Theory of Plates

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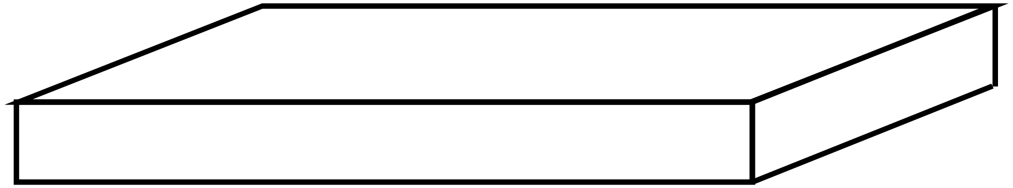
Introduction

- *"When a body is bounded by surfaces, flat in geometry, whose lateral dimensions are large compared to the separation between the surfaces is called a PLATE"*



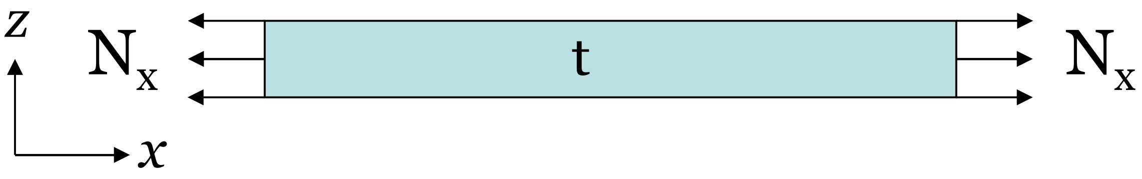
- Plates are initially flat structural elements

Introduction

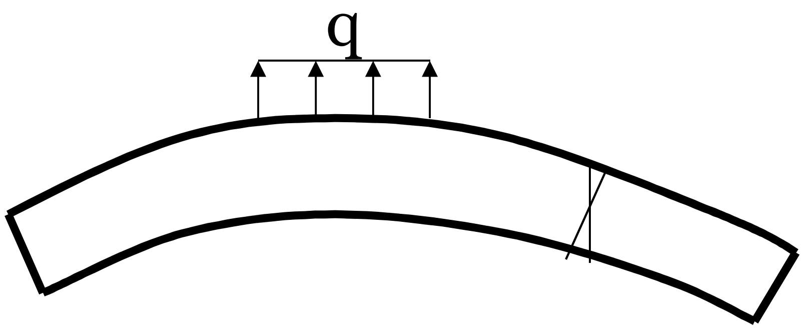
- Plates are subjected to transverse loads – loads normal to its mid-surface
- Transverse loads supported by combined bending and shear action
- Plates may be subjected to in-plane loading also => uniform stress distribution => membrane
- Membrane action – in-plane loading or pronounced curvature & slope
- Plate bending – plate's mid-surface doesn't experience appreciable stretching or contraction
- In-plane loads cause stretching and/or contraction of mid-surface

Introduction

- Plate stretching



Uniform stretching of the plate => u_o



Axial deformation due to transverse load

Net deformation = Algebraic sum of uniform stretching and axial deformation due to bending load

Introduction

- For plates - $\frac{1}{10} \geq \frac{t}{b} \geq \frac{1}{2000}$
- Thin & thick plates –
  - Thin plate => $t < 20b$ b = smallest side
  - Thick plate => $t > 20b$
- Small deflections – $w \leq \frac{t}{5}$
- Thin plate theory – Kirchoff's Classical Plate Theory (KCPT)
- Thick plate theory – Reissner – Mindlin Plate Theory (MPT)

KCPT - Assumptions

Assumptions –

- Thickness is much smaller than the other physical dimensions
  - vertical deflection $w(x, y, z) = w(x, y)$
- Displacements u, v & w are small compared to plate thickness
  - Governing equations are derived based on undeformed geometry
- In plane strains are small compared to unity – consider only linear strains
- Normal stresses in transverse direction are small compared with other stresses – neglected

KCPT - Assumptions

- Material – linear elastic – Hooke’s law holds good
- Middle surface remains unstrained during bending – neutral surface
- Normals to the middle surface before deformation remain normal to the same surface after deformation => doesn’t imply shear across section is zero – transverse shear strain makes a negligible contribution to deflections.
  - Transverse shear strains are negligible
- Rotary inertia is neglected

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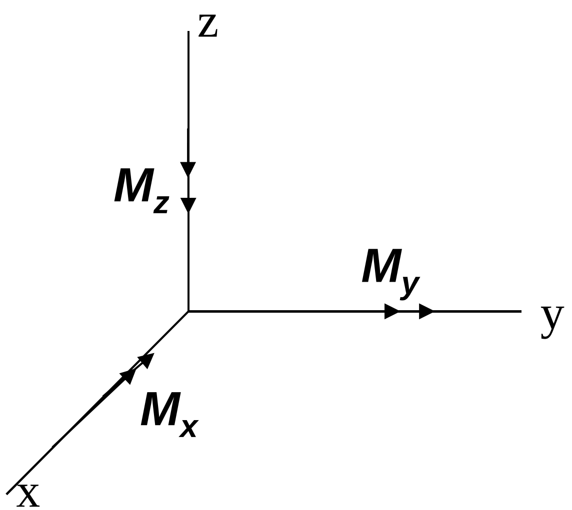
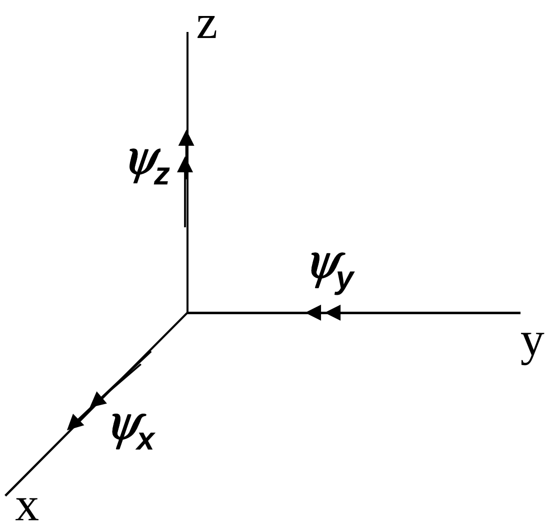
Sign convention

- Following sign convention will be followed



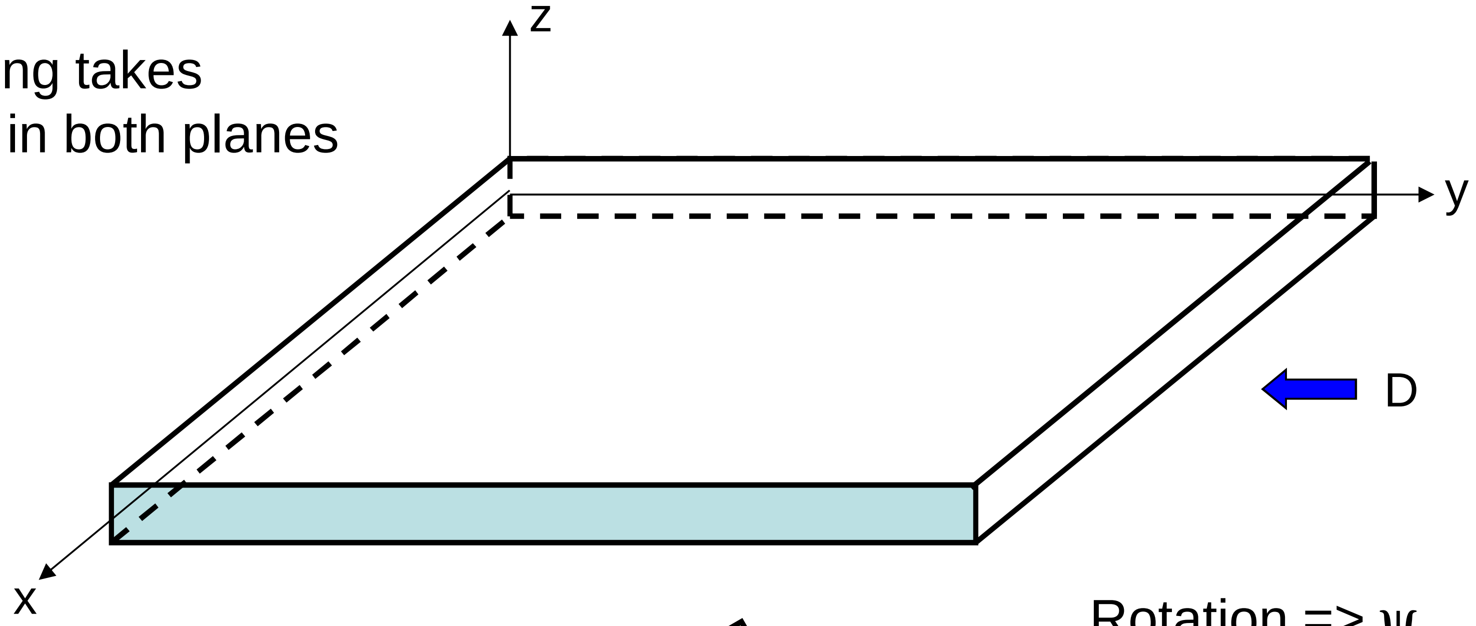
Positive moments



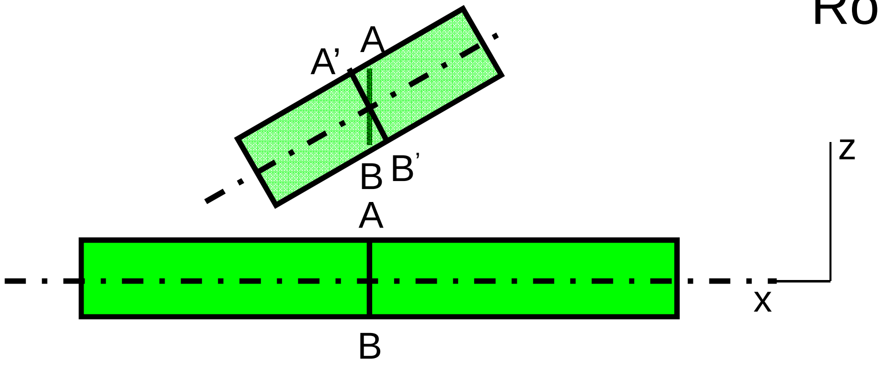
Positive rotations

Bending deformations

Bending takes place in both planes



Rotation => ψ_y



View 'D'

