AM135abc

Elasticity Notes of Eli Sternberg

1. Elements of Cartesian tensor analysis

References: Jeffreys & Jeffreys (Chapter 3), Temple, McConnell, Prager, Sokolnikoff, Chadwick, Spencer

Motivation: role in math. physics & continuum mech.

Indicial notation

Recall expansion of scalar product of two vectors

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

Range convention. All lower-case Latin indices (subscripts and superscripts) – unless otherwise specified – have the range (1,2,3).

v_i ... 3 elements, v_{ij} ... $3^2 = 9$ elements, v_{ijk} ... $3^3 = 27$ elemen

$$\underbrace{v_{ij\ldots k}}_{N} \ldots 3^N \text{ elements} \quad (N \ldots \text{number of subs.})^*$$

Summation convention. If in a monomial the same index appears twice, summation over this repeated ind is implied – unless suspended by "no sum".

---

* The objects $v_{ij\ldots k}$ are understood to be sets of real numbers, unless otherwise qualified.

Examples and remarks:

$u_i v_i = u_j u_j$, $v_{ii} = v_{jj} = v_{11} + v_{22} + v_{33}$

repeated index = "dummy index" (explain)

$a_{ij} x_j = b_i \Leftrightarrow a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 = b_i$ (explain)

$$a_{ij} x_j = c_{kki} \Leftrightarrow a_{ik} x_k = c_{kki} \Leftrightarrow a_{pk} x_k = c_{kkp}$$

NONSENSE: $a_i c_{ji} = c_k$, $a_{ikk} c_k$

(a) Same index cannot appear more than twice in same monomial

(b) Free (unrepeated) subs. on either side of an equation in indicial notation must agree

(c) Both free and dummy subs. may be altered subject to requirements (a), (b).

Def. Kronecker delta. The set of 9 numbers δ_{ij} is defined by

$$\delta_{ij} = 0 \ (i \neq j), \quad \delta_{ii} = 1 \ (\text{no sum})$$

Consequences:

$$\left.\begin{array}{l} \delta_{ij} = \delta_{ji}, \quad \delta_{ii} = 3, \quad \delta_{ij} v_j = v_i, \quad \delta_{ij} v_{jk} = v_{ik}, \\ \delta_{ij} v_{ik} = v_{jk}, \quad \delta_{ij} v_{ij} = v_{ii}, \quad \delta_{ij}\delta_{ij} = 3 \end{array}\right\} \quad (1.1)$$

Def. <u>Alternating symbols.</u> The 27 numbers ε_{ijk} are defined by

$\varepsilon_{ijk}=1$ if (i,j,k) are an <u>even</u> permutation of $(1,2,3)$
$\varepsilon_{ijk}=-1$ if (i,j,k) are an <u>odd</u> permutation of $(1,2,3)$
$\varepsilon_{ijk}=0$ otherwise, i.e. if at least two subs. coincide

E.g. $\varepsilon_{123}=\varepsilon_{231}=\varepsilon_{312}=1$, $\varepsilon_{213}=-1$, $\varepsilon_{121}=\varepsilon_{233}=\varepsilon_{222}=0$

Clearly,

$$\varepsilon_{ijk}=\varepsilon_{kij}=\varepsilon_{jki}=-\varepsilon_{jik}=-\varepsilon_{kji}=-\varepsilon_{ikj}\,,\quad \varepsilon_{ijk}\varepsilon_{ijk}=6 \qquad (1.2)$$

<u>Representation of 3x3 determinants</u>

Consider $\varepsilon_{ijk}u_iv_jw_k$. By def. of ε_{ijk} one has

$$\varepsilon_{1jk}v_jw_k = v_2w_3 - v_3w_2 = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \text{ and thus}$$

$$\varepsilon_{ijk}u_iv_jw_k = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad (1.3)$$

Adopt the notation

$$[\underset{\sim}{V}] = [V_{ij}] = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}, \quad \det[\underset{\sim}{V}] = \begin{vmatrix} V_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} \qquad (1.4)$$

<u>Claim:</u>

$$\varepsilon_{ijk}\varepsilon_{pqr}\det[\underset{\sim}{V}] = \begin{vmatrix} V_{ip} & V_{iq} & V_{ir} \\ V_{jp} & V_{jq} & V_{jr} \\ V_{kp} & V_{kq} & V_{kr} \end{vmatrix} \qquad (1.5)$$

Proof. Clearly (1.5) holds for $(i,j,k)=(p,q,r)=(1,2,3)$; also if two among indices (i,j,k) or (p,q,r) coincide; also if (i,j,k) or (p,q,r) are even or odd permutations of $(1,2,3)$ (see properties of determinants). This exhausts all possibilities. qed.

From (1.3), (1.4) and properties of determinants,

$$\left.\begin{aligned} \det[\underline{V}] &= \varepsilon_{ijk} V_{1i} V_{2j} V_{3k} = \varepsilon_{ijk} V_{i1} V_{j2} V_{k3} \\ \varepsilon_{pqr} \det[\underline{V}] &= \varepsilon_{ijk} V_{pi} V_{qj} V_{rk} = \varepsilon_{ijk} V_{ip} V_{jq} V_{kr} \end{aligned}\right\} \quad (1.6)$$

"Multiply" last of (1.6) by ε_{pqr}, use $\varepsilon_{pqr}\varepsilon_{pqr}=6$. Thus,

$$\det[\underline{V}] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr} V_{ip} V_{jq} V_{kr} \qquad (1.7)$$

<u>Relations connecting δ_{ij} and ε_{ijk}</u>

Specialize (1.5) by taking $[\underline{V}] = [V_{ij}] = [\delta_{ij}] = [\underline{1}]$. Thus,

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \qquad (1.8)$$

Expand determinant, set $i=p$ (summation!), and use (1.1) (properties of Kronecker delta) to arrive at

$$\varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq} \tag{1.9}$$

Discuss structure of (1.9). Specialize (1.9) to see that

$$\varepsilon_{ijk}\varepsilon_{ijr} = 2\delta_{kr}, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{1.10}$$

<u>Exercise 1</u>

(a) Deduce (1.9)

(b) Use results established in the lectures to show that

$$\left.\begin{aligned} & v_{ij}x_j = w_i, \ \det[\underset{\sim}{v}] \neq 0 \Longrightarrow \\ & \quad x_i = \frac{1}{2\det[\underset{\sim}{v}]}\varepsilon_{ijk}\varepsilon_{pqr}v_{qj}v_{rk}w_p \end{aligned}\right\} \tag{1.11}$$

(c) Let $[\underset{\sim}{u}], [\underset{\sim}{v}]$ be 3×3 matrices and $[\underset{\sim}{w}] = [\underset{\sim}{u}][\underset{\sim}{v}]$, i.e. $w_{ij} = u_{ik}v_{kj}$. Use results established in the lectures to show that $\det[\underset{\sim}{w}] = \det[\underset{\sim}{u}]\det[\underset{\sim}{v}]$.

<u>Connection with vector algebra</u>

Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ be vectors in $E_3 \equiv E$. <u>Notation</u>:

$\underset{\sim}{u}\cdot\underset{\sim}{v}$ ... scalar product $\qquad \underset{\sim}{u}\wedge\underset{\sim}{v}$ ... vector product

$\mathfrak{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ ... rectangular cartes. coord. frame with origin O and unit base vectors $\underset{\sim}{e}_i$

$\mathcal{F}$ ... class of all rectangular frames

$$\mathbb{X} \in \mathcal{F} \Rightarrow \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$$

$\mathcal{F}_+$ ... class of all "right-handed" rectangular frames

$$\mathbb{X} \in \mathcal{F}_+ \Rightarrow \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}\ , \quad (\underset{\sim}{e}_1 \wedge \underset{\sim}{e}_2) \cdot \underset{\sim}{e}_3 = +1$$

Let $\mathbb{X} \in \mathcal{F}_+$ and $u_i = \underset{\sim}{u} \cdot \underset{\sim}{e}_i$ so that $\underset{\sim}{u} = \underset{\sim}{e}_i u_i$ etc.

u_i ... scalar compts. of $\underset{\sim}{u}$ in $\mathbb{X}$. Remark on extension of summation convention to products of vectors and real numbers.

