SCALAR

Number and Scalar

Number. Previous notes have defined a number field as a set closed under two operations: addition and multiplication. In linear algebra we will mostly use the field of real numbers, R, and the field of complex numbers, C. Occasionally we will use the field of rational numbers, Q. In general we denote a number field by F.

Two usages of the word scalar. Textbooks of linear algebra use the word *scalar* in two ways. In the first usage, the word scalar is a synonym to the word *number*, an element in field F.

In the second usage, the word scalar appears in the definitions of linear form, bilinear form, quadratic form, and inner product. In this second usage, the scalar is an object distinct from the number. Adding two scalars results in a scalar, multiplying a scalar with a number resulting in a scalar, but multiplying two scalars does not result in a scalar in the same set.

In physics, the word scalar is used to indicate a property like mass, volume, charge, and energy. These properties are scalable and additive, and are known as extensive properties. The usage in physics is consistent with the second usage in linear algebra, but is inconsistent with the first one in several ways:

- A physical property like mass is more than just a number; it has a unit.
- The multiplication defined on a field makes no sense to a physical quantity like mass: the multiplication of two elements in F gives yet another element in F, but the multiplication of two masses does not give another mass.
- If we regard both mass and volume as elements in the field F, then we need to assign a meaning to the addition of mass and volume. What does that even mean?

Do not confuse number and scalar. Using the same word scalar in two ways, textbooks of linear algebra confuse the objects of two distinct types. In our notes, we will call each element in the field F a number, and will reserve the word scalar for an object in another set, which we call *scalar set*. As we will see, a number field and a scalar set have different algebraic structures. Numbers and scalars are distinct objects.

The importance of being scalar. Scalar is the fundamental building block of linear algebra. Being linear is being scalar. We will use scalar sets to construct vector spaces of any dimension. We will use scalar sets to define other objects, such as linear form and inner product.

Scalar sets are also used to construct models whenever linear algebra is used. In physics, for example, we use scalar sets to model mass, energy, space, and time. We then put space and time together to model spacetime. Give us scalars. We will build a world.

Scalar Set

A set S is called a scalar set over a number field F if the following conditions hold.

Adding two elements in *S* gives an element in *S*. To any two elements x and y in S there corresponds an element in S, written as $x+y$, called the addition of x and y. Addition obeys the following rules:

1) Addition is commutative: $x+y=y+x$ for every x and y in S.

2) Addition is associative: $\left(x+y\right)+z=x+\left(y+z\right)$ for every x, y and z in S.

3) There exists an element 0 (the zero element) in V such that $0+x=x$ for every x in S.

4) For every x in V there exists an element (the negative element) z in S such that $x+z=0$. We write $z=-x$.

Multiplying an element in *F* and an element in *S* gives an element in *S*. To every element α in F and every element x in V there corresponds an element in S, written as αx, called the multiplication of α and x. The multiplication obeys the following rules:

5) $1\cdot x=x$ for every x in S.

6) $\alpha\left(\beta x\right)=\left(\alpha\beta\right)x$ for every x in V and for every α, β in F.

Multiplication is distributive over addition. The multiplication of an element in F and an element in V is distributive over two the types of addition: addition of elements in F and addition of elements in V.

7) $\left(\alpha+\beta\right)x=\alpha x+\beta x$ for every x in S and for every α, β in F.

8) $\alpha\left(x+y\right)=\alpha x+\alpha y$ for every x and y in S and every α in F.

Elements in *S* are linearly dependent. For every x and y in S, there exist α and β in F, not both of which are zero, such that

9) $\alpha x+\beta y=0$.

We call each element in F a number, and each element in S a scalar.

A scalar set invokes two sets and four operations. The definition of number field invokes a single set F and two binary maps. As stipulated in the definition of a field, the set F is closed under two binary maps: adding two elements in F gives an element in F, $(F,F)\to F$; multiplying two elements in F gives an element in F, $(F,F)\to F$.

By contrast, the definition of scalar set invokes two sets F and S, as well as four binary maps. In addition to the two binary maps used to define the number field F, we invoke two other binary maps: adding two elements in S gives an element in S, $(S,S)\to S$; multiplying an element in F and an element in S gives an element in S, $(F,S)\to S$.

Given two sets S and F, we can think of many possible binary maps to combine elements in the two sets. Most binary maps, however, do not appear in the definition of scalar set. In particular, we exclude from the definition of scalar set any binary map that might represent the addition of an element in F and an element in S, or represent the multiplication of two elements in S.