Bending deformations

- Deformation in ‘x’ direction

$$u(x, y, z) = -z\psi_y(x, y)$$

- Deformation in ‘y’ direction

$$v(x, y, z) = -z\psi_x(x, y)$$

- Vertical deformation

$$w = w(x, y)$$

Strains

- Assumption – out of plane shear strain - negligible

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$\Rightarrow \gamma_{xz} = -\psi_y + \frac{\partial w}{\partial x} = 0 \Rightarrow \psi_y = \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\Rightarrow \gamma_{yz} = -\psi_x + \frac{\partial w}{\partial y} = 0 \Rightarrow \psi_x = \frac{\partial w}{\partial y}$$

Strains

- Non-zero strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial \psi_y}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

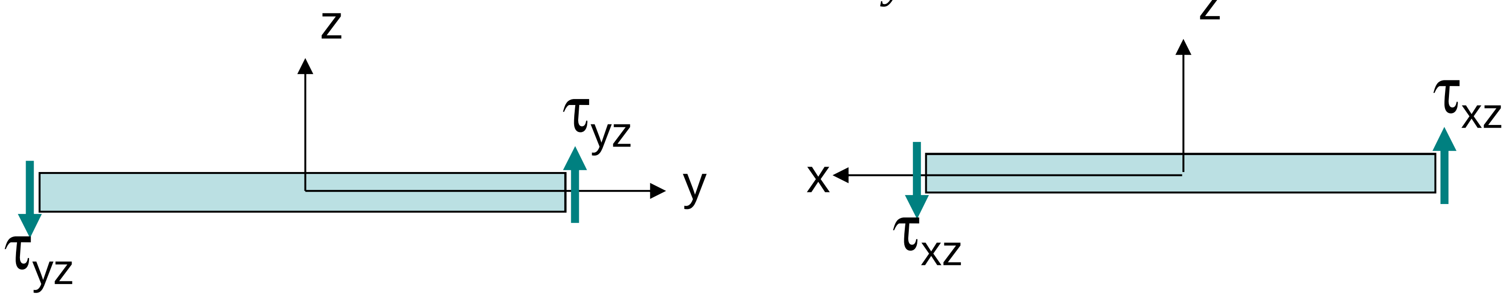
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial \psi_x}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Stresses

- Thin plate – out of plane shear strains vanish – out of plane shear stresses also vanish

$$\tau_{xz} = 0, \qquad \tau_{yz} = 0$$



- Out of plane normal stress is also assumed to be zero – logical – thin structure – plane stress conditions

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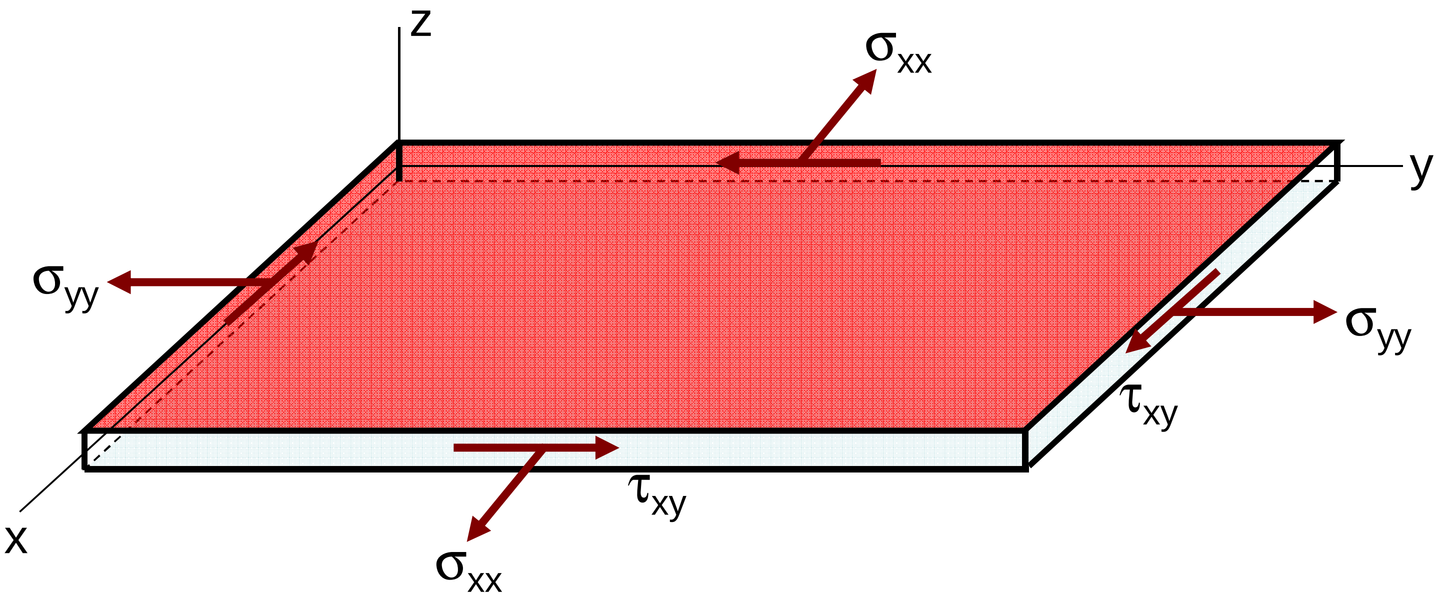
Stresses

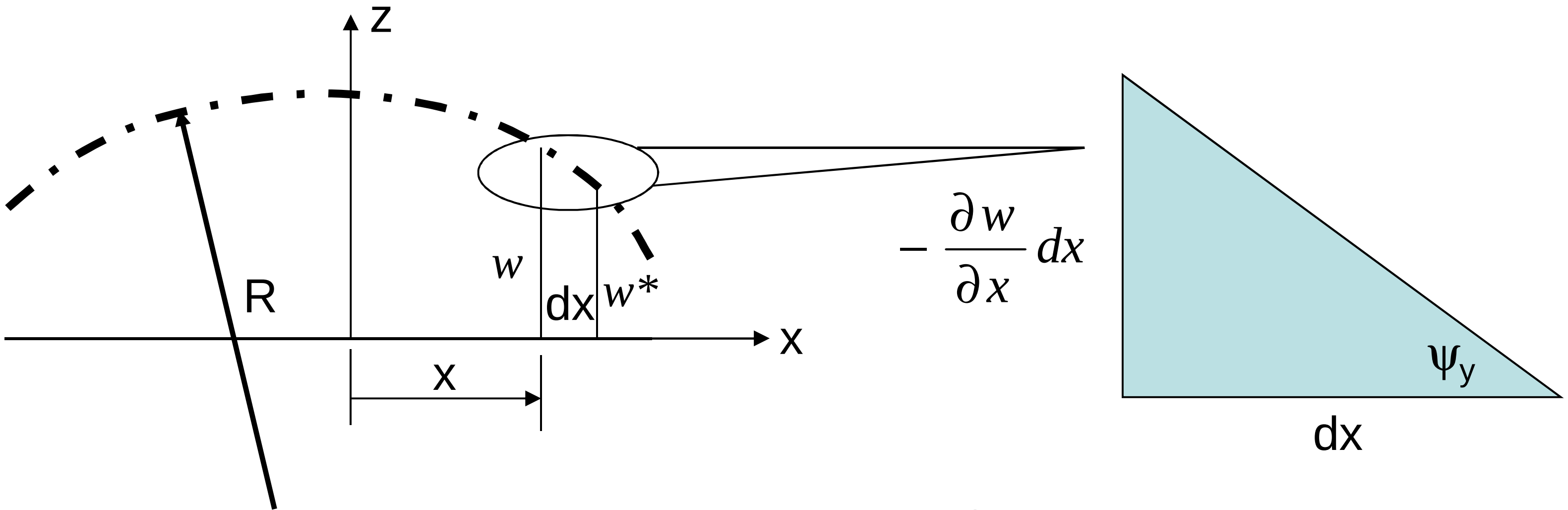
- Non-zero stress components



All three stress components, σ_{xx}, σ_{yy}, τ_{xy} – in-plane

Curvatures

- Curvature – reciprocal of radius of bending
- Rate of change of slope



$$\text{slope} = \text{rotation} = \psi_y = -\frac{\partial w}{\partial x} \qquad \text{curvatue} = \frac{\partial \psi_y}{\partial x} = -\frac{\partial^2 w}{\partial x^2} = \kappa_{xx}$$

Curvatures

- Similarly bending in *yz* plane introduces a curvature

$$\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

- Twisting of plate

$$\kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

Constitutive law

- Linear elastic isotropic – Hookean material
- Three stress and strain components

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \upsilon \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \upsilon \frac{\sigma_{xx}}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\tau_{xy}}{E} 2\left(1 + \upsilon\right)$$

Writing all three equations in matrix form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \dfrac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