By (1.3) and familiar results from vector algebra

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_i v_i\ , \quad \underset{\sim}{u} \wedge \underset{\sim}{v} = \varepsilon_{ijk} \underset{\sim}{e}_i u_j v_k\ , \quad (\underset{\sim}{u} \wedge \underset{\sim}{v}) \cdot \underset{\sim}{w} = \varepsilon_{ijk} u_i v_j w_k \qquad (1.12)$$

<u>Exercise</u> 2. Use indicial notation to prove:

(a) $(\underset{\sim}{u} \wedge \underset{\sim}{v}) \cdot \underset{\sim}{w} = \underset{\sim}{u} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{w})$

(b) $(\underset{\sim}{u} \wedge \underset{\sim}{v}) \wedge \underset{\sim}{w} = (\underset{\sim}{u} \cdot \underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{v} \cdot \underset{\sim}{w})\underset{\sim}{u}$

(c) $(\underset{\sim}{u} \wedge \underset{\sim}{v})^2 = \underset{\sim}{u}^2 \underset{\sim}{v}^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2$

<u>Change of frame: transformation of vector compts.</u>

Consider

$$\mathbb{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}\ , \quad \mathbb{X}' = \{O; \underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3\} \in \mathcal{F}$$

---

* Restriction to $\mathcal{F}_+$ not needed for first of (1.12)

$$e_i \cdot e_j = \delta_{ij} \ , \ e'_p \cdot e'_q = \delta_{pq} \ , \ e'_i \cdot e_j = A_{ij} = \cos(e'_i, e_j) \quad (1.1$$

Expand scalar product in <u>second</u> of (1.13) in $\mathbb{X}$
" " " " <u>first</u> " " " $\mathbb{X}'$

$$\left.\begin{aligned} A_{pi} A_{qi} = \delta_{pq} \ , \ A_{pi} A_{pj} = \delta_{ij} \\ \det [A_{ij}] = (e'_1 \wedge e'_2) \cdot e'_3 = \pm 1 \end{aligned}\right\} (1.14)$$

Thus $[A] \equiv [A_{ij}]$ is an orthogonal matrix obeying

$$[A][A]^T = [A]^T [A] = [1] \ , \ [A]^{-1} = [A]^T , \ \det [A] = \pm 1 \quad (1.1$$

Note: $\mathbb{X} \in \mathcal{F}_+ , \mathbb{X}' \in \mathcal{F}_+ \Rightarrow \det [A] = +1$, in which case $[A]$ is <u>proper</u> orthogonal

Notation:

$$[A] : \mathbb{X} \to \mathbb{X}'$$

Consider

$$v = e_p v_p^{\mathbb{X}} \ , \ v = e'_p v_p^{\mathbb{X}'}$$

Remark on notation for scalar components. Dot-multiply first eq. by e'_i and use (1.13) to obtain

$$v_i^{\mathbb{X}'} = A_{ip} v_p^{\mathbb{X}} \quad (1.16)$$

"Multiply" (1.16) by A_{iq}, use (1.14), change indices:

$$v_i^{\mathbb{X}} = A_{pi} v_p^{\mathbb{X}'} \quad (1.17)$$

Conversely (1.17) $\Rightarrow$ (1.16).

Let $[\underset{\sim}{A}] = [A_{ij}]$, $[\underset{\sim}{v}] = \begin{bmatrix} v_1^{\mathbb{X}} \\ v_2^{\mathbb{X}} \\ v_3^{\mathbb{X}} \end{bmatrix}$, $[\underset{\sim}{v}'] = \begin{bmatrix} v_1^{\mathbb{X}'} \\ v_2^{\mathbb{X}'} \\ v_3^{\mathbb{X}'} \end{bmatrix}$.

Then (1.16), (1.17) in matrix form become

$$[\underset{\sim}{v}'] = [\underset{\sim}{A}][\underset{\sim}{v}] \;,\quad [\underset{\sim}{v}] = [\underset{\sim}{A}]^T[\underset{\sim}{v}']$$

Remark on "tensorial" definition of vector concept and on impending generalization.

<u>Def</u>. Let $N \geq 0$ be an integer. A (cartesian) <u>tensor</u> $\underset{\sim}{v}$ <u>of</u> <u>order</u> N is a function defined ~~on defined~~ on $\mathcal{F}$ that assigns to each $\mathbb{X} \in \mathcal{F}$ an ordered set of 3^N real numbers $v_{ij\ldots\ldots k}^{\mathbb{X}}$ (N indices), called the components of $\underset{\sim}{v}$ in $\mathbb{X}$, $\ni \forall\, [\underset{\sim}{A}] : \mathbb{X} \to \mathbb{X}'$,

$$v_{ij\ldots\ldots k}^{\mathbb{X}'} = A_{ip} A_{jq} \cdots A_{kr}\, v_{pq\ldots r}^{\mathbb{X}} \,. \qquad (1.18)$$

Further, if $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are both tensors of order N and c is a real number, we adopt the definition

(i) $\underset{\sim}{u} = \underset{\sim}{v} \iff u_{ij\ldots\ldots k}^{\mathbb{X}} = v_{ij\ldots k}^{\mathbb{X}} \quad \forall\, \mathbb{X} \in \mathcal{F}$

(ii) $\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v} \iff w_{ij\ldots k}^{\mathbb{X}} = u_{ij\ldots k}^{\mathbb{X}} + v_{ij\ldots k}^{\mathbb{X}} \quad \forall\, \mathbb{X} \in \mathcal{F}$

(iii) $\underset{\sim}{w} = c\underset{\sim}{v} \iff w_{ij\ldots\ldots k}^{\mathbb{X}} = c\, v_{ij\ldots\ldots k}^{\mathbb{X}} \quad \forall\, \mathbb{X} \in \mathcal{F}$

Remarks. Clearly, $\underset{\sim}{w}$ in (ii), (iii) is a tensor of order N. Equality needs to be tested in single $\mathfrak{X}$ only.

If N = 0, $\underset{\sim}{v}$ is a "scalar" & we write v instead!

If N = 1, $\underset{\sim}{v}$ is a "vector" & (1.18) reduces to (1.16)

From (1.14), (1.18) follows easily that

$$\overset{\mathfrak{X}}{v}_{ij\ldots k} = A_{pi} A_{qj} \ldots A_{rk} \overset{\mathfrak{X}'}{v}_{pq\ldots r} \qquad (1.19)$$

Conversely, (1.19) ⇒ (1.18). Ask to confirm this.

Mention general tensor concept, the definition of which involves essentially arbitrary curvilinear coordinate transformations (rather than merely transformations from one system of orthogonal cartesian coords. to another). See Mc Connell.

We agree that "tensor" henceforth means "cartesian tensor."

~~...tensors general tensors and agree that henceforth~~
~~... means cartesian tensors~~

<u>Def.</u> A tensor $\underset{\sim}{v}$ of order $N \geq 2$ is <u>symmetric</u> (resp. <u>skew-symmetric</u>) w.r. to a pair of indices of its components if the components of $\underset{\sim}{v}$ remain unaltered (resp. change sign) under an interchange of these two indices.

