

Stress-Strain Curve of a Metal

Two length scales. A body is made of atoms, each atom is made of electrons, protons and neutrons, and each proton is made of... This kind of description is too detailed for us. We will not go very far in helping the engineer if we keep thinking of a bridge as a pile of atoms. Instead, we will develop a continuum theory. This theory is effective whenever we can identify two widely separated length scales.

The deformation of the body is in general inhomogeneous—that is, the amount of deformation varies from one part of the body to another part. We may identify two length scales:

- length scale over which the macroscopic variation of deformation occurs
- length scale over which the microscopic process of deformation occurs.

For example, when a rubber eraser is bent, the macroscopic field of deformation varies over a length scaled with the thickness of the eraser (several millimeters). The rubber is a network of molecular chains. The microscopic process of deformation occurs over the length scaled with the length of an individual molecular chain (several nanometers).

Representative elementary volume. In many applications, the two length scales are widely separated. If they are, we can describe the behavior of the material by using a volume much larger than the size characteristic of the microscopic process of deformation, but much smaller than the size characteristic of the macroscopic deformation. Such a volume is known as a representative elementary volume (REV).

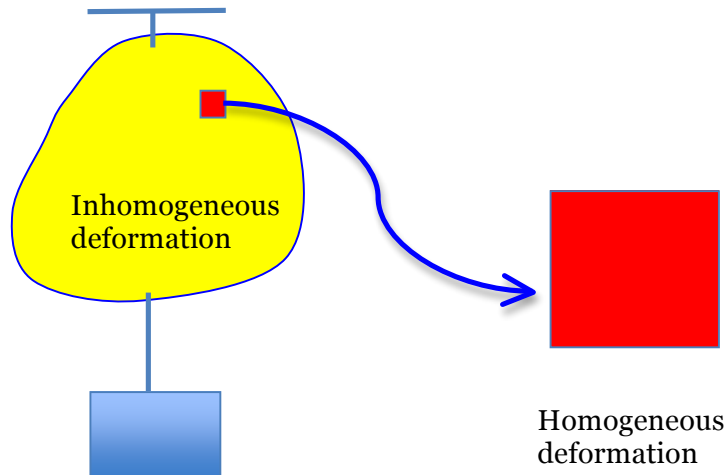
In the rubber eraser, for example, the microscopic process is the thermal motion of individual polymer chains, the body is the whole eraser, and the REV can be a small part of the eraser.

As another example, consider an airplane wing made of aluminum. The microscopic process can be activities of dislocations in aluminum, the body can be the entire wing, and the REV can be a tensile specimen of the aluminum.

The size of REV should be selected well between the two lengths scales. If a volume is too small, the volume cannot be treated as a continuum. If a volume is too large, the shape of the body affects the behavior of the volume.

Division of labor. To analyze the inhomogeneous deformation in the body, the continuum theory regards the body as a sum of many small pieces. Each small piece evolves in time through a sequence of homogeneous deformations. All the pieces are then put together to represent the inhomogeneous deformation of the entire body. The division of labor results in

two distinct tasks.



First, we describe the evolution of homogeneous deformation. We relate the history of stress to the history of deformation. Such a relation is known as a material model, constitutive law, equation of state, rheological model, or rheological equation of state. The material model is material-specific, and often relies on experimental measurements and microscopic modeling. However, the main quantities—stress and deformation—are applicable to all materials.

Second, we analyze the inhomogeneous deformation of a body. The continuum theory regards the body as a sum of many small pieces. Different small pieces communicate following general principles:

- Deformation preserves compatibility.
- Deformation conserves mass.
- Deformation conserves linear momentum.
- Deformation conserves angular momentum.