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Constitutive law

- Express strains in terms of curvatures

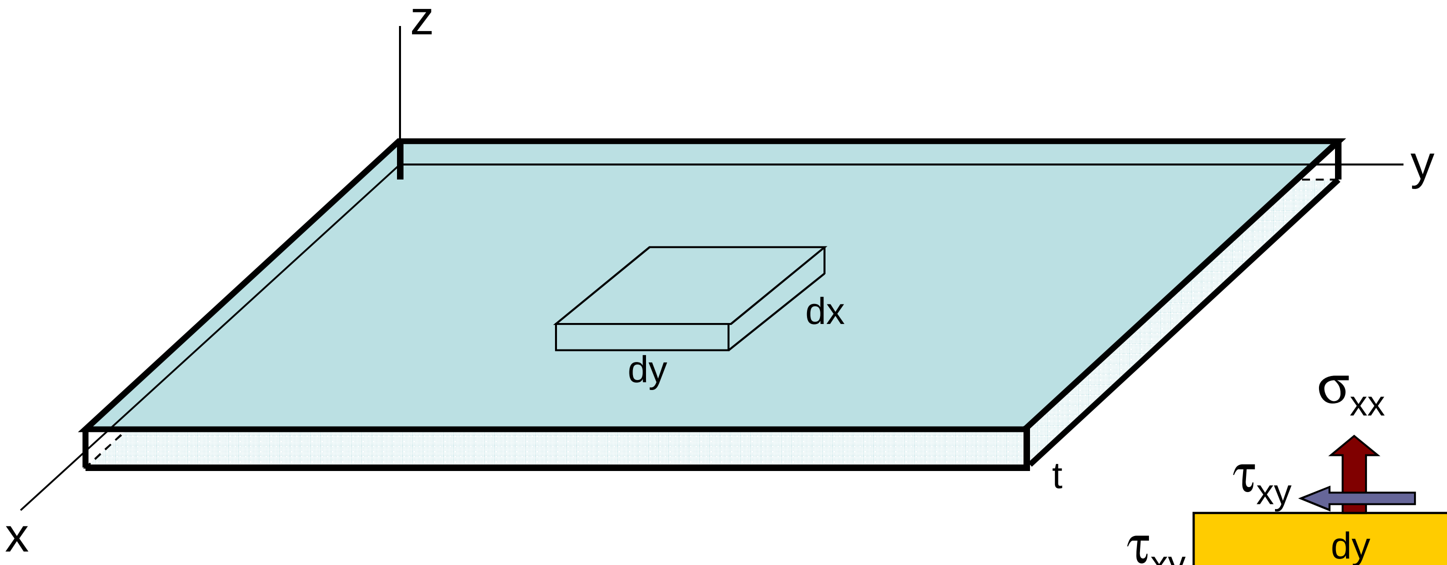
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = -\frac{Ez}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1+\upsilon}{2} \end{bmatrix}\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

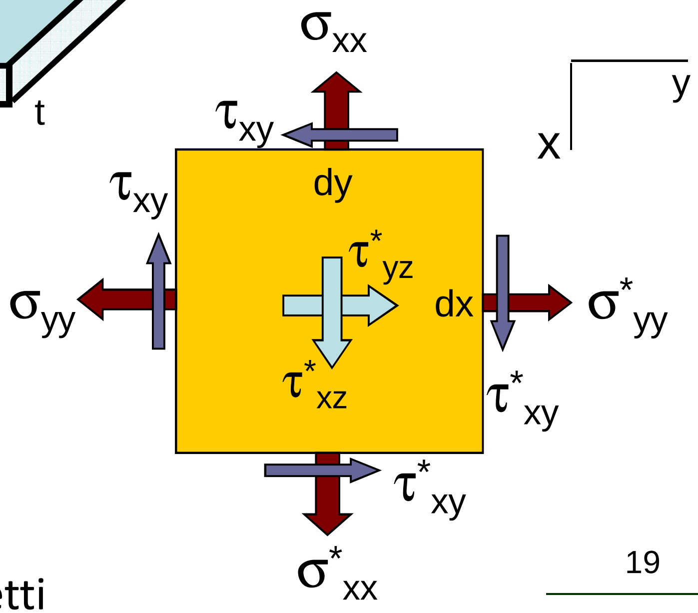
Variation of stresses across thickness is linear

Basis – thin plates – plane section remain plane after bending – variation of axial deflection is linear across thickness – strains also vary linearly

Equilibrium equations

- Equilibrium of an infinitesimal element





Forces acting on an infinitesimal element $dx\ dy\ dz$

Equilibrium equations

- Equilibrium in ‘x’ direction

$$\sigma^*_{xx}\, dydz + \tau^*_{zx}\, dxdy + \tau^*_{yx}\, dzdx$$
$$- \sigma_{xx}\, dydz - \tau_{zx}\, dxdy - \tau_{yx}\, dzdx = 0$$

$$\sigma^*_{xx} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx \right)$$

$$\tau^*_{zx} = \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right)$$

$$\tau^*_{yx} = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right)$$

Equilibrium equations

- Substitute – the following equation is obtained

$$\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}=0 \quad - \quad (1)$$

Similarly take equilibrium in ‘y’ and ‘z’ directions

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0 \quad - \quad (2)$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}=0 \quad - \quad (3)$$

Shear stresses

- From equation (1) – shear stress τ_{xz} can be computed

Use stress deflection/curvature relations

$$\frac{\partial}{\partial x}\left[-\frac{Ez}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}\right)\right]+\frac{\partial}{\partial y}\left[-\frac{Ez}{1+\upsilon}\frac{\partial^2 w}{\partial x \partial y}\right]+\frac{\partial \tau_{xz}}{\partial z}=0$$

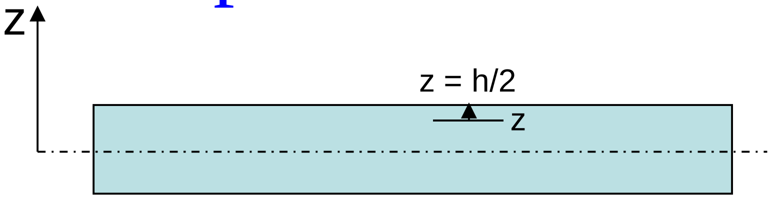
$$\frac{\partial \tau_{xz}}{\partial z}=\left(\frac{Ez}{1-\upsilon^2}\right)\left(\frac{\partial^3 w}{\partial x^3}+\upsilon\frac{\partial^3 w}{\partial x \partial y^2}+(1-\upsilon)\frac{\partial^3 w}{\partial x \partial y^2}\right)$$

$$\frac{\partial \tau_{xz}}{\partial z}=\left(\frac{Ez}{1-\upsilon^2}\right)\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

Integrate this across thickness to get shear stress

Shear stresses

- Integrate from mid plane to top surface of plate



$$\int_{\tau_{xz}}^{0} d\tau_{xz} = \int_{0}^{h/2}\left(\frac{Ez}{1-\upsilon^2}\right)\frac{\partial\left(\nabla^2 w\right)}{\partial x}dz = \frac{E}{1-\upsilon^2}\frac{\partial\left(\nabla^2 w\right)}{\partial x}\int_{z}^{h/2} z\, dz$$

$$-\tau_{xz} = \frac{E}{1-\upsilon^2}\frac{\partial\left(\nabla^2 w\right)}{\partial x}\left(\frac{z^2}{2}\right]_{z}^{h/2}$$

$$=> -\tau_{xz} = \frac{E}{2\left(1-\upsilon^2\right)}\frac{\partial\left(\nabla^2 w\right)}{\partial x}\left(\frac{h^2}{4} - z^2\right)$$

$$=> \tau_{xz} = \frac{E}{2\left(1-\upsilon^2\right)}\frac{\partial\left(\nabla^2 w\right)}{\partial x}\left(z^2 - \frac{h^2}{4}\right)$$

Parabolic variation

Moments

- Moment wrt ‘y’ axis

$$dF_{xx} = \sigma_{xx} dz$$

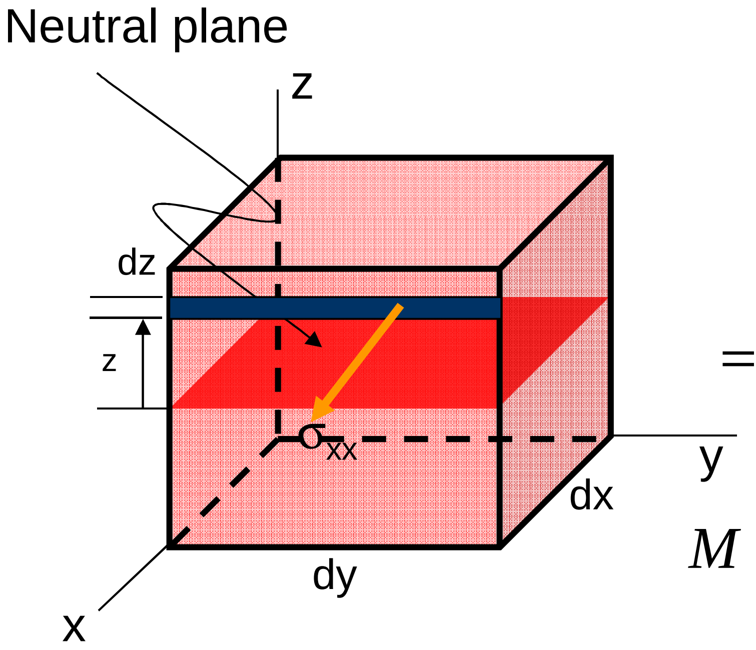
$$dM_y = z dF_{xx} = z \sigma_{xx} dz$$

$$\sigma_{xx} = -\frac{Ez}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right)$$

$$\Rightarrow \; dM_y = -\frac{Ez^2}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{E}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)\int_{-h/2}^{+h/2} z^2 dz$$

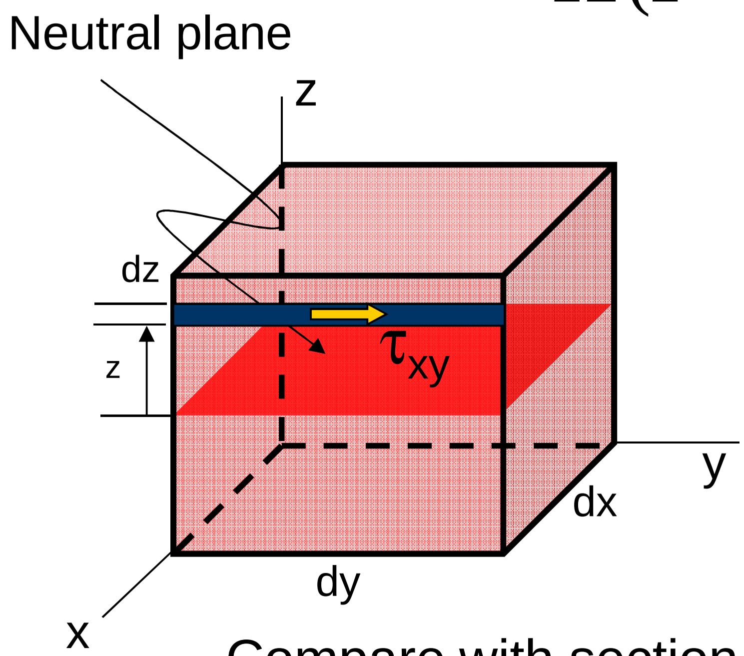
$$M_y = -\frac{Eh^3}{12\left(1-\upsilon^2\right)}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)$$



