Theories of Failure

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Summary

- Maximum principal stress theory
- Maximum principal strain theory
- Maximum strain energy theory
- Distortion energy theory
- Maximum shear stress theory
- Octahedral stress theory
Introduction

- Failure occurs when material starts exhibiting inelastic behavior
- Brittle and ductile materials – different modes of failures – mode of failure – depends on loading
- Ductile materials – exhibit yielding – plastic deformation before failure
- Yield stress – material property
- Brittle materials – no yielding – sudden failure
- Factor of safety (FS)
Introduction

- Ductile and brittle materials

Well – defined yield point in ductile materials – FS on yielding

No yield point in brittle materials sudden failure – FS on failure load

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Introduction

- Stress developed in the material < yield stress
- Simple axial load

If $\sigma_x = \sigma_Y \Rightarrow$ yielding starts – failure

Yielding is governed by single stress component, $\sigma_x$

Similarly in pure shear – only shear stress.

If $\tau_{\text{max}} = \tau_Y \Rightarrow$ Yielding in shear

Multi-axial stress state ??

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Introduction

- Various types of loads acting at the same time

Axial, moment and torque

Internal pressure and external UDL
Introduction

- Multiaxial stress state – six stress components – one representative value
- Define effective / equivalent stress – combination of components of multiaxial stress state
- Equivalents stress reaching a limiting value – property of material – yielding occurs – Yield criteria
- Yield criteria define conditions under which yielding occurs
- Single yield criteria – doesn’t cater for all materials
- Selection of yield criteria
- Material yielding depends on rate of loading – static & dynamic

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Introduction

- Yield criteria expressed in terms of quantities like stress state, strain state, strain energy etc.
- Yield function => \( f(\sigma_{ij}, Y) \), \( \sigma_{ij} = \) stress state
- If \( f(\sigma_{ij}, Y) < 0 \) => No yielding takes place – no failure of the material
- If \( f(\sigma_{ij}, Y) = 0 \) – starts yielding – onset of yield
- If \( f(\sigma_{ij}, Y) > 0 \) - ??
- Yield function developed by combining stress components into a single quantity – effective stress => \( \sigma_e \)

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Introduction

- Equivalent stress depends on stress state and yield criteria – not a property
- Compare $\sigma_e$ with yield stress of material
- Yield surface – graphical representation of yield function, $f(\sigma_{ij}, Y) = 0$
- Yield surface is plotted in principal stress space – Haiagh – Westergaard stress space
- Yield surface – closed curve
Parameters in uniaxial tension

- **Maximum principal stress**
  
  Applied stress $\Rightarrow Y$

  \[ \sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0 \]

- **Maximum shear stress**

  \[ \tau_{\text{max}} = \left| \frac{\sigma_1 - \sigma_3}{2} \right| = \frac{Y}{2} \]

- **Maximum principal strain**

  \[ \sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0 \]

  \[ \varepsilon_Y = \frac{\sigma_1}{E} - \frac{\nu}{E} \left( \sigma_2 + \sigma_3 \right) = \frac{Y}{E} \]

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Parameters in uniaxial tension

**Total strain energy density**

Linear elastic material

\[
U = \frac{1}{2} Y \varepsilon_y = \frac{1}{2} \frac{Y^2}{E}
\]

**Distortional energy**

\[
[\sigma] = \begin{bmatrix}
Y & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
Y - p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{bmatrix}
+ \begin{bmatrix}
p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p
\end{bmatrix}
\]

First invariant = 0 for deviatoric part => \( p = \frac{Y}{3} \)

\[
U = U_D + U_V
\]

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Parameters in uniaxial tension

Volumetric strain energy density, \( U_V = \frac{p^2}{2K} \)

\[
U_V = \frac{p^2}{2K} = \frac{Y^2}{18K} = \frac{(1 - 2\nu)}{6E}Y^2
\]

\[
U_D = U - U_V
\]

\[
U_D = \frac{Y^2}{2E} - \frac{(1 - 2\nu)Y^2}{6E} = \frac{Y^2}{6E}(3 - 1 + 2\nu) = \frac{Y^2}{3E}(1 + \nu)
\]

\[
U_D = \frac{Y^2}{6G}
\]