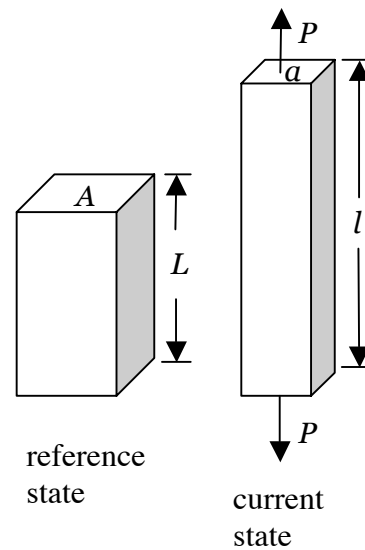
This division of labor is applicable to many materials, but not all materials. For example, some material models involve both strain and strain gradient. Materials that can be described by homogeneous deformation only are known as *simple materials*. This phrase, however, is not widely adopted. The phrase *complex material* is undefined in continuum theory, and is meaningless.

A simple type of the representative elementary volume

Subject to a load, the representative elementary volume undergoing homogeneous deformation. This homogeneous deformation is mediated by microscopic processes. Examples of simple loading conditions

- Uniaxial tension and compression
- Shear
- hydrostatic stress

A metal bar subject to a uniaxial force. When the bar is not subject to any force, the cross-sectional area is A and the length is L . We will call this state the reference state. The bar is then subject to an axial force P , and deforms to a new state, cross-sectional area a and length l . We will call this state the current state. The experimentalist records the force as a function of the length.



Strain. Define the **engineering strain** by the elongation of the bar in the current state divided by the length of the bar in the reference state:

$$e = \frac{\text{elongation}}{\text{length in the reference state}} = \frac{l - L}{L}.$$

Another type of strain is defined as follows. Deform the material from a current length l by a small amount to $l + \delta l$. Define the increment in the strain, $\delta \varepsilon$, as the increment in the length of the bar divided by the current length of the bar, namely,

$$\delta \varepsilon = \frac{\text{increment in length}}{\text{length in the current state}} = \frac{\delta l}{l}.$$

This equation defines the increment of **natural strain**. Integrating from L to l , we obtain that

$$\varepsilon = \log\left(\frac{l}{L}\right).$$

There seems to be no lack of ingenuity to invent yet another definition of strain. All these definitions contain the same information: the ratio l/L . The two definitions of strain relate to each other as

$$\varepsilon = \log(1 + e)$$

When deformation is small, namely, $e \ll 1$, the true strain and the engineering

strain are approximately equal, $\varepsilon \approx e$.

Stress. When dealing with finite deformation, we must be specific about the area used in defining the stress. Define the **nominal stress**, s , as the force applied to the bar in the current state divided by the cross-sectional area of the bar in the reference state:

$$s = \frac{\text{force in the current state}}{\text{area in the reference state}} = \frac{P}{A}.$$

The nominal stress is also known as the engineering stress, or the first Piola-Kirchhoff stress.

Define the **true stress**, σ , as the force in the current state divided by the area in the current state, namely,

$$\sigma = \frac{\text{force in the current state}}{\text{area in the current state}} = \frac{P}{a}.$$

The true stress is also known as the Cauchy stress

You should not be misled by the names. The true stress is no truer than the nominal stress. The engineering stress is no less scientific. They are just different definitions of stress, and we need to have different names for them.

Work. When the bar elongates from length l to length $l + \delta l$, the force P does work $P\delta l$. Recall one pair of definitions of stress and strain:

$$P = sA, \quad l = \lambda L.$$

Consequently the work done by the force is

$$P\delta l = ALs\delta\lambda.$$

Since AL is the volume of the bar in the reference state, we note that

$$s\delta\lambda = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

We say that the nominal stress and the stretch are *work-conjugate*. Also note that $\delta e = \delta\lambda$, so that the nominal stress is also work-conjugate to the engineering strain.

Recall another pair of definitions of stress and strain:

$$P = \sigma a, \quad \delta l = l\delta\varepsilon.$$

The work done by the force is

$$P\delta l = a l \sigma \delta\varepsilon.$$

Since al is the current volume of the bar, we note that

$$\sigma\delta\varepsilon = \frac{\text{increment of work in the current state}}{\text{volume in the current state}}.$$

That is, the true stress is work-conjugate to the natural strain.

Stress-strain curve. Loading, unloading, reloading.