Define fully symm. (fully skew symm.) tensors.
Note that symm (skew-symm) needs to be tested only in one frame. (intrinsic properties)

<u>Def.</u> A tensor $\underset{\sim}{v}$ of arbitrary order is <u>isotropic</u> if its compts. $v^{\mathfrak{X}}_{ij\ldots k}$ are independent of $\mathfrak{X}$.

Note that a scalar is an <u>isotropic</u> tensor of order zer

<u>Thm. 1.1</u>

(a) Let $v^{\mathfrak{X}}_{ij} = \delta_{ij} \ \forall \ \mathfrak{X} \in \mathfrak{F}$. Then $v^{\mathfrak{X}}_{ij}$ are the compts. of an isotropic second-order tensor $\underset{\sim}{v}$ and we write $\underset{\sim}{v} = \underset{\sim}{1}$ (Kronecker tensor or idem tensor)

(b) Let $\overset{X}{v}_{ijk} = \varepsilon_{ijk}\ \forall\ X \in \mathcal{F}$ and replace $\mathcal{F}$ in def. of a tensor by $\mathcal{F}_+$. Then $\overset{X}{v}_{ijk}$ are the components of an isotropic tensor $\underset{\sim}{v}$ of order three (alternating tensor)

Proof. Clearly suffices to show that $\overset{X}{v}_{ij}$ and $\overset{X}{v}_{ijk}$ are the compts. in X of a tensor (explain)

Re (a). By hyp. and (1.14) one has

$$A_{ip} A_{jq} \overset{X}{v}_{pq} = A_{ip} A_{jq} \delta_{pq} = A_{ip} A_{jp} = \delta_{ij} = \overset{X'}{v}_{ij} \quad \text{qed.}$$

Re (b). By hyp. and (1.14), (1.6) one has

$$A_{ip} A_{jq} A_{kr} \overset{X}{v}_{pqr} = A_{ip} A_{jq} A_{kr} \varepsilon_{pqr}$$

$$= \varepsilon_{ijk} \underbrace{\det[\underset{\sim}{A}]}_{1 \text{ if } X \in \mathcal{F}_+,\ X' \in \mathcal{F}_+} = \varepsilon_{ijk} = \overset{X'}{v}_{ijk} \quad \text{qed.}$$

Remarks: $\overset{X}{v}_{ijk}$ would not be tensor components if $\det[\underset{\sim}{A}] = -1$ (pseudo-tensor). $\underset{\sim}{v}$ is symm. in (a) fully skew-symm. in (b).

Restriction of tensor concept. Motivated by (b) in Thm. 1.1 we henceforth replace $\mathcal{F}$ by $\mathcal{F}_+$ in def. of cartesian tensors and let $\mathcal{F}$ from here on have the meaning of $\mathcal{F}_+$ (change of notation).

~~Equality of addition of tensors, multiplication by scalar~~

~~Let $\underset{\sim}{u}$, $\underset{\sim}{v}$ be tensors of same order N, let c be a scalar. Adopt the definitions:~~

~~(a) $\underset{\sim}{u} = \underset{\sim}{v} \Leftrightarrow \overset{\mathbf{X}}{u}_{ij\ldots k} = \overset{\mathbf{X}}{v}_{ij\ldots k} \;\forall\, \mathbf{X} \in \mathcal{F}$~~

~~(b) $\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v} \Leftrightarrow \overset{\mathbf{X}}{w}_{ij\ldots k} = \overset{\mathbf{X}}{u}_{ij\ldots k} + \overset{\mathbf{X}}{v}_{ij\ldots k} \;\forall\, \mathbf{X} \in \mathcal{F}$~~

~~(c) $\underset{\sim}{w} = c\,\underset{\sim}{u} \Leftrightarrow \overset{\mathbf{X}}{w}_{ij\ldots k} = c\,\overset{\mathbf{X}}{u}_{ij\ldots k} \;\forall\, \mathbf{X} \in \mathcal{F}$~~

~~Remarks. Clearly, $\underset{\sim}{w}$ is a tensor of order N.~~

~~Note that equality needs to be tested in one $\mathbf{X}$ only~~

Theorem 1.2 (Outer multiplication of tensors). Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be tensors of order M and N, respectively. Define

$$\underbrace{\overset{\mathbf{X}}{w}_{ij\ldots k\,mn\ldots r}}_{M+N \text{ subs.}} = \underbrace{\overset{\mathbf{X}}{u}_{ij\ldots\ldots k}}_{M \text{ subs.}} \underbrace{\overset{\mathbf{X}}{v}_{mn\ldots r}}_{N \text{ subs.}} \quad \forall\, \mathbf{X} \in \mathcal{F}$$

Then $\overset{\mathbf{X}}{w}_{ij\ldots r}$ are the compts. in $\mathbf{X}$ of a tensor $\underset{\sim}{w}$ of order M+N, which is called the outer product of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

<u>Proof.</u> By hyp. and def. of a tensor, if $[\underset{\sim}{A}]:\mathfrak{X}\to\mathfrak{X}'$,

$$w^{\mathfrak{X}'}_{ij\ldots kmn\ldots r} = u^{\mathfrak{X}'}_{ij\ldots k}\, v^{\mathfrak{X}'}_{mn\ldots r}$$
$$= A_{ip}\ldots A_{kq}\, u^{\mathfrak{X}}_{p\ldots q}\, A_{ms}\ldots A_{rt}\, v^{\mathfrak{X}}_{s\ldots t}$$
$$= A_{ip}\ldots A_{kq} A_{ms}\ldots A_{rt}\, w^{\mathfrak{X}}_{p\ldots qs\ldots t},$$

so that conclusion follows from def. of a tensor.

<u>Thm.</u> 1.3 (Contraction of tensors). Let $\underset{\sim}{u}$ be a tensor of order $N\geq 2$. Define

$$v^{\mathfrak{X}}_{\underbrace{ij\ldots k}_{N-2}} = u^{\mathfrak{X}}_{\underbrace{ij\ldots n\ldots n\ldots k}_{N}} \quad \forall\, \mathfrak{X}\in\mathfrak{F}.$$

Then $v^{\mathfrak{X}}_{ij\ldots k}$ are the components in $\mathfrak{X}$ of a tensor $\underset{\sim}{v}$ of order $N-2$, said to be formed by contraction of the tensor $\underset{\sim}{u}$.

The proof of this theorem will be left as an exercis

<u>Examples.</u> In what follows c is a scalar and $\underset{\sim}{u}, \underset{\sim}{v}$ tensors of appropriate positive order. We now illustrate the formation of a new tensor $\underset{\sim}{w}$ by means of outer multiplication and contraction.

Def. of $\underset{\sim}{w}$	Order of $\underset{\sim}{u}$	Order of $\underset{\sim}{v}$	Order of $\underset{\sim}{w}$	Absolute notation
$w = u_i v_i$	1	1	0	$w = \underset{\sim}{u} \cdot \underset{\sim}{v}$
$w_i = \varepsilon_{ijk} u_j v_k$	1	1	1	$\underset{\sim}{w} = \underset{\sim}{u} \wedge \underset{\sim}{v}$
$w_i = u_{ij} v_j$	2	1	1	$\underset{\sim}{w} = \underset{\sim}{u}\underset{\sim}{v}$
$w_{ij} = u_{ik} v_{kj}$	2	2	2	$\underset{\sim}{w} = \underset{\sim}{u}\underset{\sim}{v}$
$w_{ij} = u_i v_j$	1	1	2	$\underset{\sim}{w} = \underset{\sim}{u} \otimes \underset{\sim}{v}$
$w_{ijk} = u_{ij} v_k$	2	1	3	

Remark on suppression of frame label (explain,

If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are tensors of the same positive order, we define

$$\left.\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= u_{ij\ldots k} v_{ij\ldots k} \quad \text{(inner product)} \\ |\underset{\sim}{u}| &= \sqrt{\underset{\sim}{u} \cdot \underset{\sim}{u}} \quad \text{(norm)} \end{aligned}\right\} \quad (1.20$$

Note generalization of correspondg. vector notion

Raise issue of tensor division.