Moments

- Moment wrt ‘x’ axis

$$M_x = -\frac{Eh^3}{12(1-\upsilon^2)}\left(\upsilon\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$



Twisting moment due to shear stress τ_{xy}

$$M_{xy} = -\frac{Eh^3}{12(1-\upsilon^2)}(1-\upsilon)\frac{\partial^2 w}{\partial x \partial y}$$

Compare with section modulus of beam

$$D = \frac{Eh^3}{12(1-\upsilon^2)}$$

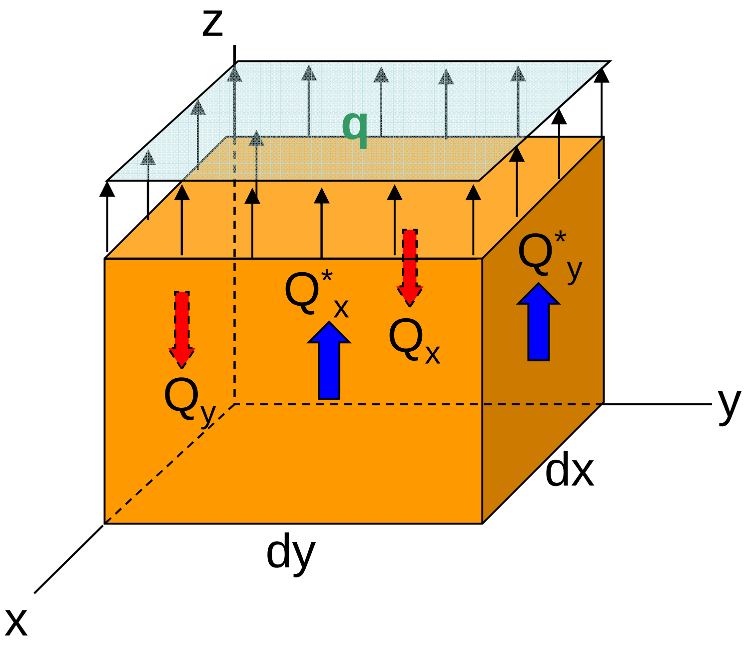
Shear forces

- Vertical equilibrium of plate

$$Q_x^* dy + Q_y^* dx + q\,dx\,dy - Q_x dy - Q_y dx = 0$$

$$Q_x^* = Q_x + \frac{\partial Q_x}{\partial x} dx$$

$$Q_y^* = Q_y + \frac{\partial Q_y}{\partial y} dy$$



Substitute these and simplify

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

Shear forces

- Shear forces across thickness can be computed by integrating shear stress across thickness

$$Q_y = \int_{-h/2}^{+h/2} \tau_{yz}\, dz \quad \text{per unit width}$$

$$\tau_{yz} = \frac{E}{2(1-\upsilon^2)} \frac{\partial(\nabla^2 w)}{\partial y}\left(z^2 - \frac{h^2}{4}\right)$$

$$Q_y = \frac{E}{2(1-\upsilon^2)} \frac{\partial(\nabla^2 w)}{\partial y} \int_{-h/2}^{+h/2}\left(z^2 - \frac{h^2}{4}\right) dz$$

$$Q_y = -\frac{Eh^3}{12\,(1-\upsilon^2)} \frac{\partial(\nabla^2 w)}{\partial y} = -D\,\frac{\partial(\nabla^2 w)}{\partial y}$$

Governing equation

- Similar expression for Q_x

$$Q_x = -D\frac{\partial(\nabla^2 w)}{\partial x}$$

Substitute Q_x and Q_y in the following expression – vertical equilibrium

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

$$\frac{\partial}{\partial x}\left(-D\frac{\partial}{\partial x}\left(\nabla^2 w\right)\right) + \frac{\partial}{\partial y}\left(-D\frac{\partial}{\partial y}\left(\nabla^2 w\right)\right) = -q$$

$$=> \frac{\partial^2}{\partial x^2}\left(\nabla^2 w\right) + \frac{\partial^2}{\partial y^2}\left(\nabla^2 w\right) = \frac{q}{D}$$

$$=> \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\nabla^2 w\right) = \frac{q}{D}$$

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Governing equation

- Governing equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\nabla^2 w = \frac{q}{D}$$

$$=> \nabla^2\left(\nabla^2 w\right) = \frac{q}{D}$$

$$=> \nabla^4 = \frac{q}{D}$$

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}}$$

Bi-harmonic equation

Compare with beam equation

Boundary conditions

- Well posed problem – Governing equations and boundary conditions
- Three basic boundary conditions
  - Simply supported
  - Clamped and
  - Free edge
- Vertical deflection and their derivatives

Simply supported

- Simply supported – for eg beam => vertical deflection = 0 and moment = 0

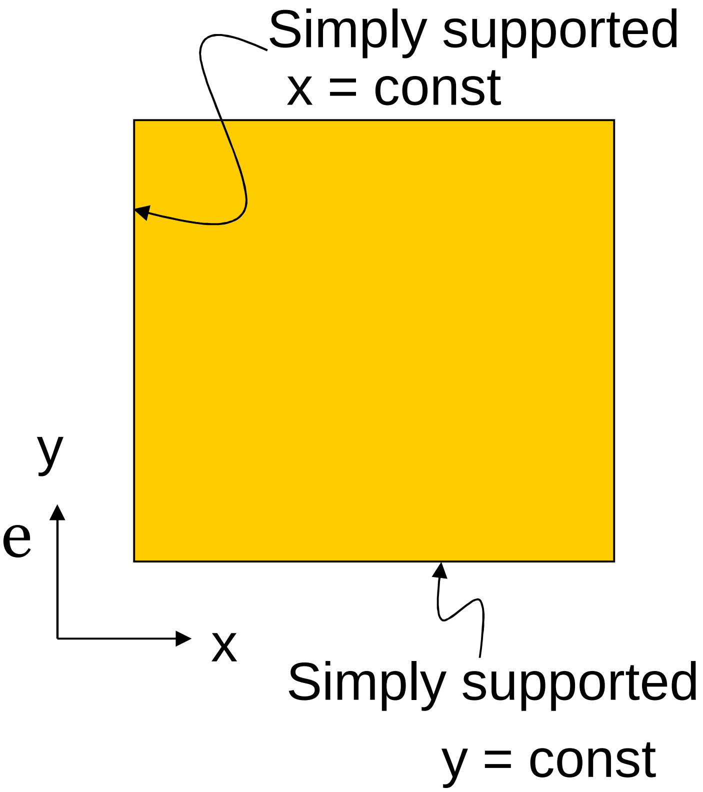
For edge, x = const

$$w(x, y) = 0$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right) = 0$$

$w = 0$ implies second derivative in the direction tangent to this line is zero

$$\frac{\partial^2 w}{\partial x^2} = 0$$


Simply supported
x = const

Simply supported
y = const

Simply supported

- Simply supported condition along edge y = const

$$w = 0$$

$$M_x = -D\left(\upsilon \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

w = *o* implies second derivative in the direction tangent to this line is zero

$$\frac{\partial^2 w}{\partial y^2} = 0$$

Clamped

- Deflection and slope in normal directions vanish

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial x} &= 0 \end{aligned}\right\} \quad x = \text{constant}$$

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial y} &= 0 \end{aligned}\right\} \quad y = \text{constant}$$

Free edges

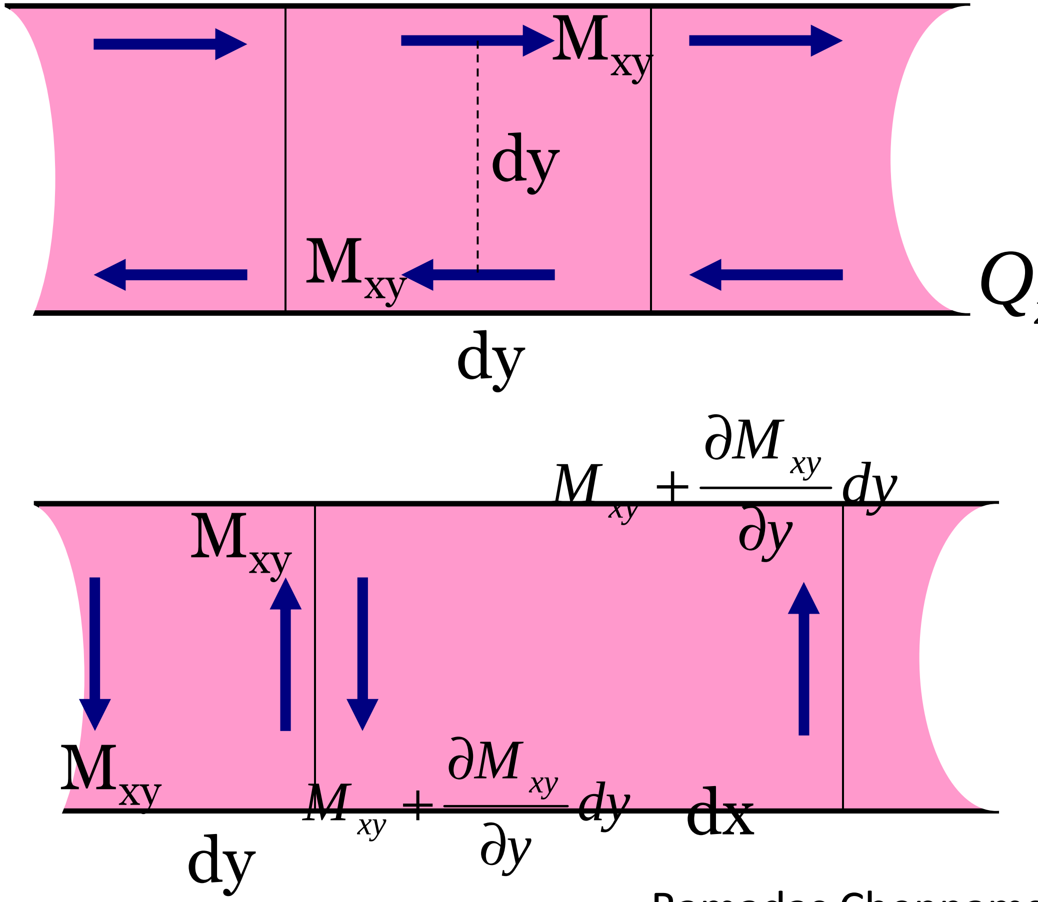
- Free edge – Free from any external loads – Natural boundary conditions
- Bending moment and Shear force vanish
- Bending moment

$$M_y = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right) = 0 \ \text{ at } x = \text{const}$$

$$M_x = -D\left(\upsilon\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \text{ at } y = \text{const}$$