Similarly for pure shear also

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Failure theories

- **Failure mode** –
  - Mild steel (M. S) subjected to pure tension
  - M. S subjected to pure torsion
  - Cast iron subjected to pure tension
  - Cast iron subjected to pure torsion

- **Theories of failure**
  - Max. principal stress theory – Rankine
  - Max. principal strain theory – St. Venants
  - Max. strain energy – Beltrami
  - Distortional energy – von Mises
  - Max. shear stress theory – Tresca
  - Octahedral shear stress theory
Max. principal stress theory

- Maximum principal stress reaches tensile yield stress (Y)
- For a given stress state, calculate principle stresses, $\sigma_1$, $\sigma_2$ and $\sigma_3$
- Yield function

$$f = \max \left( |\sigma_1|, |\sigma_2|, |\sigma_3| \right) - Y$$

If, $f < 0$ no yielding
$f = 0$ onset of yielding
$f > 0$ not defined

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Max. principal stress theory

- Yield surface –

\[
\begin{align*}
\sigma_1 &= \pm Y \Rightarrow \sigma_1 + Y = 0, \quad \sigma_1 - Y = 0 \\
\sigma_2 &= \pm Y \Rightarrow \sigma_2 + Y = 0, \quad \sigma_2 - Y = 0 \\
\sigma_3 &= \pm Y \Rightarrow \sigma_3 + Y = 0, \quad \sigma_3 - Y = 0
\end{align*}
\]

Represent six surfaces

Yield strength – same in tension and compression
Max. principal stress theory

- In 2D case, $\sigma_3 = 0$ – equations become

  $\sigma_1 = \pm Y \Rightarrow \sigma_1 + Y = 0, \quad \sigma_1 - Y = 0$
  
  $\sigma_2 = \pm Y \Rightarrow \sigma_2 + Y = 0, \quad \sigma_2 - Y = 0$

  Closed curve

  Stress state inside – elastic, outside $\Rightarrow$ Yielding

  Pure shear test $\Rightarrow \sigma_1 = + \tau_Y, \sigma_2 = - \tau_Y$

  For tension $\Rightarrow \sigma_1 = + \sigma_Y$

  From the above $\Rightarrow \sigma_Y = \tau_Y$

  Experimental results – Yield stress in shear is less than yield stress in tension

  Predicts well, if all principal stresses are tensile

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Max. principal strain theory

- “Failure occurs at a point in a body when the maximum strain at that point exceeds the value of the maximum strain in a uniaxial test of the material at yield point”
- ‘Y’ – yield stress in uniaxial tension, yield strain, $\varepsilon_y = Y/E$
- The maximum strain developed in the body due to external loading should be less than this
- Principal stresses $\Rightarrow \sigma_1, \sigma_2$ and $\sigma_3$ strains corresponding to these stress $\Rightarrow \varepsilon_1, \varepsilon_2$ and $\varepsilon_3$
Max. principal strain theory

Strains corresponding to principal stresses -

\[
\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E} (\sigma_2 + \sigma_3)
\]

\[
\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\nu}{E} (\sigma_1 + \sigma_3)
\]

\[
\varepsilon_3 = \frac{\sigma_3}{E} - \frac{\nu}{E} (\sigma_2 + \sigma_1)
\]

For onset of yielding

\[
|\varepsilon_1| = \frac{Y}{E} \Rightarrow \sigma_1 - \nu(\sigma_2 + \sigma_3) = \pm Y
\]

\[
|\varepsilon_2| = \frac{Y}{E} \Rightarrow \sigma_2 - \nu(\sigma_3 + \sigma_1) = \pm Y
\]

\[
|\varepsilon_3| = \frac{Y}{E} \Rightarrow \sigma_3 - \nu(\sigma_1 + \sigma_2) = \pm Y
\]

Maximum of this should be less than \( \varepsilon_y \)

There are six equations – each equation represents a plane

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Max. principal strain theory

- **Yield function**

\[
f = \max_{i \neq j \neq k} |\sigma_i - \nu \sigma_j - \nu \sigma_k| - Y, \quad i, j, k = 1, 2, 3
f = \sigma_e - Y
\]

\[
\sigma_e = \max_{i \neq j \neq k} |\sigma_i - \nu \sigma_j - \nu \sigma_k|
\]