<u>Thm.</u> 1.4 (Quotient rule). Let 9 real numbers $\overset{X}{w}_{ij}$ be defined $\forall X \in \mathcal{F}$ and suppose for <u>every</u> unit vector $\underset{\sim}{u}$ the numbers $\overset{X}{v}_i$ defined by

$$\overset{X}{v}_i = \overset{X}{w}_{ij} \overset{X}{u}_j \quad \forall \; X \in \mathcal{F}$$

are the components in X of a vector $\underset{\sim}{v}$. Then $\overset{X}{w}_{ij}$ are the components of a second-order tensor $\underset{\sim}{w}$.

<u>Proof</u>. By hyp. and (1.19), if $[\underset{\sim}{A}] : X \to X'$, one has

$$\overset{X}{v}_i = A_{qi} \overset{X'}{v}_q = \overset{X}{w}_{ij} A_{qj} \overset{X'}{u}_q$$

"Multiply" by A_{pi} to obtain

$$A_{pi} A_{qi} \overset{X'}{v}_q = \overset{X}{w}_{ij} A_{pi} A_{qj} \overset{X'}{u}_q$$ or, since $A_{pi} A_{qi} = \delta_{pq}$,

$$\overset{X'}{v}_p = A_{pi} A_{qj} \overset{X}{w}_{ij} \overset{X'}{u}_q \qquad (1)$$

By hyp.,

$$\overset{X'}{v}_p = \overset{X'}{w}_{pq} \overset{X'}{u}_q \qquad (2)$$

$(1),(2) \Rightarrow (\overset{X'}{w}_{pq} - A_{pi} A_{qj} \overset{X}{w}_{ij}) \overset{X'}{u}_q = 0$ for <u>every</u> $\underset{\sim}{u} \ni |\underset{\sim}{u}| =$

Choose $\underset{\sim}{u} \ni \overset{X'}{u}_q = \delta_{qs}$ with s fixed. Thus,

$$\overset{X'}{w}_{ps} = A_{pi} A_{sj} \overset{X}{w}_{ij}$$

q.e.d.

Remark on generalizations of above quotient rule and other versions (see following exercise).

Exercise 3

(a) Prove Thm 1.3 on contraction of tensors.

(b) Prove the subsequent theorem. Let 3^N real numbers $\overset{\mathcal{X}}{t}_{ij\ldots k}$ (N subs., $N \geq 1$) be defined $\forall\, \mathcal{X} \in \mathcal{F}$ and suppose for every choice of N unit vectors $\underset{\sim}{u}, \underset{\sim}{v}, \ldots \underset{\sim}{w}$,

$$\overset{\mathcal{X}}{t}_{ij\ldots k}\, \overset{\mathcal{X}}{u}_i \overset{\mathcal{X}}{v}_j \ldots \overset{\mathcal{X}}{w}_k = c \quad \forall\, \mathcal{X} \in \mathcal{F},$$

where c is a real number (indept. of $\mathcal{X}$). Then $\overset{\mathcal{X}}{t}_{ij\ldots k}$ are the components in $\mathcal{X}$ of a tensor of order N.

Second-order tensors ("two-tensors")

Remark on important role of two-tensors in mechan (e.g. inertia tensor, deformation and strain tensors, stress tensors, etc)

Notation: Throughout what follows $\underset{\sim}{U}, \underset{\sim}{V}$, etc. two-tensors $\underset{\sim}{u}, \underset{\sim}{v}$, etc. are vectors, unless otherwise specified.

Recall from (1.18), (1.19) that if $[A]: X \to X'$

$$W^{X'}_{ij} = A_{ip} A_{jq} W^{X}_{pq}, \quad W^{X}_{ij} = A_{pi} A_{qj} W^{X'}_{pq} \quad (*)$$

Let $[\underset{\sim}{W}] = [W^{X}_{ij}]$, $[\underset{\sim}{W}'] = [W^{X'}_{ij}]$. Then (*) is equivalent to the matrix equations

$$[\underset{\sim}{W}'] = [\underset{\sim}{A}][\underset{\sim}{W}][\underset{\sim}{A}]^T, \quad [\underset{\sim}{W}] = [\underset{\sim}{A}]^T[\underset{\sim}{W}'][\underset{\sim}{A}] \quad (**)$$

Remark on obvious equivalence of first & second in (**)

Product of a two-tensor and a vector

If $\underset{\sim}{W}$ is a two-tensor and $\underset{\sim}{u}$ a vector, define

$$\underset{\sim}{v} = \underset{\sim}{W}\,\underset{\sim}{u} \iff v_i = W_{ij} u_j \qquad (1.21)$$

Mention distributivity of this multiplication
Refer to Thms. 1.2, 1.3 and remark on omission of frame labels. Set

$$[\underset{\sim}{u}] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad [\underset{\sim}{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad [\underset{\sim}{W}] = [W_{ij}]$$

Then (1.21) $\iff$

$$[\underset{\sim}{v}] = [\underset{\sim}{W}][\underset{\sim}{u}]$$

INSERT

Connection between two-tensors & linear vector funcs

Recall definition of a linear vector function:

A <u>linear vector function</u> $\underset{\sim}{L}$ is a mapping that associates with each vector $\underset{\sim}{u} \in E_3$ a vector $\underset{\sim}{v} = \underset{\sim}{L}(\underset{\sim}{u}) \in$ such that

$$(i)\ \underset{\sim}{L}(\underset{\sim}{u}_1 + \underset{\sim}{u}_2) = \underset{\sim}{L}(\underset{\sim}{u}_1) + \underset{\sim}{L}(\underset{\sim}{u}_2)\ , \quad (ii)\ \underset{\sim}{L}(c\underset{\sim}{u}) = c\underset{\sim}{L}(\underset{\sim}{u})$$

$\forall$ real number c.

<u>Thm.</u> 1.5. Let $\underset{\sim}{W}$ be a two-tensor and define a vector-valued function $\underset{\sim}{L}$ through

$$\underset{\sim}{v} = \underset{\sim}{L}(\underset{\sim}{u}) = \underset{\sim}{W}\underset{\sim}{u} \quad \forall\ \underset{\sim}{u} \in E_3 . \ (*)$$

Then $\underset{\sim}{L}$ is a <u>linear</u> vector function.

Conversely, given a linear vector function $\underset{\sim}{L}$, $\exists$ a unique two-tensor $\underset{\sim}{W} \ni$ (*) holds true.

<u>Proof.</u> First claim is trivial (explain).

<u>Re converse</u>. Let $\underset{\sim}{L}$ be given. Choose $\mathfrak{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathfrak{F}$

Let $W_{ji}^{\mathfrak{X}}$ be the j-th scalar comp. of $\underset{\sim}{L}(\underset{\sim}{e}_i)$ in $\mathfrak{X}$, whence

$$\underset{\sim}{L}(\underset{\sim}{e}_i) = W_{ji}^{\mathfrak{X}} \underset{\sim}{e}_j \quad (**)$$

Then from the linearity of $\underset{\sim}{L}$ one has $\forall\ \underset{\sim}{u} \in E_3$,

$$\underset{\sim}{v} = \underset{\sim}{L}(\underset{\sim}{u}) = \underset{\sim}{L}(u_i^{\mathfrak{X}} \underset{\sim}{e}_i) = u_i^{\mathfrak{X}} \underset{\sim}{L}(\underset{\sim}{e}_i) = u_i^{\mathfrak{X}} W_{ji}^{\mathfrak{X}} \underset{\sim}{e}_j = v_j^{\mathfrak{X}} \underset{\sim}{e}_j$$

Hence

$$V_j^{X} = W_{ji}^{X} u_i^{X} \quad \text{or} \quad V_i^{X} = W_{ij}^{X} u_j^{X} \quad \forall\ X \in \mathcal{F}, \forall\ \underset{\sim}{u} \in E_3$$

Now Thm. 1.4 (quotient rule) assures that W_{ij}^{X} are the compts. in X of a two-tensor $\underset{\sim}{W}$. Thus, by (1.21'),

$$\underset{\sim}{v} = \underset{\sim}{W}\,\underset{\sim}{u} \quad \forall\ \underset{\sim}{u} \in E_3$$

and (*)' holds. Uniqueness of $\underset{\sim}{W}$ is trivial. qed.