Zeros in the two sets. The element zero in the set F is an object different from the element zero in the set S. The two objects have the same notation, 0. We tell them apart by seeing them in context.

Likewise, we distinguish the addition of two numbers from the addition of two scalars, and distinguish the multiplication of two numbers from the multiplication a number and a scalar.

A scalar set is a one-dimensional vector space. The algebraic structure of a scalar set is clearly different from that of a number field. For people who know the definition of vector space, it is evident that a scalar set is, by definition, a one-dimensional vector space. Axioms 1)-8) define a vector space of any dimensions, whereas Axiom 9) makes the vector space one-dimensional. We will talk about vector space later.

Examples and Counterexamples

Example. The set of all numbers of form $p\sqrt{2}$, where p is a rational number, is a scalar set over the field of rational numbers.

Counterexample. The set of all numbers of form $p\sqrt{2}+q\sqrt{3}$, where p are q are rational numbers, is not a scalar set over the field of rational numbers. The set violates Axim 9).

Example. The set of all numbers of form bi, where b is a real number and $i=\sqrt{-1}$, is a scalar set over the field of real numbers.

Counterexample. The set of all numbers of form $a+bi$, where a are b are real numbers and $i=\sqrt{-1}$, is not a scalar set over the field of real numbers. The set violates Axim 9).

Gold. We have collected pieces and pieces of gold. We can define the addition of the pieces, but we do not have a sensible definition of the multiplication of the pieces. Thus, this set of gold is not a number field.

This set of gold, however, is a scalar set over the field of real numbers. Adding two pieces of gold corresponds to a piece of gold. Multiplying a real number r and a piece of gold corresponds to a piece of gold r times the amount. For any two pieces of gold x and y, there exist α and β in R, not both of which are zero, such that $\alpha x+\beta y=0$.

Money. The set of all different amounts of money is a scalar set. We regard US dollars and Euros as different scalar sets. Within the set of the US dollars, the addition of two amounts means putting the two amounts together. Multiplying a real number r and an amount of money corresponds to an amount of money r times. For any two amounts of money x and y, there exist α and β in R, not both of which are zero, such that $\alpha x+\beta y=0$.

Mass. The set of all different amounts of mass is a scalar set.

Temperature is not a scalar. Since the time of Galileo, people have ordered levels of hotness—temperature—by numbers. Many physicists call the temperature as a scalar. This usage is inconsistent with our definition of scalar. The set of all temperatures is *not* a scalar set. The addition of two temperatures is meaningless.

Linear Algebraic Equation

Homogeneous linear equation. Let us state the simplest algebraic equation: a homogeneous linear equation of a single unknown. Let S be a scalar set over a number field F. Given an element α in F, find an element x in S that satisfies the equation

$$\alpha x=0 .$$

Because αx is an element in S, the "0" on the right side of the equation is the element 0 in S.

The equation has a single unknown: the x element in S. The equation is linear in the unknown x. The equation is called homogeneous because the right side is the element zero in S. In solving this equation, we distinguish two cases: $\alpha=0$ and $\alpha\neq 0$.

The case of $\alpha = 0$. If $\alpha = 0$, any $x \in S$ is a solution to the equation. That is $0x = 0$ for every element *x* in *S*. We prove this statement using the axioms that define the field and the scalar set. For every $x \in S$, note that

$$0x = (0+0)x = 0x + 0x.$$

Subtract $0x$ from both sides of the equation, we obtain that $0 = 0x$.

The case of $\alpha \neq 0$. If $\alpha \neq 0$, the equation has a unique solution, $x = 0$. We first show that $x = 0$ is a solution to the equation $\alpha x = 0$. For every $\alpha \in F$, note that

$$\alpha 0 = \alpha(0+0) = \alpha 0 + \alpha 0.$$

Subtract $\alpha 0$ from both sides of the equation, we obtain that $0 = \alpha 0$. Thus, the zero element in *S* times any number in *F* gives the zero element in *S*.

We next show that, if α is a nonzero element in *F*, then $x = 0$ is a unique solution to the equation $\alpha x = 0$. Because, $\alpha \neq 0$, another element α^{-1} exists in *F*, such that $\alpha^{-1}\alpha = 1$. Not that

$$x = 1x = (\alpha^{-1}\alpha)x = \alpha^{-1}(\alpha x) = \alpha^{-1}0 = 0.$$

The last equation holds because we have already shown that the zero element in *S* times any number in *F* gives the zero element in *S*.