- **For 2D case**

\[
|\sigma_1 - \nu \sigma_2| = Y \Rightarrow \sigma_1 - \nu \sigma_2 = \pm Y
\]
\[
|\sigma_2 - \nu \sigma_1| = Y \Rightarrow \sigma_2 - \nu \sigma_1 = \pm Y
\]

There are four equations, each equation represents a straight line in 2D stress space

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Max. principal strain theory

Equations –

\[ \sigma_1 - \nu \sigma_2 = Y, \quad \sigma_1 - \nu \sigma_2 = -Y \]
\[ \sigma_2 - \nu \sigma_1 = Y, \quad \sigma_2 - \nu \sigma_1 = -Y \]

Plotting in stress space

Failure – equivalent stress falls outside yield surface
Max. principal strain theory

- **Biaxial loading**

  For onset of yielding –

  \[ Y = \sigma_1 - \nu \sigma_2 = \sigma (1 + \nu) \]

  \[ Y = \sigma (1 + \nu) \]

  Maximum principal stress theory –

  \[ Y = \sigma \]

  Max. principal strain theory predicts smaller value of stress than max. principal stress theory

  Conservative design

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Max. principal strain theory

- Pure shear

Principal stresses corresponding to shear yield stress

\[ \sigma_1 = +\tau_y, \quad \sigma_2 = -\tau_y \]

For onset of yielding – max. principal strain theory

\[ Y = \tau_y + \nu \tau_y = \tau_y (1 + \nu) \]

Relation between yield stress in tension and shear

\[ \tau_y = Y / (1 + \nu) \text{ for } \nu = 0.25 \]

\[ \tau_y = 0.8Y \text{ Not supported by experiments} \]

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Strain energy theory

- “Failure at any point in a body subjected to a state of stress begins only when the energy density absorbed at that point is equal to the energy density absorbed by the material when subjected to elastic limit in a uniaxial stress state”

- In uniaxial stress (yielding)

\[ \sigma = E \varepsilon \Rightarrow \text{Hooke’s law} \]

Strain energy density,

\[ U = \int \sigma_{ij} d \varepsilon_{ij} \Rightarrow U = \int_{0}^{\varepsilon_y} \sigma d \varepsilon \]

\[ U = \frac{1}{2} \frac{Y^2}{E} \]

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Strain energy theory

- Body subjected to external loads => principal stresses

Strain energy associated with principal stresses

\[ U = \frac{1}{2} \left( \sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3 \right) \]

\[ \varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E} \left( \sigma_2 + \sigma_3 \right) \]
\[ \varepsilon_2 = \frac{\sigma_2}{E} - \frac{\nu}{E} \left( \sigma_3 + \sigma_1 \right) \]
\[ \varepsilon_3 = \frac{\sigma_3}{E} - \frac{\nu}{E} \left( \sigma_1 + \sigma_2 \right) \]

For onset of yielding,

\[ U = \frac{1}{2E} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu (\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_2\sigma_3) \right] \]

\[ \frac{Y^2}{2E} = \frac{1}{2E} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu (\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_2\sigma_3) \right] \]
Strain energy theory

- **Yield function** –

\[ f = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \nu(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) - Y^2 \]

\[ f = \sigma_e^2 - Y^2 \]

Equivalent stress \( \Rightarrow \sigma_e^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \nu(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \)

Yielding \( \Rightarrow f = 0 \), safe \( f < 0 \)

- **For 2D stress state** \( \Rightarrow \sigma_3 = 0 \) – Yield function becomes

\[ f = \sigma_1^2 + \sigma_2^2 - \nu \sigma_1 \sigma_2 - Y^2 \]

For onset of yielding \( \Rightarrow f = 0 \)

\[ \sigma_1^2 + \sigma_2^2 - \nu \sigma_1 \sigma_2 - Y^2 = 0 \]

Plotting this in principal stress space

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Rearrange the terms –

\[
\frac{\sigma_1}{Y}^2 + \frac{\sigma_2}{Y}^3 - 2\nu \frac{\sigma_1 \sigma_2}{Y^2} = 1
\]