Remark on coord.-free def. of two-tensors as l.v.f. Components then defined by (1.21'). (eg. see Gurtin)

<u>Transpose of a two-tensor</u>

Trivially, if U_{ij}^{X} are the compts. in X of a two-tensor $\forall\ X \in \mathcal{F}$, so are $V_{ij}^{X} = U_{ji}^{X}$ (explain). Define,

$$\underset{\sim}{V} = \underset{\sim}{U}^{T} \Leftrightarrow V_{ij} = U_{ji} \tag{1.22}$$

Thus $[\underset{\sim}{V}] = [\underset{\sim}{U}]^{T}$. Clearly, $\underset{\sim}{U}$ is <u>symm.</u> iff $\underset{\sim}{U}^{T} = \underset{\sim}{U}$ and <u>skew-symm.</u> iff $\underset{\sim}{U}^{T} = -\underset{\sim}{U}$.

If $\underset{\sim}{U}$ is a two-tensor and $\underset{\sim}{m}, \underset{\sim}{n}$ are vectors, one has the "reciprocal relation":

$$\underset{\sim}{m} \cdot (\underset{\sim}{U}\,\underset{\sim}{n}) = (\underset{\sim}{U}^{T}\underset{\sim}{m}) \cdot \underset{\sim}{n} \tag{1.23}$$

since

$$m_i U_{ij} n_j = m_j U_{ji} n_i = U_{ji} m_j n_i$$

<u>Product of two second-order tensors</u>

If $\underset{\sim}{U}, \underset{\sim}{V}$ are two-tensors, define

$$\underset{\sim}{W} = \underset{\sim}{U}\,\underset{\sim}{V} \iff W_{ij} = U_{ik} V_{kj} \qquad (1.24)$$

Clearly, $\underset{\sim}{W}$ is a two-tensor. Product non-commutati

$$[\underset{\sim}{W}] = [\underset{\sim}{U}][\underset{\sim}{V}] \qquad \text{(elaborate)}$$

$\underset{\sim}{W} = \underset{\sim}{U}\,\underset{\sim}{V} \Rightarrow \underset{\sim}{W}^T = \underset{\sim}{V}^T \underset{\sim}{U}^T$, as follows from (1.24), (1.22)

If $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ are vectors and $\underset{\sim}{U}, \underset{\sim}{V}$ two-tensors,

$$\underset{\sim}{y} = \underset{\sim}{V}\underset{\sim}{x}, \quad \underset{\sim}{z} = \underset{\sim}{U}\underset{\sim}{y} \Rightarrow \underset{\sim}{z} = \underset{\sim}{W}\underset{\sim}{x}, \quad \underset{\sim}{W} = \underset{\sim}{U}\,\underset{\sim}{V}.$$

<u>Tensor (dyadic) product of two vectors</u>

If $\underset{\sim}{u}, \underset{\sim}{v}$ are vectors, define

$$\underset{\sim}{W} = \underset{\sim}{u} \otimes \underset{\sim}{v} \iff W_{ij} = u_i v_j \qquad (1.25)$$

Clearly, $\underset{\sim}{W}$ is a two-tensor and this product too is non-commutative; it <u>is</u> distributive.

INSERT

<u>Trace of a two-tensor</u>

$$c \doteq \operatorname{tr} \underset{\sim}{V} \iff c = V_{ii} \qquad (1.25')$$

Fundamental scalar invariants of a 2-tensor

$$\left.\begin{aligned}
I_1(\underset{\sim}{V}) &= V_{ii} = \operatorname{tr}\underset{\sim}{V} = V_{11}+V_{22}+V_{33}\\
I_2(\underset{\sim}{V}) &= \tfrac{1}{2}(V_{ii}V_{jj} - V_{ij}V_{ji}) = \tfrac{1}{2}[(\operatorname{tr}\underset{\sim}{V})^2 - \operatorname{tr}(\underset{\sim}{V}^2)]\\
&= \begin{vmatrix} V_{11} & V_{12}\\ V_{21} & V_{22}\end{vmatrix} + \begin{vmatrix} V_{22} & V_{23}\\ V_{32} & V_{33}\end{vmatrix} + \begin{vmatrix} V_{33} & V_{31}\\ V_{13} & V_{11}\end{vmatrix}\\
I_3(\underset{\sim}{V}) &= \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}V_{ip}V_{jq}V_{kr} = \det[V_{ij}] = \det\underset{\sim}{V}
\end{aligned}\right\}\ (1.2$$

The scalar (invariant) character of $I_i(\underset{\sim}{V})$ is immediate from Thms. 1.2, 1.3. Describe invariants in terms of $[V_{ij}^{X}]$.

A two-tensor $\underset{\sim}{V}$ is <u>nonsingular</u> if $\det\underset{\sim}{V} \neq 0$.

If $\underset{\sim}{U}, \underset{\sim}{V}$ are two-tensors, (1.20), (1.24) & Exercise 1 ⇒

$$\left.\begin{aligned}
\underset{\sim}{U}\cdot\underset{\sim}{V} = U_{ij}V_{ij} = \operatorname{tr}(\underset{\sim}{U}\underset{\sim}{V}^T),\ |\underset{\sim}{U}| = \sqrt{\operatorname{tr}(\underset{\sim}{U}\underset{\sim}{U}^T)},\\
\det(\underset{\sim}{U}\underset{\sim}{V}) = \det\underset{\sim}{U}\det\underset{\sim}{V}
\end{aligned}\right\}\ (1.27)$$

<u>Exercise</u> 4 (Inverse of a two-tensor).

(a) With the aid of (1.11)* and without recourse to matrix algebra, show that if $\underset{\sim}{U}$ is a nonsingular two-tensor, ∃ a unique two-tensor $\underset{\sim}{V} = \underset{\sim}{U}^{-1}$ ∋

---

* See (b) in Exercise 1.

$\underset{\sim}{V}\,\underset{\sim}{U} = \underset{\sim}{U}\,\underset{\sim}{V} = \underset{\sim}{1}$. Show further that

$$V_{ij} = (\underset{\sim}{U}^{-1})_{ij} = \frac{1}{2\det \underset{\sim}{U}}\, \varepsilon_{ipq}\, \varepsilon_{jrs}\, U_{rp} U_{sq} \qquad (1.28)$$

(b) Assuming $\underset{\sim}{U}, \underset{\sim}{V}$ to be nonsingular two-tensors and $\underset{\sim}{W} = \underset{\sim}{U}\,\underset{\sim}{V}$, show by means of (1.28) that

$$(\underset{\sim}{U}^{-1})^T = (\underset{\sim}{U}^T)^{-1} \equiv \underset{\sim}{U}^{-T}, \quad \underset{\sim}{W}^{-1} = \underset{\sim}{V}^{-1}\underset{\sim}{U}^{-1} \qquad (1.29)$$

Note that $[\underset{\sim}{V}] = [\underset{\sim}{U}]^{-1}$ in (a), so that (1.28) supplies a formula in index notation for the inverse of a nonsingular 3×3 matrix.

