scale with  $v$ . We write

$$gu = g_M v .$$

This expression maps the unit  $u$  of  $S$  to the unit  $v$  of  $T$ . We call the number  $g_M$  in  $F$  as the magnitude of the linear map with respect to the units  $u$  and  $v$ .

A combination of the above expressions, we obtain that

$$t_M v = g_M s_M v .$$

The components must equal:

$$t_M = g_M s_M$$

This expression relates the three magnitudes, the numbers in  $F$ .

**Example.** Let  $S$  be the set of all numbers of form  $q\sqrt{2}$ , and  $T$  be the set of all numbers of form  $bi$ , where  $q$  and  $b$  are rational numbers, and  $i = \sqrt{-1}$ . A particular a linear map  $g: S \rightarrow T$  associates an element  $q\sqrt{2}$  in  $S$  to an element  $1.6qi$  in  $T$ ; that is,  $g(q\sqrt{2}) = 1.6qi$ . This description is independent of the choice of units of the two scalar sets.

We can also choose units for the two scalar sets, say  $u = 2\sqrt{2}$  as a unit of  $S$ , and  $v = 3i$  as a unit of  $T$ . Write an element  $s$  in  $S$  as  $s = \hat{s}2\sqrt{2}$ , and an element  $t$  in  $T$  as  $t = \hat{t}3i$ , where  $\hat{s}$  and  $\hat{t}$  are rational numbers, and are the magnitudes of the two scalars relative the two units.

Recall that  $g(u) = \hat{g}v$  defines the rational number  $\hat{g}$  as the magnitude of the linear map  $g$  relative to the two units  $u$  and  $v$ . Write

$$g(2\sqrt{2}) = (1.6)2i = (1.6)(2/3)(3i),$$

so that the magnitude of the linear map  $g$  is  $\hat{g} = (1.6)(2/3)$ .

**Change of units of the two scalar sets.** Let  $u$  and  $\tilde{u}$  be two non-zero scalars in  $S$ . The two scalars are proportional to each other:

$$\tilde{u} = pu ,$$

where  $p$  is a number in  $F$ , and is the magnitude of the scalar  $\tilde{u}$  relative to the scalar  $u$ . Let  $v$  and  $\tilde{v}$  be two non-zero scalars in  $S$ . The two scalars are proportional to each other:

$$\tilde{v} = rv,$$

where  $r$  is a number in  $F$ , and is the magnitude of the scalar  $\tilde{v}$  relative to the scalar  $v$ .

Recall that  $gu = g_M v$  defines the number  $g_M$  in  $F$  as the magnitude of the linear map  $g$  relative to the two units  $u$  and  $v$ . Similarly,  $g(\tilde{u}) = \tilde{g}_M \tilde{v}$  defines the number  $\tilde{g}_M$  in  $F$  as the magnitude of the linear map  $g$  relative to the two units  $\tilde{u}$  and  $\tilde{v}$ . The two magnitudes  $g_M$  and  $\tilde{g}_M$  are related as

$$\tilde{g}_M = rg_M / p.$$

Thus the magnitude of the linear map is covariant with respect to the unit of  $T$ , but contravariant with respect to the unit of  $S$ .

**The collection of all linear maps from one scalar set to the other scalar.** Let  $L(S, T)$  be the collection of all linear maps from a scalar set  $S$  to a scalar set  $T$ , both over the number field  $F$ . For every element  $s$  in  $S$ , every number  $\alpha$  in  $F$ , and every linear map  $g: S \rightarrow T$ ,  $\alpha(gs)$  is an element in  $T$ . The multiplication of  $\alpha$  and  $g$ , written as  $\alpha g$ , is defined by

$$(\alpha g)s = \alpha(gs).$$

That is, the collection  $L(S, T)$  is also a scalar set space over the number field  $F$ .

**Example.** Let  $S$  be the set of all numbers of form  $q\sqrt{2}$ , and  $T$  be the set of all numbers of form  $bi$ , where  $q$  and  $b$  are rational numbers, and  $i = \sqrt{-1}$ . Each linear from  $S$  to  $T$  associates an element  $q\sqrt{2}$  in  $S$  to an element  $rqi$  in  $T$ , where  $r$  is a particular rational number. That is, each rational number  $r$  corresponds to a distinct linear map from  $S$  to  $T$ . All these linear maps constitute  $L(S, T)$ .

**Successive linear maps.** Let  $S$ ,  $T$  and  $U$  be three scalar sets over a number field  $F$ . Consider two successive linear maps,  $g: S \rightarrow T$  and  $h: T \rightarrow U$ . We can use the successive maps to define a map  $hg: S \rightarrow U$ .