This represents an ellipse –

Transform to \(\zeta-\eta\) csys

\[
\sigma_1 = \zeta \cos 45 - \eta \sin 45 = \frac{1}{\sqrt{2}} (\zeta - \eta)
\]

\[
\sigma_2 = \zeta \sin 45 + \eta \cos 45 = \frac{1}{\sqrt{2}} (\zeta + \eta)
\]

Equivalent stress inside – no failure

Substitute these in the above expression
Strain energy theory

Simplifying,
\[
\frac{\zeta^2}{Y^2} + \frac{\eta^2}{(1-\nu)}\frac{Y^2}{(1+\nu)} = 1 \Rightarrow \frac{\zeta^2}{a^2} + \frac{\eta^2}{b^2} = 1
\]

Semi major axis – OA \(\Rightarrow a = \frac{Y}{\sqrt{1-\nu}}\)

Semi minor axis – OB \(\Rightarrow b = \frac{Y}{\sqrt{1+\nu}}\)

Higher Poisson ratio – bigger major axis, smaller minor axis

If \(\nu = 0 \Rightarrow\) circle of radius ‘\(Y\)’

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Strain energy theory

- Pure shear

Principal stresses corresponding to shear yield stress

\[ \sigma_1 = +\tau_y, \quad \sigma_2 = -\tau_y \]

\[ \varepsilon_1 = \frac{\tau_y}{E} (1 + \nu), \quad \varepsilon_2 = -\frac{\tau_y}{E} (1 + \nu) \]

Strain energy,

\[ U_\tau = \frac{(1 + \nu)}{2E} 2\tau_y^2 = \frac{1}{2E} Y^2 \Rightarrow Y = \sqrt{2(1 + \nu)\tau_y} \]

\[ \tau_y = 0.632 Y \]

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Distortional energy theory (von-Mises)

- Hydrostatic loading
  - applying uniform stress from all the directions on a body
  - Large amount of strain energy can be stored
  - Experimentally verified
  - Pressures beyond yield stress – no failure of material
  - Hydrostatic loading – change in size – volume

Pressure ‘p’ applied from all sides

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von-Mises theory

- Energy associated with volumetric change – volumetric strain energy
- Volumetric strain energy – no failure of material
- Strain energy causing material failure – distortion energy – associated with shear – First invariant of deviatoric stress = 0
- For a given stress state estimate distortion energy – this should be less than distortion energy due to uniaxial tensile – safe
von-Mises theory

- Given stress state referred to principal co-ordinate system –

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix} = \begin{bmatrix}
\sigma_1 - p & 0 & 0 \\
0 & \sigma_2 - p & 0 \\
0 & 0 & \sigma_3 - p
\end{bmatrix} + \begin{bmatrix}
p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p
\end{bmatrix}
\]

First invariant, \( J_1 = 0 \)

\[(\sigma_1 - p) + (\sigma_2 - p) + (\sigma_3 - p) = 0 \]

\[\Rightarrow p = \frac{1}{3} \sigma_{ii}\]

Principal strains \( \Rightarrow \varepsilon_1, \varepsilon_2, \varepsilon_3 \)

Volumetric strain \( \Rightarrow \varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \)

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von-Mises theory

This gives –

\[ \varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1}{E} \left\{ (\sigma_1 + \sigma_2 + \sigma_3) - 2\nu(\sigma_1 + \sigma_2 + \sigma_3) \right\} \]

\[ \varepsilon_v = \frac{(1-2\nu)}{E} (\sigma_1 + \sigma_2 + \sigma_3) = \frac{3(1-2\nu)}{E} p \]

Volumetric strain energy, \[ U_v = \frac{1}{2} p \varepsilon_v \]

\[ U_v = \frac{1}{2} p \frac{3(1-2\nu)}{E} p = \frac{3(1-2\nu)}{2E} p^2 = \frac{(1-2\nu)}{6E} (\sigma_1 + \sigma_2 + \sigma_3)^2 \]

\[ U = \text{strain energy due to principal stresses} \ & \text{strains} \]

\[ U = \frac{1}{2E} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{\nu}{E} \left( \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \right) \]
von-Mises theory