Inhomogeneous linear equation. Given an element α in *F* and given an element *y* in *S*, find an element *x* in *S* that satisfies the equation

$$\alpha x = y.$$

When the right side is a nonzero element in *S*, the equation is called *inhomogeneous*.

We again distinguish two cases: $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ and $y = 0$, we already know that any *x* in *S* is a solution to the equation $\alpha x = y$. If $\alpha = 0$ but $y \neq 0$, the equation $\alpha x = y$ has no solution.

If $\alpha \neq 0$, the equation $\alpha x = y$ has a unique solution $x = \alpha^{-1}y$.

Unit and Magnitude

Elements in a scalar set *scale* with one another. Let *u* be a nonzero element in *S*, and let *s* be any element in *S*. Axiom 9) stipulates that there exist two elements α and β in *F*, not both of which are zero, such that

$$\alpha s + \beta u = 0.$$

Because *u* is a nonzero element in *S*, α must be a nonzero element in *F*; otherwise, β would be the element zero in *F*, which would make both α and β the element zero in *F*.

Now that $\alpha \neq 0$, we write equation $\alpha s + \beta u = 0$ as

$$s = \alpha^{-1}\beta u.$$

Denote $s_M = \alpha^{-1}\beta$ and write

$$s = s_M u,$$

where s_M is a number in *F*. We call *u* a unit of the scalar set *S*, and s_M the magnitude of the scalar *s* relative to the unit *u*. We say that elements in *S* scale with one another.

Remark. For people know about the vector space of any dimension, we note the following. A scalar set is, by definition, a one-dimensional vector space. Nonetheless it is helpful to replace generic terms for a vector space with specific terms for a scalar set. We replace the phrase "one-dimensional vector space" with the phase "scalar set". We call an element in the vector space a vector, and call an element in the scalar set a scalar. A basis of an *n*-dimensional vector space consists of *n* linearly independent elements in the vector space, whereas a unit of a scalar set is a nonzero element in the scalar set. We replace the phrase "components of a vector with respect to a basis" with the phrase "magnitude of a scalar relative to a unit".

Do not confuse a scalar with its magnitude. For a scalar set *S* over a number field *F*, given a unit *u*, a scalar *s* in *S* scales with the unit, $s = s_M u$, where the magnitude s_M is a number in *F*. For the set of gold, the scalar is an object, a piece of gold, and the magnitude is a real number. We do not confuse a piece of gold with a number.

The unit of mass. Various amounts of mass form a scalar set. For this scalar set, the unit mass, kilogram, is the mass of a block metal, called the International Prototype Kilogram (IPK), preserved in a vault located in Sevres, France. Any other mass equals this unit times a real number. For example, 1.7 kg means a mass, which is 1.7 times the mass of the IPK. The mass of the IPK is an element in the scalar set of masses, and by an international convention we agree to call it a unit of mass. The mass 1.7 kg is another element in the set.

The unit of money. In the United States, the unit of money is a dollar, with symbol $.

Change of Unit

Given a scalar set *S* over a number field *F*, once we choose a nonzero element *u* in *S* as a unit, any other element *s* in *S* scales with the unit, $s = s_M u$. Here s_M denotes the magnitude of the scalar *s* relative to the unit *u*. For given *u* and *s*, the value of s_M is unique.

We can, of course, choose any nonzero element in S as a unit. Let u and $\tilde{u}$ be two non-zero scalars in a scalar set S over a number field F. The two scalars are proportional to each other:

$$\tilde{u} = pu,$$

where p is a number in F, and is the magnitude of the scalar $\tilde{u}$ relative to the scalar u.

A scalar s in S scales with either unit:

$$s = s_M u = \tilde{s}_M \tilde{u},$$

where the number s_M in F and is the magnitude of the scalar s relative to the unit u, and the number $\tilde{s}_M$ is the magnitude of the scalar s relative to the unit $\tilde{u}$.

A combination of the above expressions gives that

$$s_M = p\tilde{s}_M.$$

This expression relates three numbers in F. The magnitude of a scalar converts in a way opposite to the way in which the unit of the scalar set converts. Thus, the scalar set is contravariant.

Example. Kilogram and pound are two units of mass. They convert to each other by 1 kilogram = 2.20462 pounds. Thus, a 10-pound turkey is 4.5 kilograms.

Example. Money in the United States comes in many units: cent, nickel, dime, quarter, and dollar. These units are represented by distinct physical objects.