Elasticity. When the applied stress is low, dislocations are immobile. Elastic deformation is due to distortion of atomic bonds, and atoms do not change neighbors. Upon unloading, the distortion disappears, and the stress-strain curve exhibits no hysteresis. Atomic bonds behave like springs. Elastic modulus reflects the stiffness of the atomic bonds.

Yield. When the applied stress exceeds some value, dislocations are mobile. Plastic deformation is due to the slips of crystalline lattice mediated by the movements of the dislocations. Atoms across the slip plane change neighbors. Upon unloading, the slips remain, and the stress-strain curve exhibits hysteresis.

Non-hardening. After yielding, the slope of the stress-strain curve is much smaller than Young's modulus. Plastic deformation of metal is similar to viscous flow of a liquid. Atoms change neighbors, but new neighbors do not change the kind of atomic bonds. Thus the amount of deformation has (nearly) no effect on stress needed for deformation. An excellent model for most applications is non-hardening model.

True stress accounts for the change in area in tension and compression. **The increment of the natural strain** disregards the initial reference state. Using true stress and natural strain, the tension and compression stress-strain curves often look identical. In the ideal plasticity, they look exactly the same.

Work hardening (or strain hardening). Now we add a refinement for the description. Plastic deformation of a metal is not exactly the same as viscous flow of a liquid. After yielding, the applied stress increases somewhat in order to increase strain. More and more dislocations are generated. The movements of one dislocation is resisted by other dislocations. That is, the microstructure of a metal evolves during plastic deformation. By contrast, the microstructure of a liquid remains the same during viscous flow.

Power law hardening. A model to fit experimentally measured stress-strain curve. For a metal undergoing large, plastic deformation, the stress-strain curve (without unloading) is often fit to a power law in terms of the true stress and the natural strain:

$$\sigma = K\varepsilon^N,$$

where K and N are parameters to fit experimental data. Some representative values: $N = 0.15-0.25$ for aluminum, $N = 0.3-0.35$ for copper, $N = 0.45-0.55$ for stainless steel. K has the dimension of stress; it represents the true stress at strain $\varepsilon = 1$. Representative values for K are 100 MPa – 1GPa.

Unloading after plastic deformation. Dislocations remain immobile. The spacing between dislocations is much larger than the spacing between atoms. Dislocation is a line defect, and only affect the behavior of atoms in the core of the dislocation. Elastic modulus is the same as measured during loading.

Hysteresis. On the stress-strain diagram, the unloading curve does not go along the loading curve. The area enclosed by the hysteresis curve is the work done by the loading machine. This work is mostly dissipated as heat, through motion of dislocations. Only tiny fraction of the work is used to create new dislocations.

Plastic strain. Offset yield point. In most experiments, we don't try to watch the motion of dislocations. We just record the stress-strain curves. It is often hard to identify the yield point on the stress-strain curve. A practical procedure is to unload, and define the yield point by plastic strain of, say, 0.2%.

Reverse plastic flow. Isotropic hardening. Bauschinger effect. Reloading. The yield strength in the opposite direction is less than it would be if the strain had continued in the initial direction. Dislocations pile up at barriers. Back stress.

Effect of loading rate. Effect of temperature. Modulus is insensitive to loading rate and temperature, but strength can be sensitive. We should look at numbers. For many applications, the ranges of loading rate and temperature are not large, and we may neglect the effect of the loading rate and temperature on the stress-strain curve.

Hydrostatic stress does not affect plastic deformation

- Hydrostatic stress only causes elastic deformation. Hydrostatic stress cannot move dislocation.
- Plastic deformation changes the shape, not the volume.
- When a hydrostatic stress is superimposed upon a uniaxial stress, the yield strength and plastic strain remain unchanged. Hydrostatic stress does not even change the mobility of dislocations.

- Under hydrostatic tension, a metal undergoes no plastic flow until cavities form.

These experimental observations may be interpreted in terms of dislocation picture. In most engineering applications, the hydrostatic component of the stress is at most a few times of yield strength. This stress is too small to affect the dislocation core.