- Distortional energy –

\[ U_D = U - U_V \]

\[ U_D = \frac{1}{2E} \left[ (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2\nu(\sigma_2\sigma_1 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \right] - \left( \frac{1-2\nu}{6E} \right) (\sigma_1 + \sigma_2 + \sigma_3)^2 \]

Simplifying this

\[ U_D = \frac{1}{12G} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

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von-Mises theory

- Compare this with distortion in uniaxial tensile stress

\[ U_D = \frac{Y^2}{6G} = \frac{1}{12G} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

\[ \Rightarrow 2Y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \]

Yield function,

\[ f = \sigma_e^2 - Y^2 \]

Equivalent stress,

\[ \sigma_e^2 = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

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von-Mises theory

Principal stresses of deviatoric shear stress, $S_{ii}$

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - p & 0 & 0 \\ 0 & \sigma_2 - p & 0 \\ 0 & 0 & \sigma_3 - p \end{bmatrix} + \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$[\sigma] = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} + \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$S_{ii} = \sigma_{ii} - p \Rightarrow \sigma_{ii} = S_{ii} + p$$

$$2Y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

$$2Y^2 = ((S_1 + p) - (S_2 + p))^2 + ((S_2 + p) - (S_3 + p))^2 + ((S_3 + p) - (S_1 + p))^2$$

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Simplifying this expression –

\[ 2Y^2 = (S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2 \]

Hydrostatic pressure does not appear in the expression

- von-Mises criteria has square terms – result independent of signs of individual stress components
- Von-Mises equivalent stress => +ve stress
von-Mises theory

- 2D stress state $\Rightarrow \sigma_3 = 0$

  Yield function, $f = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 - Y^2$

  Onset of yielding, $\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = Y^2$

Re-arrange the terms –

\[
\left(\frac{\sigma_1}{Y}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 - \left(\frac{\sigma_1 \sigma_2}{Y^2}\right) = 1
\]

This represents an ellipse

Semi - major axis, $OA = \sqrt{2Y}$

Semi - minor axis, $OB = \sqrt{\frac{2}{3}}Y$

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von-Mises theory

- **Pure shear** –

Principal stresses corresponding to shear yield stress

\[ \sigma_1 = +\tau_y, \quad \sigma_2 = -\tau_y \]

\[ Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = 3\tau_y^2 \implies \tau_y = 0.577Y \]

Shear yield = 0.577 * Tensile yield

Suitable for ductile materials

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von-Mises theory

- Plot yield function in 3D principal stress space

\[ f = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2Y^2 = 0 \]

Cylinder, with hydrostatic stress as axis

Axis makes equal DCs with all axes

\[ n = \frac{1}{\sqrt{3}} \left( i + j + k \right) \]

\[ OA = \sigma_1 i + \sigma_2 j + \sigma_3 k \]

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Projection of OA on hydrostatic axis

\[ OA \cdot n = |OA||n|\cos\theta \Rightarrow OA \cos\theta = \frac{OA \cdot n}{|n|} = OB \]

\[ OB = \frac{1}{\sqrt{3}} (\sigma_1 + \sigma_2 + \sigma_3) \]

\[ \Rightarrow OB = OB \quad n = \frac{1}{\sqrt{3}} (\sigma_1 + \sigma_2 + \sigma_3) \frac{1}{\sqrt{3}} (i + j + k) \]

\[ OB = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3} (i + j + k) = p(i + j + k) \]

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\[ OA = OB + BA \]

\[ BA = OA - OB \]

\[ BA = r = \text{radius of cylinder} \]
von-Mises theory

- Radius of cylinder

\[
BA = R = OA - OB = \left( \sigma_1 i + \sigma_2 j + \sigma_3 k \right) - p \left( i + j + k \right) \\
R = \left( \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} \right) i + \left( \frac{2\sigma_2 - \sigma_3 - \sigma_1}{3} \right) j + \left( \frac{2\sigma_3 - \sigma_1 - \sigma_2}{3} \right) k \\
R = S_1 i + S_2 j + S_3 k
\]

Radius => \( R = \sqrt{S_1^2 + S_2^2 + S_3^2} \)

First invariant of deviatoric stress tensor, \( J_1 = 0 \) => \( S_1 + S_2 + S_3 = 0 \)

\[
(S_1 + S_2 + S_3)^2 = 0 = S_1^2 + S_2^2 + S_3^2 = -2(S_1 S_2 + S_2 S_3 + S_3 S_1)
\]