<u>Def</u>. A two-tensor $\underset{\sim}{Q}$ is <u>orthogonal</u> if $\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \underset{\sim}{1}$.

Thus,

$$\underset{\sim}{Q} \text{ orthog.} \Rightarrow \underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{1}, \quad \underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T, \quad \det \underset{\sim}{Q} = \pm 1$$

$\underset{\sim}{Q}$ is <u>proper orthogonal</u> if it is orthogonal and $\det \underset{\sim}{Q} = +1$

<u>Def</u>. Let $\underset{\sim}{V}$ be a two-tensor. A real number $\lambda = \lambda(\underset{\sim}{V})$ is a <u>principal value</u> of $\underset{\sim}{V}$ if $\exists$ a vector $\underset{\sim}{n} \ni$

$$\underset{\sim}{V}\,\underset{\sim}{n} = \lambda \underset{\sim}{n}, \quad \underset{\sim}{n}^2 = 1 \quad \text{or} \quad V_{ij} n_j = \lambda n_i, \quad n_i n_i = 1 \qquad (1.30)$$

If so, $\underset{\sim}{n}$ is a <u>princ. direction vector</u> of $\underset{\sim}{V}$ belonging to the p. value λ and $(\underset{\sim}{n}, -\underset{\sim}{n})$ determine a <u>principal axis</u> of $\underset{\sim}{V}$ belonging to λ.

<u>Remarks</u>. Interpret (1.30) geometrically in terms of l.v.f. generated by $\underset{\sim}{V}$. Raise existence and uniqueness issue.

<u>Thm. 1.6'</u>. Let $\underset{\sim}{V}$ be a two-tensor. Then λ is a p. value of $\underset{\sim}{V}$ iff it is a real root of the cubic equation

$$F(\lambda) \equiv \det(\underset{\sim}{V} - \lambda \underset{\sim}{1}) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0, \qquad (1.31)^{\S}$$

where $I_i = I_i(\underset{\sim}{V})$ are the scalar invariants of $\underset{\sim}{V}$. If λ is a p. value of $\underset{\sim}{V}$; the p. direction vectors belonging to λ are the solutions for $\underset{\sim}{n}$ of the equations

$$(V_{ij} - \lambda \delta_{ij}) n_j = 0, \quad n_i n_i = 1 \qquad (1.32)$$

<u>Proof</u>. By def., λ is p. value of $\underset{\sim}{V}$ iff $\exists$ real $n_i \ni$ (1.32) hold. For this it is nec. & suff. that

$$\det[V_{ij} - \lambda\delta_{ij}] \equiv \det[\underset{\sim}{V} - \lambda\underset{\sim}{1}] = 0 \quad (*)^{\S\S}$$

and (*) is easily confirmed to be equivalent to (1.31) with the aid of (1.7), (1.9), (1.10), (1.26).

§ Characteristic equation of $\underset{\sim}{V}$

§§ $(*) \Leftrightarrow \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}(V_{ip} - \lambda\delta_{ip})(V_{jq} - \lambda\delta_{jq})(V_{kr} - \lambda\delta_{kr}) = 0$

Remark. Clearly, (1.31) has at least one real root and at most three distinct real roots (explain and interpret).

Thm. 1.7. Let $\underset{\sim}{V}$ be a symmetric two-tensor. Then all three roots $(\lambda_1, \lambda_2, \lambda_3)$ of its characteristic equation (1.31) are real. $\exists$ at least three mutually $\perp$ p. axes of $\underset{\sim}{V}$ and at least one p. frame $X' = \{O; \underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3\} \in \mathcal{F}$. Also,

$$[V_{ij}^{X'}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{if } \underset{\sim}{V}\underset{\sim}{e}'_i = \lambda_i \underset{\sim}{e}'_i \text{ (no sum)} \tag{1.33}$$

Moreover, the following classification is exhaustive:

Case I. If $(\lambda_1, \lambda_2, \lambda_3)$ are distinct (i.e. all roots of (1.31) are simple), then $\exists$ a unique p. axis belonging to each λ the three p. axes being mutually $\perp$.

Case II. If, say, $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_*$ (i.e. (1.31) has a double and a simple root), then $\exists$ a unique p. axis belonging to λ_1 and every $\perp$ axis is a p. axis belonging to λ_*.

Case III. If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_*$ (i.e. (1.31) has a triple root), then every axis is a p. axis belonging to λ_* and $\underset{\sim}{V} = \lambda_* \underset{\sim}{1}$ so that $\underset{\sim}{V}$ is isotropic.

Lemma 1. If $\underset{\sim}{n}^{(1)}, \underset{\sim}{n}^{(2)}$ are p. direction vectors belonging to two distinct p. values of a symm. two-tensor, then $\underset{\sim}{n}^{(1)} \perp \underset{\sim}{n}^{(2)}$.

Proof. Let $\underset{\sim}{n}^{(1)}, \underset{\sim}{n}^{(2)}$ belong to p. values λ_1, λ_2 $(\lambda_1 \neq \lambda_2)$.

Then

$$\underset{\sim}{V}\underset{\sim}{n}^{(1)} = \lambda_1 \underset{\sim}{n}^{(1)}, \quad \underset{\sim}{V}\underset{\sim}{n}^{(2)} = \lambda_2 \underset{\sim}{n}^{(2)} \qquad (*)$$

By (*), (1.23) and since $\underset{\sim}{V} = \underset{\sim}{V}^T$,

$$\underset{\sim}{V}\underset{\sim}{n}^{(1)} \cdot \underset{\sim}{n}^{(2)} = \underset{\sim}{V}\underset{\sim}{n}^{(2)} \cdot \underset{\sim}{n}^{(1)}, \quad \lambda_1 \underset{\sim}{n}^{(1)} \cdot \underset{\sim}{n}^{(2)} = \lambda_2 \underset{\sim}{n}^{(2)} \cdot \underset{\sim}{n}^{(1)}$$

$$\Rightarrow (\lambda_1 - \lambda_2)\, \underset{\sim}{n}^{(1)} \cdot \underset{\sim}{n}^{(2)} = 0 \Rightarrow \underset{\sim}{n}^{(1)} \cdot \underset{\sim}{n}^{(2)} = 0 \quad \text{qed.}$$

Lemma 2. Let λ be a p. value of a symmetric two-tensor $\underset{\sim}{V}$ and $\underset{\sim}{n}$ a p. direction vector belonging to λ. For fixed k $(k = 1,2,3)$ choose $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F} \ni \underset{\sim}{e}_k = \underset{\sim}{n}$. Then $V_{ki}^X = V_{ik}^X = \lambda \delta_{ik}$.

Proof. Choose and fix k. By hyp.,

$$\underset{\sim}{V}\underset{\sim}{e}_k = \lambda \underset{\sim}{e}_k, \quad V_{ij}^X \delta_{jk} = \lambda \delta_{ik}, \quad V_{ik}^X = V_{ki}^X = \lambda \delta_{ik} \quad \text{qed.}$$

Proof of Thm. 1.7.

By Thm. 1.6, the p. values of $\underset{\sim}{V}$ are the real roots of (1.31) and if λ_j is such a root, the associated p. direct vectors $\underset{\sim}{n}^{(j)}$ are obtained by solving (1.32) for $\underset{\sim}{n}$ after replacing λ by λ_j.