Free edges

- Moment M_{xy} and shear forces



The net force acting on the face

$$Q_x^{'} = -M_{xy} - \frac{\partial M_{xy}}{\partial y} + M_{xy}$$

$$\Rightarrow Q_x^{'} = -\frac{\partial M_{xy}}{\partial y}$$

Total shear force,

$$V_x = Q_x + Q_x^{'}$$

Free edges

- Total shear force,

$$V_x = Q_x + Q_x^{'}$$

$$\Rightarrow V_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right) - D(1-\upsilon)\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$\Rightarrow V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

Free edges

- The forces, V_x and V_y => reduced, or Kirchoff's or effective shear forces
- In case of a free edge,

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x \partial y^2}\right] = 0 \quad \text{at } x = \text{const}$$

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x^2 \partial y}\right] = 0 \quad \text{at } y = \text{const}$$

Bending of plates

- Governing equation of plate rectangular plate bending

$$\nabla^2\left(\nabla^2 w\right) = \frac{q}{D}$$

w = vertical deflection = $w(x, y)$

external loading, $q = q(x, y)$

- Solution to this equation – product of two functions – Assume

$$w = w\left(x, y\right) = F\left(x\right)G\left(y\right)$$

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Bending of plates

- Choice of functions – algebraic, trigonometric, hyperbolic etc or combination of these function
- Selection of a function – depends on boundary conditions
- Simply supported edges – trigonometric function – Navier solution
- Deflection of a plate can be written as sum of infinite trigonometric functions

Bending of plates

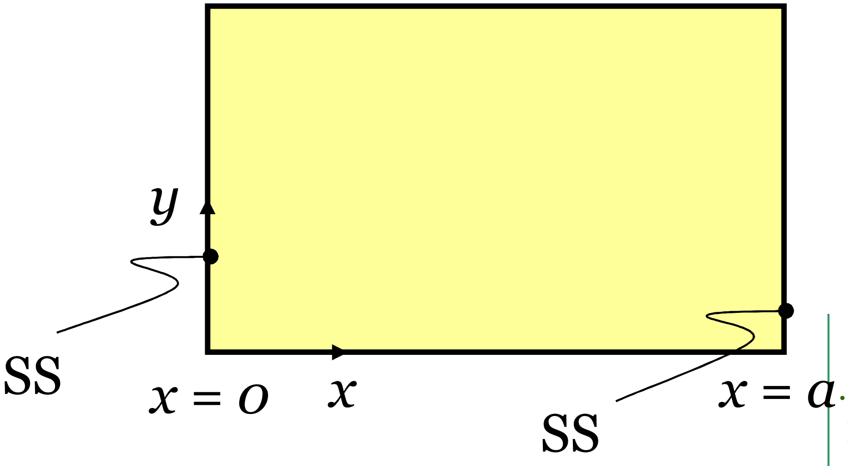
- Edges, $x = 0$ and $x = a$ simply supported

Vertical deflection vanish

$w(x=0, y)=0$, $w(x=a, y)=0$

Possible form of solution

$$F(x) = \sum_{m}^{\infty} F_m \sin\left(\frac{m\pi\, x}{a}\right)$$



F_1, F_2,…….F_∞ are coefficients

Symmetric loading wrt x = a/2
Maximum deflection at x = a/2

Bending of plates

- Other edges simply supported

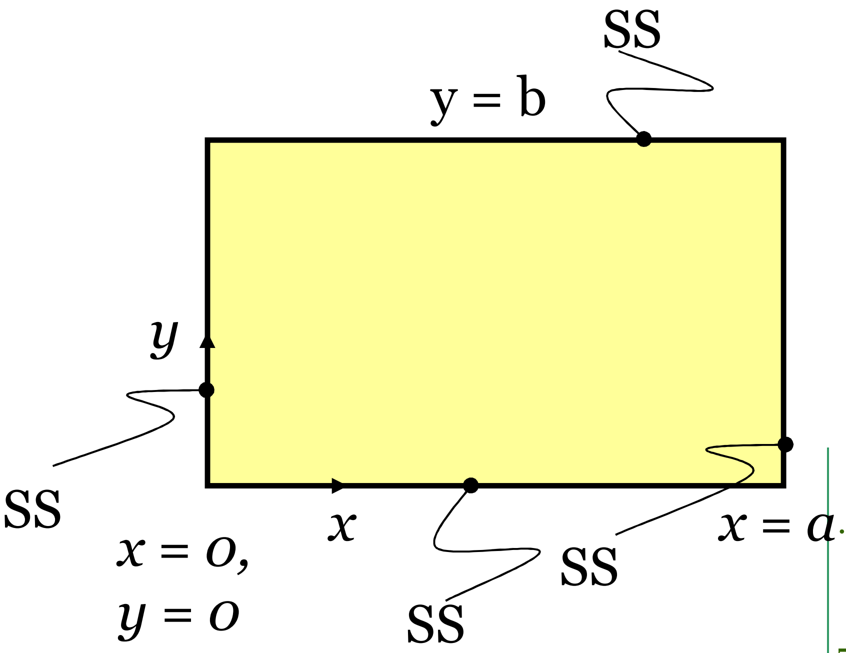
Vertical deflection vanish

$w(x, y=0)=0, w(x, y=b)=0$

Possible form of solution

$$G(y)=\sum_{n}^{\infty} G_n \sin\left(\frac{n\pi y}{b}\right)$$

$G_1, G_2, \ldots\ldots G_\infty$ are coefficients



Selection of functions based on BCs

Bending of plates

- Final solution,

$$w(x,y)=\sum_{m}^{\infty}\sum_{n}^{\infty}F_m G_n \sin\left(\frac{m\pi\, x}{a}\right)\sin\left(\frac{n\pi\, y}{b}\right)$$

$$\Rightarrow w(x,y)=\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\sin\alpha_m x\ \sin\beta_n y$$

$$\alpha_m=\frac{m\pi}{a},\quad \beta_n=\frac{n\pi}{b}$$

Coefficients, F_m, G_n and w_{mn} computed using Fourier Series

Coefficients

- Any periodic function can be expanded into a sine or cosine function using Fourier expansion

Function of one variable

$$f(x) = \sum_{m}^{\infty} f_m \sin\frac{m\pi x}{a}$$

$$\Rightarrow f(x)\sin\frac{m^{'}\pi x}{a} = f_m \sin\frac{m\pi x}{a}\ \sin\frac{m^{'}\pi x}{a}$$

Integrate the above from limits 0 to a

Coefficients

- Integration

$$\int_0^a f(x)\sin\frac{m'\pi x}{a}\,dx = f_m\int_0^a \sin\frac{m\pi x}{a}\ \sin\frac{m'\pi x}{a}\,dx$$

$$=>\frac{f_m}{2}\int_0^a 2\sin\frac{m\pi x}{a}\ \sin\frac{m'\pi x}{a}\,dx=\frac{f_m}{2}\int_0^a\left[\cos\frac{\pi x}{\mathrm{a}}\left(m-m'\right)-\cos\frac{\pi x}{\mathrm{a}}\left(m+m'\right)\right]dx$$

$$=>\frac{f_m}{2}\left[\frac{\sin\frac{\pi x}{\mathrm{a}}\left(m-m'\right)}{\frac{\pi}{\mathrm{a}}\left(m-m'\right)}-\frac{\sin\frac{\pi x}{\mathrm{a}}\left(m+m'\right)}{\frac{\pi}{\mathrm{a}}\left(m+m'\right)}\right]_0^a$$

Lower limit vanishes, evaluate upper limit

Coefficients

- Upper limit

$$\int_0^a f(x)\sin\frac{m'\pi x}{a}dx=\frac{f_m}{2}\left.\frac{\sin\frac{\pi x}{a}(m-m')}{\frac{\pi}{a}(m-m')}\right|_{x=a}$$

$$=>\int_0^a f(x)\sin\frac{m'\pi x}{a}dx=\frac{f_m a}{2}\qquad \text{for } m=m'$$

$$=0\qquad \text{for } m\neq m'$$

$$f_m=\frac{2}{a}\int_0^a f(x)\sin\frac{m\pi x}{a}dx$$

Coefficients

- Function of variable ‘y’

$$g(y)=\sum_{n}^{\infty} g_n \sin\frac{n\pi y}{b}$$

$$\Rightarrow g(y)\sin\frac{n'\pi y}{b}=g_n\sin\frac{n\pi y}{b}\;\sin\frac{n'\pi y}{b}$$

$$g_n=\frac{2}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}dy$$

Coefficients

- Function of two variables

$$w(x,y)=\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$=>\int_{x=0}^{x=a}w(x,y)\sin\left(\frac{m'\pi x}{a}\right)dx=\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\sin\left(\frac{n\pi y}{b}\right)\int_{x=0}^{x=a}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m'\pi x}{a}\right)dx$$

$$\int_{x=0}^{x=a}w(x,y)\sin\left(\frac{m'\pi x}{a}\right)dx=\frac{a}{2}\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\sin\left(\frac{n\pi y}{b}\right)$$

$$=>\int_{y=0}^{y=b}\int_{x=0}^{x=a}w(x,y)\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)dxdy=\frac{a}{2}\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\int_{y=0}^{y=b}\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n'\pi y}{b}\right)dy$$

$$=>\boxed{w_{mn}=\frac{4}{ab}\int_{y=0}^{y=b}\int_{x=0}^{x=a}w(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy}$$