Yield criteria => \( (S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2 = 2Y^2 \)

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Yield criteria,

\[ Y^2 = S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1 \]

Use,

\[ S_1^2 + S_2^2 + S_3^2 = -2(S_1S_2 + S_2S_3 + S_3S_1) \]

\[ Y^2 = S_1^2 + S_2^2 + S_3^2 + \frac{1}{2}(S_1^2 + S_2^2 + S_3^2) \]

\[ Y^2 = \frac{3}{2}(S_1^2 + S_2^2 + S_3^2) = \frac{3}{2}R^2 \]

\[ Y = \sqrt{\frac{3}{2}}R \]

Yielding depends on deviatoric stresses
Hydrostatic stress has no role in yielding
von-Mises theory

- Second invariant of deviatoric stress

\[
\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} \Rightarrow J_2 = \begin{vmatrix} S_1 & 0 \\ 0 & S_2 \end{vmatrix} + \begin{vmatrix} S_2 & 0 \\ 0 & S_3 \end{vmatrix} + \begin{vmatrix} S_1 & 0 \\ 0 & S_3 \end{vmatrix}
\]

\[J_2 = S_1S_2 + S_2S_3 + S_3S_1\]

\[S_1S_2 + S_2S_3 + S_3S_1 = -\frac{1}{2} \left( S_1^2 + S_2^2 + S_3^2 \right)\]

\[J_2 = -\frac{1}{2} \left( S_1^2 + S_2^2 + S_3^2 \right) \Rightarrow |J_2| = \frac{1}{2} \left( S_1^2 + S_2^2 + S_3^2 \right) = \frac{R^2}{2}\]

\[Y^2 = \frac{3}{2} R^2 \Rightarrow Y^2 = 3|J_2|\]

Redefining yield function \( f = 3|J_2| - Y^2 \)

J2 Materials
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Max. shear stress theory (Tresca)

“Yielding begins when the maximum shear stress at a point equals the maximum shear stress at yield in a uniaxial tension”

\[ τ_{\text{max}} = K_T = \left| \frac{\sigma_1 - \sigma_2}{2} \right| \Rightarrow τ_{\text{max}} = \frac{Y}{2} = K_T \]

If maximum shear stress < Y/2 => No failure occurs

For pure shear, \( σ_1 = +τ_y \), \( σ_2 = -τ_y \)

\[ τ_{\text{max}} = K_T = \left| \frac{σ_1 - σ_2}{2} \right| \Rightarrow τ_{\text{max}} = τ_y = K_T \]

Shear yield = 0.5 Tensile yield

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Tresca theory

- In 3D stress state – principal stresses $\Rightarrow \sigma_1, \sigma_2$ and $\sigma_3$
- Maximum shear stress

$$\max \left\{ \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2} \right\}$$

- Yield function

$$f = \max \left\{ \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2} \right\} - K_T \left( = \frac{Y}{2} \right)$$

$f < 0 \Rightarrow$ No yielding  
$f = 0 \Rightarrow$ Onset of yielding

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Tresca theory

- Following equations are obtained

\[
\left| \frac{\sigma_1 - \sigma_2}{2} \right| = K_T \Rightarrow \frac{\sigma_1 - \sigma_2}{2} = \pm K_T
\]

\[
f_1(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2 - 2K_T; \quad f_2(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2 + 2K_T
\]

\[
\left| \frac{\sigma_2 - \sigma_3}{2} \right| = K_T \Rightarrow \frac{\sigma_2 - \sigma_3}{2} = \pm K_T
\]

\[
f_3(\sigma_2, \sigma_3) = \sigma_2 - \sigma_3 - 2K_T; \quad f_4(\sigma_2, \sigma_3) = \sigma_2 - \sigma_3 + 2K_T
\]

\[
\left| \frac{\sigma_3 - \sigma_1}{2} \right| = K_T \Rightarrow \frac{\sigma_3 - \sigma_1}{2} = \pm K_T
\]

\[
f_5(\sigma_3, \sigma_1) = \sigma_3 - \sigma_1 - 2K_T; \quad f_6(\sigma_3, \sigma_1) = \sigma_3 - \sigma_1 + 2K_T
\]
Redefining yield function as,