Scalar Set as a Model of Reality

Apple. We have piles and piles of apples. We perform operations of two types. The addition of any two piles corresponds to another pile having the same quantity of apples as the two piles put together. The multiplication of any pile by any real number α corresponds to another pile α times the quantity of applies. The multiplication requires us to multiply apples by number, but does not require us to multiply apples by apples.

The piles and piles of apples form a set. If we wish to emphasize, we write the set as APPLE. Each element in the set is a pile containing a distinct quantity of apples. We model this set as a scalar set over the field of real numbers. As we have just learned in the formal definition, a scalar set is a set *closed* under two operations. The addition of any two elements in the set corresponds to another element in the set. The multiplication of any element in the set and any real number corresponds to another element in the set.

The piles and piles of apples form a scalar set. Denote a particular pile of apples by u. Any other pile of apples, x, is pile u multiplying a real number α, namely,

$$x = \alpha u.$$

All elements in the scalar set *scale* with one another by real numbers.

Model and reality. The definition of scalar set requires that an element in the set multiplying any real number be an element in the set. If the real number is too large, we do not have that many apples. If the real number is too small, we reach subatomic dimension, in which case the "pile" no longer contains any apple. Also, the definition of the vector space will require that negative quantities of apples be in the set.

In representing the reality with a model, we ignore inconvenient truths. (All models are wrong, but some are useful.) But we do check what the model predicts against the reality. If the model predicts a negative quantity of apples, it means that we are in deficit. If the model predicts a non-integer quantity of apples, we cut apples in pieces, or just round up. If a piece is too small to preserve its appleness, we may approximate the piece as the vector "zero".

A model achieves the economics of abstraction. A model is not the reality. Why do we link the reality to a model then? A map of a city is not the city, but the map lets us plan a tour without walking through the city. The map would be useless if it were as large and as detailed as the city. The model abstracts: it subtracts most details, retains a few, and idealizes them. Abstraction is value. A sketch of a shrimp by Qi Baishi is worth a lot more than a photo of a shrimp, or the shrimp itself.

The model of the piles of apples lets us reason something about apples without having apples. The reasoning is about the quantities of apple. This model ignores all other aspects of apples—the smell, the taste, the color, etc.

This particular model—the scalar set, and its generalization, the vector space—has been reasoned thoroughly in mathematics. The model applies to piles of apples, and to piles of oranges. The model applies to piles containing both apples and oranges. The model applies to durations of time. The model, with one more feature (the metric), applies to displacements in space. The model, with yet one more feature (the constancy of the speed of light), applies to spacetime. The model achieves the economics of abstraction, as well as the economics of scale.

Algebraic Physics

Like calculus and geometry, algebra has long been used to build physics. In these notes of linear algebra, we will highlight the roles of linear algebra in physics. Here we begin with using scalar sets to model nature.

Extensive property. A piece of a substance, such as gold, has many physical properties, including volume, shape, color, temperature, mass, and energy. A physical property is extensive if it is proportional to the amount of the substance. Volume, mass, and energy are extensive properties. Shape, color, temperature are not extensive properties.

We can use a one-dimensional vector space S to model an extensive physical property such as mass. In this model, F is the field of real numbers. We define the addition of two masses by lumping them together. We define the multiplication of a mass and a real number r by another mass r times the amount.

Like mass, each of extensive physical properties, including energy, entropy, and electric charge, forms a scalar set.

Distance. The set of points on a straight line is not a scalar set, because it is unclear how we define the addition of two points, or the multiplication of a point and a number. However, we can form a scalar set by the following procedure. Mark a particular point on the line as the origin. The position of any point on the line relative to the origin defines a directed segment. The set of all directed segments is a scalar set over the field of real numbers. The addition of two directed segments x and y is a directed segment, formed by placing the tail of x at the origin, and placing the tail of y at the tip of x. The multiplication of a real number r and a segment x is a segment of length r times that of x.

Time. Similarly, the set of all times is not a scalar set, because it is unclear how we define the addition of two times, or the multiplication of a time and a number. However, we can form a scalar set by following a similar procedure. Mark a particular time as the reference. The difference of any other time relative to this reference defines a directed interval. The set of all directed intervals is a scalar set over the field of real numbers. The addition of two directed intervals x and y is a directed interval, formed by placing the tail of x at the reference, and placing the tail of y at the tip of x. The multiplication of a real number r and a n interval x is a segment of length r times that of x. We can use the same procedure to construct a scalar set by using differences in energy with respect to a reference point.