Simply supported plate

- Assume loading over plate

$$q = q(x, y) = \sum_{m}^{\infty}\sum_{n}^{\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Solution

$$w = w(x, y) = \sum_{m}^{\infty}\sum_{n}^{\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Governing equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Simply supported plate

- Differentiating vertical displacement

$$\frac{\partial^4 w}{\partial x^4} = \sum_m^\infty \sum_n^\infty \left(\frac{m\pi}{a}\right)^4 w_{mn} \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right)$$

$$\frac{\partial^4 w}{\partial y^4} = \sum_m^\infty \sum_n^\infty \left(\frac{n\pi}{b}\right)^4 w_{mn} \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right)$$

$$2\frac{\partial^4 w}{\partial x^2 \partial y^2} = 2\sum_m^\infty \sum_n^\infty \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 w_{mn} \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right)$$

Simply supported plate

- Plug in governing equation

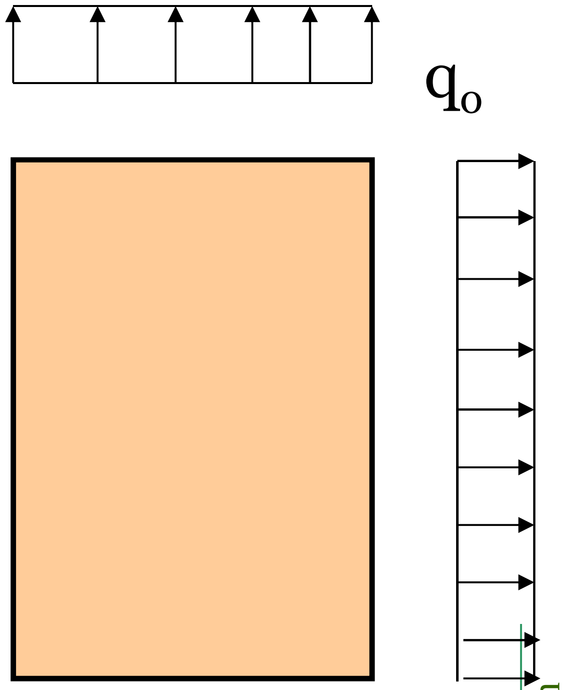
$$\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] w_{mn} = \frac{q_{mn}}{D}$$

$$\Rightarrow w_{mn} = \frac{q_{mn}}{D\pi^4\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}$$

$$w = w(x, y) = \frac{1}{D\pi^4}\sum_m^\infty\sum_n^\infty \frac{q_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}\sin\left(\frac{m\pi\, x}{a}\right)\sin\left(\frac{n\pi\, y}{b}\right)$$

UDL

- Uniformly distributed load



Computation of coefficients

$$q(x,y)=\sum_{m}^{\infty}\sum_{n}^{\infty}q_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$q(x,y)=q_o$$

$$q_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy$$

UDL

- Coefficients q_{mn}

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q_o \sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b}dxdy$$

$$q_{mn} = \frac{4q_o}{ab}\int_0^a \sin\frac{m\pi x}{a}dx\int_0^b \sin\frac{m\pi y}{b}dy$$

$$=> q_{mn} = \frac{4q_o}{ab}\left(\frac{4ab}{mn\pi^2}\right) = \frac{16q_o}{mn\pi^2}$$

UDL

- Vertical deflection

$$w = w(x,y) = \frac{1}{\pi^4 D}\sum_m^\infty \sum_n^\infty \frac{q_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

$$\Rightarrow w(x,y) = \frac{16}{\pi^6}\frac{q_o}{D}\sum_m^\infty \sum_n^\infty \frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

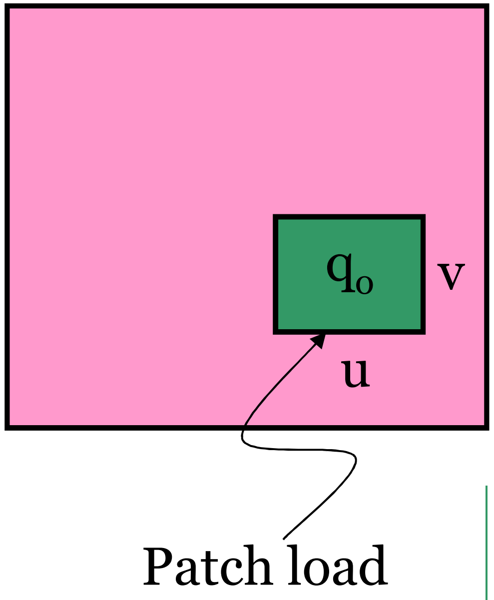
Patch load

- A patch load applied over an area *u* x *v*

Centroid at (x_o, y_o)

$$q(x,y) = \sum_{m}^{\infty}\sum_{n}^{\infty} q_{mn} \sin\left(\frac{m\pi\, x}{a}\right)\sin\left(\frac{n\pi\, y}{b}\right)$$

$$\Rightarrow q_o = \sum_{m}^{\infty}\sum_{n}^{\infty} q_{mn} \sin\left(\frac{m\pi\, x}{a}\right)\sin\left(\frac{n\pi\, y}{b}\right)$$



$$q_{mn} = \frac{4q_o}{ab}\int_{x_o-\frac{u}{2}}^{x_o+\frac{u}{2}}\int_{y_o-\frac{v}{2}}^{y_o+\frac{v}{2}} \sin\frac{m\pi\, x}{a}\sin\frac{m\pi\, y}{b}\,dxdy$$

Patch load

- Evaluating integrals

$$q_{mn} = \frac{4q_o}{ab} \int_{x_o - \frac{u}{2}}^{x_o + \frac{u}{2}} \int_{y_o - \frac{v}{2}}^{y_o + \frac{v}{2}} \sin \frac{m\pi\, x}{a} \sin \frac{m\pi\, y}{b}\, dxdy$$

$$\int_{x_o - \frac{u}{2}}^{x_o + \frac{u}{2}} \sin \frac{m\pi\, x}{a}\, dx = \frac{2a}{m\pi} \sin\left(\frac{m\pi\, x_o}{a}\right) \sin\left(\frac{m\pi\, u}{2a}\right)$$

$$=> q_{mn} = \frac{16}{mn} \frac{q_o}{\pi^2} \sin\left(\frac{m\pi\, x_o}{a}\right) \sin\left(\frac{m\pi\, u}{2a}\right) \sin\left(\frac{n\pi\, y_o}{b}\right) \sin\left(\frac{m\pi\, v}{2b}\right)$$

Patch load

- Deflection

$$w(x,y)=\frac{16q_o}{\pi^6 D}\sum_m^{\infty}\sum_n^{\infty}\frac{S_{mn}\sin\left(\dfrac{m\pi\, x}{a}\right)\sin\left(\dfrac{n\pi\, y}{b}\right)}{mn\left[\left(\dfrac{m}{a}\right)^2+\left(\dfrac{n}{b}\right)^2\right]}$$

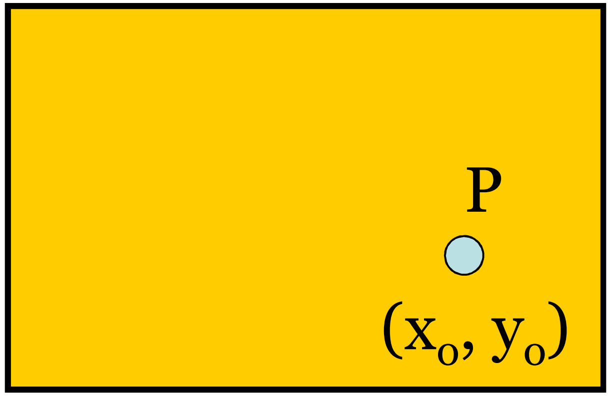
where,

$$S_{mn}=\sin\left(\frac{m\pi\, x_o}{a}\right)\sin\left(\frac{m\pi\, u}{2a}\right)\sin\left(\frac{n\pi\, y_o}{b}\right)\sin\left(\frac{m\pi\, v}{2b}\right)$$

Point load

- Point load

Assume the point load acts over an infinitesimal area *u* x *v*



Corresponding UDL

$$q_o = \frac{P}{uv}$$

From the earlier analysis

$$q_{mn} = \frac{16}{mn}\frac{q_o}{\pi^2}\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}\sin\frac{m\pi u}{2a}\sin\frac{n\pi v}{2b}$$

$$\Rightarrow q_{mn} = \frac{16}{mn}\frac{P}{uv\pi^2}\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}\sin\frac{m\pi u}{2a}\sin\frac{n\pi v}{2b}$$

Point load

- Simplifying

$$\Rightarrow q_{mn} = \frac{4P}{ab}\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}\left(\frac{\sin\frac{m\pi u}{2a}}{\frac{m\pi u}{2a}}\right)\left(\frac{\sin\frac{n\pi v}{2b}}{\frac{m\pi v}{2b}}\right)$$

$$\Rightarrow q_{mn} = \frac{4P}{ab}\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}$$