\[ f(\sigma_1, \sigma_2, \sigma_3) = f_1(\sigma_1, \sigma_2), f_2(\sigma_1, \sigma_2), f_3(\sigma_2, \sigma_3), f_4(\sigma_2, \sigma_3), f_1(\sigma_1, \sigma_2) \]

\[ f(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_2 - 2K_T)(\sigma_1 - \sigma_2 + 2K_T) \]
\[ (\sigma_2 - \sigma_3 - 2K_T)(\sigma_2 - \sigma_3 + 2K_T) \]
\[ (\sigma_3 - \sigma_1 - 2K_T)(\sigma_3 - \sigma_1 + 2K_T) \]

Each function represents a plane in 3D principal stress space

\[ f(\sigma_1, \sigma_2, \sigma_3) = \left( (\sigma_1 - \sigma_2)^2 - 4K_T^2 \right) \left( (\sigma_2 - \sigma_3)^2 - 4K_T^2 \right) \left( (\sigma_3 - \sigma_1)^2 - 4K_T^2 \right) \]

No effect of hydrostatic pressure in Tresca criteria
**Tresca theory**

- **Yield function in principal stress space**

![Diagram of Tresca yield surface and its properties.](image)

- **Tresca yield surface**
- **View 'A' – along hydrostatic axis**

Hydrostatic axis

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**Tresca theory**

- **Yield surface intersects principal axes at** $2K_T$

  - **Hydrostatic c axis** $\Rightarrow \sigma_1 = \sigma_2 = \sigma_3$
    - $\cos \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 54.73^0$
    - $\alpha + \theta = 90^0 \Rightarrow \theta = 35.26^0$
    - $OA = 2K_T$
    - $OB = OA \cos \theta = 2K_T \sqrt{\frac{2}{3}}$

  - **OB** – projection of **OA** on deviatoric plane

  ![Diagram](image)
**Tresca theory**

- **Tresca hexagon**

\[ OC = OD \cos 30 \]

\[ \Rightarrow OC = 2K_T \sqrt{\frac{2}{3}} \frac{\sqrt{3}}{2} \]

\[ \Rightarrow OC = \sqrt{2}K_T \]
Tresca theory

- 2D stress state - $\sigma_3 = 0$

Each equation represents two lines in 2D stress space

\[
\begin{align*}
\sigma_1 - \sigma_2 &= \pm 2 K_T \\
\sigma_2 &= \pm 2 K_T \\
\sigma_1 &= \pm 2 K_T
\end{align*}
\]

Yield curve – elongated hexagon

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Tresca theory

- 2D stress state - $\sigma_3 = 0$

Each equation represents two lines in 2D stress space

$$\sigma_1 - \sigma_2 = \pm 2K_T$$

$$\sigma_2 = \pm 2K_T$$

$$\sigma_1 = \pm 2K_T$$

Yield curve – elongated hexagon

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von-Mises – Tresca theories

- Pure tension – \( \sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0 \)

von-Mises criteria => \( J_2 = K_M^2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \)

Tresca’s criteria => \( K_T = \max \left\{ \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2} \right\} \)

\( K_M = \frac{1}{\sqrt{3}} Y, \quad K_T = \frac{1}{2} Y \)

Pure shear => \( \sigma_1 = +\tau_y, \sigma_2 = -\tau_y \Rightarrow K_M = \tau_y = K_T \)

\( K_M = \frac{1}{\sqrt{3}} Y = \tau_y, \quad K_T = \frac{1}{2} Y = \tau_y \)

\( \tau_y = 0.577 Y \) (von – Mises), \( \tau_y = 0.5Y \) (Tresca)

von-Mises criteria predicts 15% higher shear stress than Tresca

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von-Mises – Tresca theories

- 2D stress space – von-Mises and Tresca

\[ Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 \]
\[ Y = \max \left\{ |\sigma_1 - \sigma_2|, |\sigma_1|, |\sigma_2| \right\} \]

Yielding in uniaxial tension
Tresca – conservative

\[ \frac{\tau_y}{2} = \max \left\{ \left| \frac{\sigma_1 - \sigma_2}{2} \right|, \left| \frac{\sigma_1}{2} \right|, \left| \frac{\sigma_2}{2} \right| \right\} \]