We show first that all three roots of (1.31) are real if $\underline{V}$ is symm. Since I_i is real, one real root, say λ_1, is assured. Let $\underset{\sim}{n}^{(1)}$ be a p. dir. vector belonging to λ_1. Choose $X' = \{O; \underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3\} \in \mathcal{F} \ni \underset{\sim}{e}'_1 = \underset{\sim}{n}^{(1)}$. Write $V'_{ij} \equiv V_{ij}^{X'}$. By Lemma 2,

$$V'_{11} = \lambda_1, \quad V'_{12} = V'_{21} = V'_{13} = V'_{31} = 0 \qquad (a)$$

Hence (1.31) becomes

$$\det[V'_{ij} - \lambda\delta_{ij}] = \begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & V'_{22} - \lambda & V'_{23} \\ 0 & V'_{23} & V'_{33} - \lambda \end{vmatrix} = 0 \quad \text{or}$$

$$(\lambda_1 - \lambda)[\lambda^2 - (V'_{22} + V'_{33})\lambda + V'_{22}V'_{33} - (V'_{23})^2] = 0 \quad (b)$$

The solutions of (b) are:

$$\lambda = \lambda_1, \quad \lambda = \lambda_{2,3} = \frac{V'_{22} + V'_{33}}{2} \pm \sqrt{\frac{(V'_{22} - V'_{33})^2}{4} + (V'_{23})^2} \quad (c)$$

Thus all three λ_i are real.

Lemma 2 $\Rightarrow$ if $\exists$ a p. frame X' for $\underset{\sim}{V} \ni \underset{\sim}{e}'_i$ is a p. direction vector for λ_i, then (1.33) holds. The existence of at least one such frame is assured by what follows.

Re Case I: $\lambda_1, \lambda_2, \lambda_3$ distinct

Here, by Lemma 1, the p. axes are unique and mutually $\perp$ (unique since $\nexists$ more than one axis through $\odot$ that is $\perp$ to a given plane in E_3).

Re Case II: Say $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_*$

Choose $X' \ni \underset{\sim}{e}'_1 = \underset{\sim}{n}^{(1)}$. Note there are infinitely many such choices. For each such choice, from (c),

$$V'_{22} = V'_{33} = \lambda_* \;,\; V'_{23} = V'_{32} = 0 \qquad (d)$$

Now (a), (d) $\Rightarrow$

$$[V'_{ij}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_* & 0 \\ 0 & 0 & \lambda_* \end{bmatrix} \Rightarrow \underset{\sim}{V}\,\underset{\sim}{e}'_2 = \lambda_* \underset{\sim}{e}'_2 \,,\; \underset{\sim}{V}\,\underset{\sim}{e}'_3 = \lambda_* \underset{\sim}{e}'_3$$

Thus $\underset{\sim}{e}'_2$ and $\underset{\sim}{e}'_3$ are both p. direction vectors belonging to the double root λ_* and hence every axis $\perp$ $\underset{\sim}{e}'_1 = \underset{\sim}{n}^{(1)}$ is a p. axis belonging to λ_*. Uniqueness of p. axis belonging to λ_1 is clear from Lemma 1.

Case III: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_*$

Here, from (a), (d), $\exists\, \mathbb{X}' \in \mathcal{F} \ni V'_{ij} = \lambda_* \delta_{ij}$. Hence by Thm. 1.1,

$$\underset{\sim}{V} = \lambda_* \underset{\sim}{1} \qquad (\underset{\sim}{V} \text{ is isotropic two-tensor})$$

and thus

$$\underset{\sim}{V}\,\underset{\sim}{n} = \lambda_* \underset{\sim}{1}\,\underset{\sim}{n} = \lambda_* \underset{\sim}{n} \quad \forall \text{ unit vector } \underset{\sim}{n}$$

Every unit vector is a p. direction vector belonging to λ_*, every $\mathbb{X} \in \mathcal{F}$ is a p. frame for $\underset{\sim}{V}$./

If $\underset{\sim}{V}$ is a symm. two-tensor, $\mathbb{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ is a p. frame for $\underset{\sim}{V}$, and $(\lambda_1, \lambda_2, \lambda_3)$ are the corresponding p. values, one has from (1.25′), (1.26), Thm. 1.6,

$$\left.\begin{aligned} &\underset{\sim}{V} = \sum_{i=1}^{3} \lambda_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_i\,, \\ &I_1(\underset{\sim}{V}) = \lambda_1 + \lambda_2 + \lambda_3\,,\ I_2(\underset{\sim}{V}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\,,\ I_3(\underset{\sim}{V}) = \lambda_1\lambda_2\lambda_3 \end{aligned}\right\} (3.$$

Exercise 5.

(a) Let $\mathbb{X} \in \mathcal{F}$ and let $\underset{\sim}{V}$ be a two-tensor such that

(i) $[V^{\mathbb{X}}_{ij}] = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ (ii) $[V^{\mathbb{X}}_{ij}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

In each case find the p. values and all associated p. direction vectors. Use (1.34) to check on p. values and confirm the p. values by means of a suitable rotation of $\mathbb{X}$.

(b) If $\underset{\sim}{V}$ is a symm. two-tensor and I_i are its scalar invariants, show that $\underset{\sim}{V}$ satisfies the tensor equation

$$\underset{\sim}{W} = \underset{\sim}{F}(\underset{\sim}{V}) = -\underset{\sim}{V}^3 + I_1 \underset{\sim}{V}^2 - I_2 \underset{\sim}{V} + I_3 \underset{\sim}{1} = \underset{\sim}{0} \quad (*) \text{ (Caley-Hamilton)}$$

Compare (*) with the characteristic equation of $\underset{\sim}{V}$.

HINT: Use p. axes of $\underset{\sim}{V}$.

Thm. 1.8 (Extremum properties of p. values). Let $\underset{\sim}{V}$ be a symm two-tensor with p. values λ_i and assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then

$$\lambda_1 = \max_{\mathfrak{X} \in \mathfrak{F}} \overset{\mathfrak{X}}{V_{ii}}, \quad \lambda_3 = \min_{\mathfrak{X} \in \mathfrak{F}} \overset{\mathfrak{X}}{V_{ii}} \quad \text{(no sums)}$$

Omit proof. See discussion of stress tensor.

Def. A symm. two-tensor $\underset{\sim}{W}$ is positive definite if

$$\underset{\sim}{v} \cdot (\underset{\sim}{W}\underset{\sim}{v}) \equiv W_{ij} v_i v_j > 0 \quad \forall \text{ vector } \underset{\sim}{v} \neq \underset{\sim}{0},$$

i.e. if $[\overset{\mathfrak{X}}{W_{ij}}]$ is pos. def. $\forall\, \mathfrak{X} \in \mathfrak{F}$.

Thm. 1.9 A nec. and suff. condition that a symm. two-tensor be pos. def. is that all of its p. values be positive

Proof. Let $\underset{\sim}{W}$ be a symm. two-tensor with p. values λ_i. By Thm. 1.7 $\exists$ p. frame $\mathfrak{X}' \in \mathfrak{F}$ for $\underset{\sim}{W}$ and

$$W_{ij}^{\mathbb{X}} v_i^{\mathbb{X}} v_j^{\mathbb{X}} = \lambda_1 (v_1^{\mathbb{X}'})^2 + \lambda_2 (v_2^{\mathbb{X}'})^2 + \lambda_3 (v_3^{\mathbb{X}'})^2 \quad \forall\, \mathbb{X} \in \mathcal{F}$$

The desired conclusion is now immediate. Explain.

Skew-symmetric two-tensors

Let $\underset{\sim}{V}$ be a skew-symm. two-tensor. Recall that then

$$\underset{\sim}{V}^T = -\underset{\sim}{V} \iff V_{ji} = -V_{ij}$$

Hence

$$[V_{ij}] = \begin{bmatrix} 0 & V_{12} & V_{13} \\ -V_{12} & 0 & V_{23} \\ -V_{13} & -V_{23} & 0 \end{bmatrix}$$

Thus $\det \underset{\sim}{V} = 0$, so that every skew two-tensor is singular.