Point load

- Deflection due to point load

$$w(x,y)=\frac{4P}{\pi^{4}abD}\sum_{m}^{\infty}\sum_{m}^{\infty}\frac{S_{mn}^{'}\sin\frac{m\pi\,x}{a}\sin\frac{m\pi\,y}{b}}{\left[\left(\frac{m}{a}\right)^{2}+\left(\frac{n}{b}\right)^{2}\right]^{2}}$$

$$S_{mn}^{'}=\sin\frac{m\pi\,x_{o}}{a}\sin\frac{m\pi\,y_{o}}{b}$$

Bending & in-plane loading

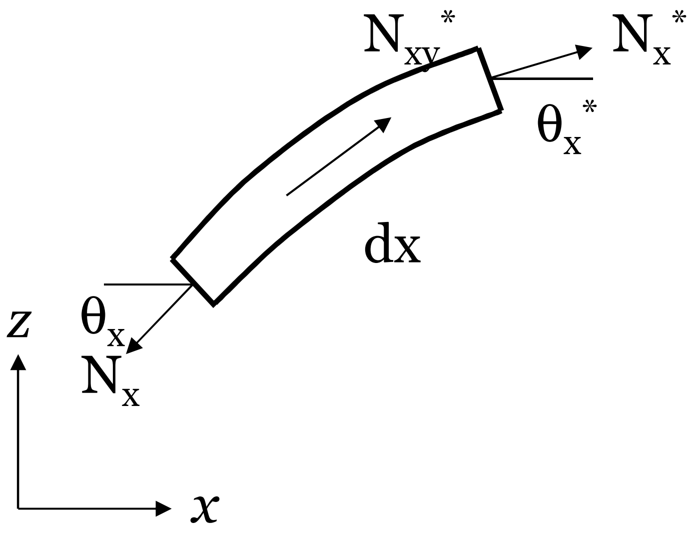
- Plates are subjected to in-plane loading also – in addition to lateral / transverse loads
- In-plane loading – tensile or compressive
- Large in-plane compressive loads – Buckling takes place
- Buckling – non-linear phenomenon – disproportionate increase of displacement with load
- Critical load – ability to resist axial load ceases – change in deformation shape

Bending & in-plane loading

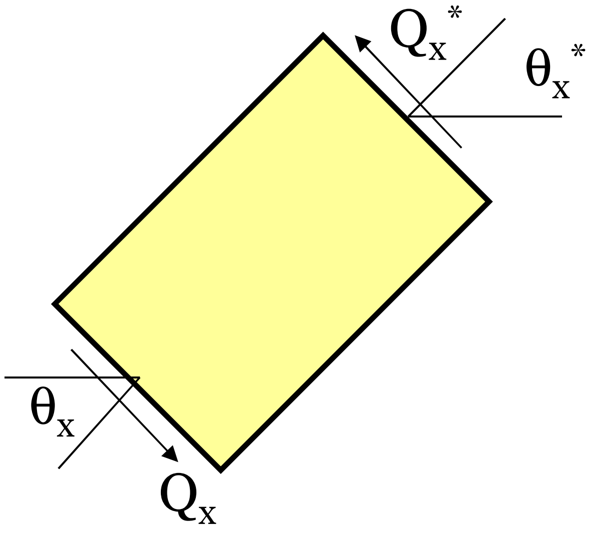
- Thin walled members – cross-sections like ‘I’, ‘L’, ‘H’, ‘C’ etc – undergo buckling – thin plates of small widths
- Combined loading of a rectangular plate – loads
  - In-plane forces: N_x, N_y, N_{xy} and N_{yx}
  - Transverse forces / moments: M_x, M_y, M_{xy}, M_{yx}, Q_x and Q_y
- Small deformation and large deformation

In-plane forces

- Infinitesimal element => $dA = dx\ dy$



In-plane loading



Transverse/out-of-plane loading

For small angles => $\sin\theta \approx \theta$ and $\cos\theta \approx 1$

In-plane forces

- Force equilibrium in x-direction

$$-N_x dy\cos\theta_x + \left(N_x + \frac{\partial N_x}{\partial x}dx\right)dy\cos\theta_x^* - N_{xy}dx\cos\theta_y +$$

$$\left(N_{xy} + \frac{\partial N_{xy}}{\partial y}dy\right)dx\cos\theta_y^* - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy\sin\theta_x^*$$

$$+Q_x dy\sin\theta_x - \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx\sin\theta_y^* + Q_y dx\sin\theta_y = 0$$

$$=> \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

If there are no in-plane forces – equation vanishes

In-plane forces

- Force equilibrium in y-direction

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{y}}{\partial y} = 0$$

- Angle is not equal to zero, but, small

$$\sin\,\theta \approx \theta \quad \text{and} \quad \cos\,\theta \approx 1$$

$$\theta_x = \frac{\partial w}{\partial x}, \qquad \theta_y = \frac{\partial w}{\partial y}$$

In-plane forces

- Force equilibrium in x-direction

$$-N_x dy + \left(N_x + \frac{\partial N_x}{\partial x}dx\right)dy - N_{xy}dx +$$

$$\left(N_{xy} + \frac{\partial N_{xy}}{\partial y}dy\right)dx - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx\right)$$

$$+Q_x dy\frac{\partial w}{\partial x} - \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx\left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2}dy\right) + Q_y dx\frac{\partial w}{\partial y} = 0$$

Neglect higher order terms

$$=> \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

No change in in-plane equilibrium equations

In-plane forces

- Similar expression for force equilibrium in y-direction
- Force equilibrium in z-direction
  - Contribution from in-plane normal forces, N_x and N_y and shear force, N_{xy}
  - Contribution from shear force, Q_x and Q_y
  - Contribution from externally applied load, q

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Z-direction

$$-N_x dy \sin\frac{\partial w}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x}dx\right)dy \sin\left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)dx\right)$$

$$-N_y dx \sin\frac{\partial w}{\partial y} + \left(N_y + \frac{\partial N_y}{\partial y}dy\right)dx \sin\left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)dy\right)$$

$$-N_{xy} dx \sin\frac{\partial w}{\partial x} + \left(N_{xy} + \frac{\partial N_{xy}}{\partial y}dy\right)dx \sin\left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)dy\right)$$

$$-N_{xy} dy \sin\frac{\partial w}{\partial y} + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x}dx\right)dy \sin\left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)dx\right)$$

$$-Q_x dy \cos\frac{\partial w}{\partial x} + \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy \cos\left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)dx\right)$$

$$-Q_y dx \cos\frac{\partial w}{\partial y} + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)\cos\left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)dy\right) + q\,dxdy = 0$$

Z-direction

- If no in-plane forces acting

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=-q$$

- Presence of in-plane forces

$$N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}+\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0$$

Z-direction

- Substituting Q_x and Q_y

$$Q_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right), \quad Q_y = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

$$N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}$$

$$+\frac{\partial}{\partial x}\left(-D\frac{\partial}{\partial x}\left(\nabla^2 w\right)\right) + \frac{\partial}{\partial y}\left(-D\frac{\partial}{\partial y}\left(\nabla^2 w\right)\right) + q = 0$$

$$=> \boxed{\nabla^4 w = \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + q\right)}$$

Plate buckling

- Buckling of thin plate

$$\nabla^4 w = \frac{1}{D}\left[q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial y \partial x} + N_y \frac{\partial^2 w}{\partial y^2}\right]$$

Assume, q = 0, in-plane load, $N_x = N_1$, rest zero

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}$$

Assume all four edges are simply supported – Navier solution to plate bending

Plate buckling

- Plate deflection

$$w = w(x,y) = \sum_{m}^{\infty}\sum_{n}^{\infty} w_{mn} \sin \alpha_m x \; \sin \beta_n y$$

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}$$

Substitute in governing equation

$$\alpha_m^4 + 2\alpha_m^2\beta_n^2 + \beta_n^4 - \frac{N_1}{D}\alpha_m^2 = 0$$

Characteristic equation

$$=> \left(\alpha_m^2 + \beta_n^2\right)^2 = \frac{N_1}{D}\alpha_m^2$$

Plate buckling

- From characteristic equation

$$\left(\alpha_m^2 + \beta_n^2\right)^2 = \frac{N_1}{D}\alpha_m^2$$

$$=> \frac{N_1}{D}\left(\frac{m\pi}{a}\right)^2 = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2$$

$$=> \; N_1 = \frac{D\pi^2 a^2}{m^2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2$$

$$=> \; N_1 = \frac{D\pi^2}{b^2}\left[\frac{m}{c} + \frac{c}{m}n^2\right]^2$$

$c = a / b$ - Aspect ratio

Plate buckling

- Critical load – smallest value
- Increase in N_1 with n^2 – Minimum value of *n* is equal to one – buckled shape in y-direction – single half sine wave

$$N_1 = \frac{D\pi^2}{b^2}\left[\frac{m}{c} + \frac{c}{m}\right]^2$$

N_1 is a function of variable ‘m’ – for minimum value of ‘m’, differentiate N_1 wrt ‘m’

Plate buckling

- Differentiate

$$\frac{dN_1}{dm} = \frac{2\pi^2 D}{b^2}\left(\frac{m}{c}+\frac{c}{m}\right)\left(\frac{1}{c}-\frac{c}{m^2}\right)=0$$

$$=> \left(\frac{1}{c}-\frac{c}{m^2}\right)=0$$

$$=> m = c = \frac{a}{b}$$ Whole number

$$\therefore\ N_{1cr} = \frac{4D\pi^2}{b^2}$$

Plate buckling

- Number of half sine waves can't be a whole number – it should be an integer
- Equation for critical load for $n = 1$

$$N_1 = K \frac{D\pi^2}{b^2} \qquad K = \left[\frac{m}{c} + \frac{c}{m}\right]^2$$

Plotting 'K' vs aspect ratio = c = a/b
for various integer values of 'm'