Yielding in shear
von-Mises – conservative
von-Mises – Tresca theories

- Experiments by Taylor & Quinney**
- Thin walled tube subjected to axial and torsional loads

\[
\sigma_1 = \sigma_{xx} + \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}
\]

\[
\sigma_2 = \frac{\sigma_{xx}}{2} - \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}
\]

von-Mises – Tresca theories

- **Tresca criteria**

\[ Y = \left| \sigma_1 - \sigma_2 \right| = 2 \sqrt{\left( \frac{\sigma_{xx}}{2} \right)^2 + \tau_{xy}^2} \Rightarrow Y^2 = \sigma_{xx}^2 + 4\tau_{xy}^2 \]

No yielding if, \( \left( \frac{\sigma_{xx}}{Y} \right)^2 + \left( \frac{\tau_{xy}}{Y / 2} \right)^2 < 1 \)

- **von-Mises criteria**

\[ Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 \]

\[ Y^2 = \sigma_{xx}^2 + 3\tau_{xy}^2 \]

No yielding if, \( \left( \frac{\sigma_{xx}}{Y} \right)^2 + \left( \frac{\tau_{xy}}{Y / \sqrt{3}} \right)^2 < 1 \)

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von-Mises – Tresca theories

- Plotting these two criteria –

![Graph showing von-Mises and Tresca criteria](image)

Experimental data shows good agreement with von-Mises theory.

Tresca – conservative

von-Mises theory more accurate – generally used in design

Experiments show that for ductile materials yield in shear is 0.5 to 0.6 times of yield in tensile.
Octahedral shear stress theory

- Octahedral plane – makes equal angles with all principal stress axes – direction cosines same
- Shear stress acting on this plane – octahedral shear

\[ \tau_{oct} = \frac{1}{9} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

Body subjected to pure tension, \( \sigma_1 = Y, \sigma_2 = \sigma_3 = 0 \)

\[ \tau_{oct} = \frac{2}{9} Y^2 \]

\[ 2Y^2 = \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

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Octahedral shear stress theory

- Comparing this with von-Mises theory => both are same
- Pure shear

\[ \sigma_1 = \tau_y, \sigma_2 = -\tau_y, \sigma_3 = 0 \]

\[ \tau_{oct}^2 = \frac{1}{9} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] = \frac{6}{9} \tau_y^2 \]

Same as von-Mises theory in pure shear

\[ \tau_{oct}^2 = \frac{2}{3} \tau_y^2 \]

\[ 6\tau_y^2 = \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

Octahedral shear stress theory => von-Mises theory

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Each failure theory gives a relation between yielding in tension and shear ($\nu = 0.25$)

<table>
<thead>
<tr>
<th>Theory</th>
<th>Loading</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum principal stress</td>
<td>$\sigma_{\text{max}} = \sigma_{\text{YP}}$</td>
<td>$\tau_{\text{YP}} = \sigma_{\text{YP}}$</td>
</tr>
<tr>
<td>Maximum principal strain</td>
<td>$\varepsilon_{\text{max}} = \sigma_{\text{YP}} / E$</td>
<td>$\tau_{\text{YP}} = 0.8 \sigma_{\text{YP}}$</td>
</tr>
<tr>
<td>Maximum octahedral shear stress</td>
<td>$\tau_{\text{oct}} = \sigma_{\text{YP}} \sqrt{\frac{2}{3}}$</td>
<td>$\tau_{\text{YP}} = 0.577 \sigma_{\text{YP}}$</td>
</tr>
<tr>
<td>Maximum distortional energy density</td>
<td>$\tau_{\text{max}} = \sigma_{\text{YP}} / 2$</td>
<td>$\tau_{\text{YP}} = 0.577 \sigma_{\text{YP}}$</td>
</tr>
<tr>
<td>Maximum shear stress</td>
<td>$\tau_{\text{max}} = \tau_{\text{YP}}$</td>
<td>$\tau_{\text{YP}} = 0.5 \sigma_{\text{YP}}$</td>
</tr>
</tbody>
</table>
Failure theories in a nutshell

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