Plate buckling

- K vs aspect ratio

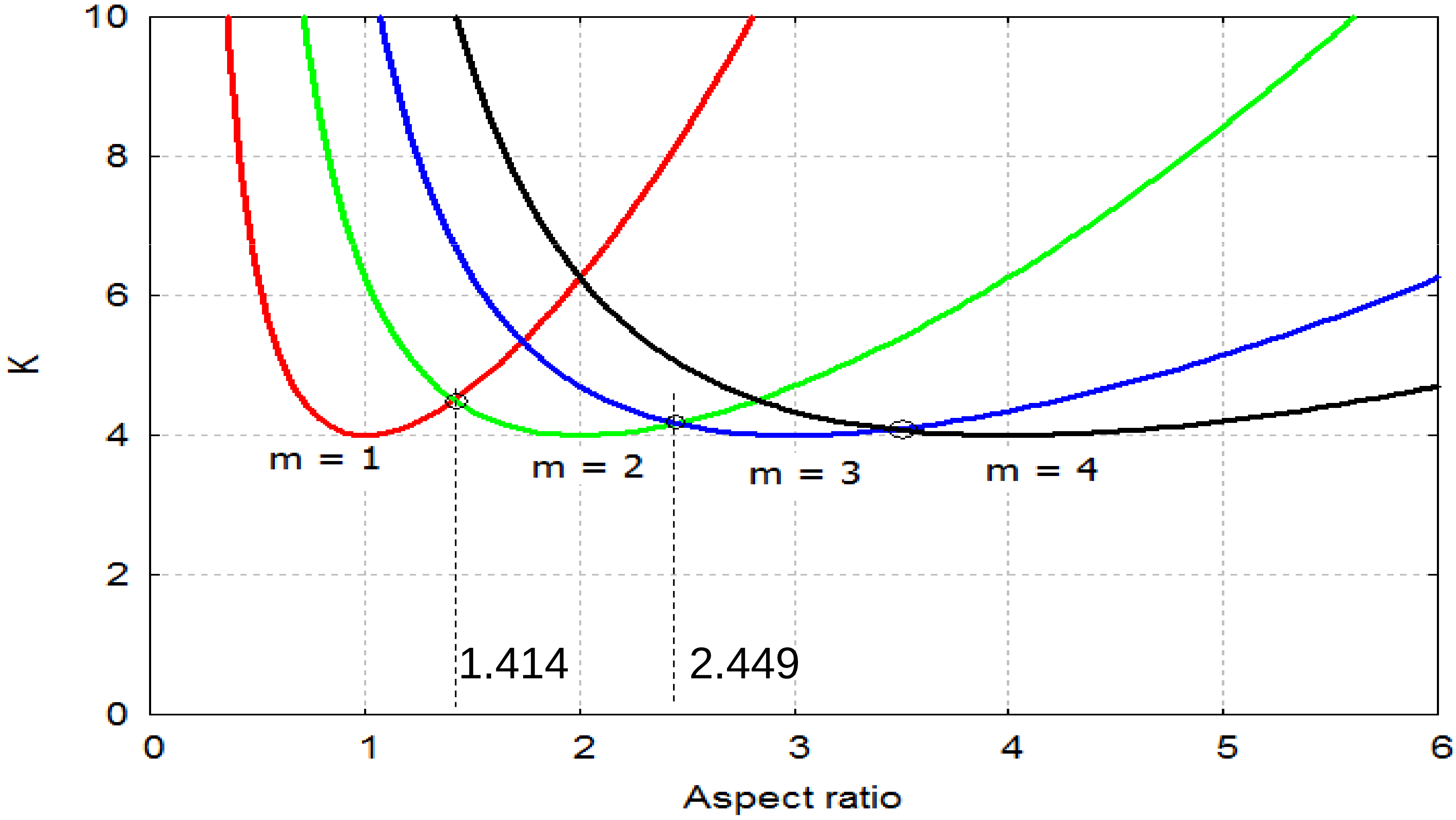

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Plate buckling

- From figure, value of ‘K’ is same as intersection points of ‘m’ and ‘m+1’

N_1 critical load at m – when load is increased, buckled form changes from ‘m’ to ‘m+1’

At transition from ‘m’ to ‘m+1’

$$\left(\frac{m}{c}+\frac{c}{m}\right)^2=\left(\frac{m+1}{c}+\frac{c}{m+1}\right)^2$$

$$\Rightarrow c^2=m(m+1)$$

$$c=\sqrt{m(m+1)}$$

Curves for $m = 1$ and $m = 2$ meet at $c = \sqrt{2}$

Aspect ratio less than $\sqrt{2} \Rightarrow m = 1$

Aspect ratio from $\sqrt{2}$ to $\sqrt{6}$, $m = 2$

$C>4 \Rightarrow K \approx 4$

Plate buckling

- Estimation of buckling load

t = 1 cm, a = 2.3 m, b = 1 m, E = 200 GPa, υ = 0.30

Flexural modulus

$$D=\frac{Et^{3}}{12\left(1-\upsilon^{2}\right)}=\frac{200\times10^{9}\times\left(1\times10^{-2}\right)^{2}}{12\left(1-0.3^{2}\right)}=17.96\,kNm$$

$$c=\frac{a}{b}=\frac{2.3}{1}=2.3=>m=2$$

$$=>K=\left[\frac{m}{c}+\frac{c}{m}\right]^{2}=\left[\frac{2}{2.3}+\frac{2.3}{2}\right]^{2}=4.0786$$

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Plate buckling

- Minimum critical load

$$N_{1cr} = K\frac{D\pi^2}{b^2} = 4.086 \times \frac{D\pi^2}{b^2} = 73.72MN/m$$

$$N = N_{1cr} \times b = 73.72MN$$

In the above expression, for a given width and elastic properties of plate, critical load depends on ‘K’. In turn ‘K’ depends on ‘m’ for a given aspect ratio, *c*

$$c = 2.3,\ \ m = 1, \quad K = 7.4790$$

$$m = 2, \quad K = 4.0786$$

$$m = 3, \quad K = 4.2890$$

Minimum value of ‘K’ is considered for estimation of N_{cr}

Plate and column buckling

- Uniaxial load for plate buckling

$$N_{1crp} = K\frac{D\pi^2}{b^2}$$

$$\sigma_{1crp} = \frac{N_{1crp}}{t} = K\frac{D\pi^2}{b^2 t} = K\frac{\pi^2}{b^2 t}\frac{Et^3}{12\left(1-\upsilon^2\right)}$$

$$\Rightarrow \sigma_{1crp} = \frac{K}{12\left(1-\upsilon^2\right)}\frac{\pi^2 E}{\left(b/t\right)^2}$$

$$\Rightarrow \sigma_{1crp} = C_1\frac{\pi^2 E}{\left(b/t\right)^2}, \qquad C_1 = \frac{K}{12\left(1-\upsilon^2\right)}$$

Plate and column buckling

- Buckling of a column

$$P_{cr} = C_2 \frac{\pi^2 EI}{l^2}$$

$$\Rightarrow P_{cr} = C_2 \frac{\pi^2 EAk^2}{l^2} = C_2 \frac{\pi^2 EA}{\left(l/k\right)^2}$$

$$\Rightarrow \boxed{\sigma_{crc} = \frac{P_{cr}}{A} = C_2 \frac{\pi^2 E}{\left(l/k\right)^2}} \quad \text{Column}$$

$$\boxed{\sigma_{crp} = C_1 \frac{\pi^2 E}{\left(b/t\right)^2}} \quad \text{Plate}$$

Plate and column buckling

- Critical stress for plate depends on thinness ratio = t/b – not on the length
  - depends on width
- More thinner plate – lesser bucking load
- Critical stress in column depends on slenderness ratio
- Longer columns – lower critical load

Strain energy

- In thin plate theory, out-of-plane shear stresses vanish => τ_{xz}, τ_{yz} and τ_{zz}
- Stress components contributing to strain energy => σ_{xx}, σ_{yy} and τ_{xy}
- Strain energy,

Linear elastic material

$$U = \frac{1}{2}\int_{\forall}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}\right)d\forall$$

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Strain energy

- Strain energy,

$$U=\frac{1}{2}\int_{\forall}\{\sigma_{xx}\quad\sigma_{yy}\quad\tau_{xy}\}\{\varepsilon_{xx}\quad\varepsilon_{yy}\quad\gamma_{xy}\}^{T}\,d\forall$$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}=-\frac{Ez}{1-\upsilon^{2}}\begin{bmatrix}1 & \upsilon & 0\\ \upsilon & 1 & 0\\ 0 & 0 & \dfrac{1+\upsilon}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=-\frac{Ez}{1-\upsilon^{2}}\begin{bmatrix}1 & \upsilon & 0\\ \upsilon & 1 & 0\\ 0 & 0 & \dfrac{1+\upsilon}{2}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^{2}w}{\partial x^{2}}\\ \dfrac{\partial^{2}w}{\partial y^{2}}\\ 2\dfrac{\partial^{2}w}{\partial x\partial y}\end{Bmatrix}$$

$$=>\{\sigma\}=[C]\{\varepsilon\}$$

Strain energy

- Strain energy,

$$U=\frac{1}{2}\int_{\mathcal{V}}\{\sigma\}^{T}\{\varepsilon\}d\mathcal{V}=\frac{1}{2}\int_{\mathcal{V}}\{\varepsilon\}^{T}[C]\{\varepsilon\}d\mathcal{V}$$

$$=>U=\frac{E}{2\left(1-\upsilon^{2}\right)}\int_{\mathcal{V}}\left[\begin{array}{l}\left(\frac{\partial^{2}w}{\partial x^{2}}+\upsilon\frac{\partial^{2}w}{\partial y^{2}}\right)\frac{\partial^{2}w}{\partial x^{2}}\\+\left(\upsilon\frac{\partial^{2}w}{\partial x^{2}}+\frac{\partial^{2}w}{\partial y^{2}}\right)\frac{\partial^{2}w}{\partial x^{2}}\\+2\left(1-\upsilon\right)\left(\frac{\partial^{2}w}{\partial x\partial y}\right)^{2}\end{array}\right]z^{2}d\mathcal{V}$$

Strain energy

- Infinitesimal volume, $dV = dxdydz$
- Carry out integration over thickness => dz

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} z^2 dz = \frac{h^3}{12}$$

$$D = \frac{Eh^3}{12\left(1-\upsilon^2\right)}$$

Strain energy

- Simplify

$$U = \frac{D}{2}\int_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right)^2 - 2\left(1-\upsilon\right)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dxdy$$


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