This suggests representability of $\underset{\sim}{V}$ in terms of a vector. Define

$$\underset{\sim}{w} = \operatorname{dual} \underset{\sim}{V} \iff w_i = \tfrac{1}{2} \varepsilon_{ijk} V_{kj} \tag{1.35}$$

One calls $\underset{\sim}{w}$ the "vector dual" of $\underset{\sim}{V}$. "Multiply" the second of (1.35) by $-\varepsilon_{ipq}$, use (1.9), and $V_{ji} = -V_{ij}$ to obtain

$$V_{ij} = \varepsilon_{ikj} w_k \quad \text{or} \quad [V_{ij}] = \begin{bmatrix} 0 & -w_3 & +w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{1.36}$$

If $\underset{\sim}{V}$ is a skew two-tensor, $\underset{\sim}{w} = \operatorname{dual} \underset{\sim}{V}$ and $\underset{\sim}{u}$ a vector, then

$$\underset{\sim}{q} = \underset{\sim}{V}\underset{\sim}{u} \iff \underset{\sim}{q} = \underset{\sim}{w} \wedge \underset{\sim}{u} \tag{1.37}$$

since by (1.36), (1.12),

$$\underset{\sim}{q} = \underset{\sim}{V}\underset{\sim}{u} \iff q_i = V_{ij} u_j \iff q_i = \varepsilon_{ikj} w_k u_j \iff \underset{\sim}{q} = \underset{\sim}{w} \wedge \underset{\sim}{u}.$$

Remark on def. of $\underset{\sim}{w}$ through $\underset{\sim}{V}\underset{\sim}{u} = \underset{\sim}{w} \wedge \underset{\sim}{u} \;\; \forall\, \underset{\sim}{u} \in E_3$

Decomposition theorems for second-order tensors

Thm. 1.10. Every two-tensor $\underset{\sim}{W}$ admits a unique decomposition $\ni$

$$\underset{\sim}{W} = \underset{\sim}{U} + \underset{\sim}{V}, \quad \underset{\sim}{U} = \underset{\sim}{U}^T, \quad \underset{\sim}{V} = -\underset{\sim}{V}^T \quad (*)$$

and this resolution is given by

$$\left.\begin{array}{c} \underset{\sim}{U} \equiv \operatorname{sym} \underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{W} + \underset{\sim}{W}^T), \quad \underset{\sim}{V} \equiv \operatorname{skw} \underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{W} - \underset{\sim}{W}^T) \\ \text{or} \quad W_{ij} = \frac{1}{2}(W_{ij} + W_{ji}), \quad V_{ij} = \frac{1}{2}(W_{ij} - W_{ji}) \end{array}\right\} \quad (1.38)$$

Proof. Assume first the desired decomposition $(*)$ exists. $(*) \Rightarrow$

$$\underset{\sim}{W} = \underset{\sim}{U} + \underset{\sim}{V} \quad (a) \qquad \underset{\sim}{W}^T = \underset{\sim}{U} - \underset{\sim}{V} \quad (b)$$

Add & subtract (a), (b) to arrive at (1.38). Hence if $\exists\ \underset{\sim}{U}, \underset{\sim}{V}$ satisfying $(*)$, they are unique and given by (1.38). Conversel $\underset{\sim}{U}, \underset{\sim}{V}$ defined by (1.38) clearly satisfy $(*)$. qed.

Notation

$$U_{ij} = \frac{1}{2}(W_{ij} + W_{ji}) \equiv W_{(ij)}, \quad V_{ij} = \frac{1}{2}(W_{ij} - W_{ji}) \equiv W_{[ij]} \quad (1.39)$$

As a prerequisite for another kind of decomposition of two-tensor we require the following result, which was mentioned earlier.

[illegible] 1.11 (Isotropic tensors of first and second order).

(a) If $\underset{\sim}{v}$ is an isotropic vector, then $\underset{\sim}{v} = \underset{\sim}{0}$ (null-vector)

(b) If $\underset{\sim}{V}$ is an isotropic skew two-tensor, then $\underset{\sim}{V} = \underset{\sim}{0}$

(c) If $\underset{\sim}{V}$ is an isotropic two-tensor, then $\underset{\sim}{V} = \alpha \underset{\sim}{1}$, where α is a real number.

Proof. Recall definition of an isotropic tensor $\underset{\sim}{W}$, i.e. $W^{\mathcal{X}}_{ij\ldots k}$ independent of $\mathcal{X}$ $\forall \mathcal{X} \in \mathcal{F}$.

Re (a). Let $\underset{\sim}{v}$ be an isotropic vector and suppose $\mathcal{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$. Write $v_i = v_i^{\mathcal{X}} = \underset{\sim}{v} \cdot \underset{\sim}{e}_i$. Consider:

$\mathcal{X}' = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_3, -\underset{\sim}{e}_2\}$ ... rot. through $\pi/2$ about $\underset{\sim}{e}_1$-axis

$\mathcal{X}'' = \{O; -\underset{\sim}{e}_3, \underset{\sim}{e}_2, \underset{\sim}{e}_1\}$ ... rot through $\pi/2$ about $\underset{\sim}{e}_2$-axis

By hyp. one has

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_3 \\ -v_2 \end{bmatrix} = \begin{bmatrix} -v_3 \\ v_2 \\ v_1 \end{bmatrix} \Rightarrow v_2 = v_3 = -v_3 = 0, \; v_1 = -v_3 = 0$$

Hence $\underset{\sim}{v} = \underset{\sim}{0}$.

Re (b). Let $\underset{\sim}{V}$ be an isotropic skew two-tensor. Let $\underset{\sim}{w} = \text{dual}\, \underset{\sim}{V}$. Thus $\underset{\sim}{w}$ is isotropic by (1.35). Hence $\underset{\sim}{w} =$ by (a). Therefore $\underset{\sim}{V} = \underset{\sim}{0}$ by (1.36).

Re (c). Let $\underset{\sim}{V}$ be an isotropic two-tensor. Then sym $\underset{\sim}{V}$ and skw $\underset{\sim}{V}$ must each be isotropic (explain). Hence skw $\underset{\sim}{V} = \underset{\sim}{0}$ by (b). So $\underset{\sim}{V} = \underset{\sim}{V}^T$, i.e. $\underset{\sim}{V}$ is symmetric. By Thm. 1.7 $\exists$ p. frame $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ for $\underset{\sim}{V} \ni$

$$[V_{ij}^{X}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \underset{\sim}{V}\underset{\sim}{e}_i = \lambda_i \underset{\sim}{e}_i \text{ (no sum)}$$

Let $X' = \{O; \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_1\}$. Clearly $X' \in \mathcal{F}$ and by (1.33),

$$[V_{ij}^{X'}] = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}.$$ Thus X' is also p. for.

Since $\underset{\sim}{V}$ is isotropic, $\lambda_1 = \lambda_2 = \lambda_3 \equiv \alpha$ and hence $\underset{\sim}{V} = \alpha \underset{\sim}{1}$ according to Case III in Theorem 1.7. qed.

<u>Thm.</u> 1.12. Every two-tensor $\underset{\sim}{W}$ admits a unique decomposition $\ni$

$$\underset{\sim}{W} = \underset{\sim}{U} + \underset{\sim}{V}, \quad \operatorname{tr}\underset{\sim}{U} = 0, \quad \underset{\sim}{V} \text{ is isotropic} \quad (*)$$

and this resolution is given by

$$\left.\begin{aligned} &\underset{\sim}{U} = \underset{\sim}{W} - \underset{\sim}{V}, \quad \underset{\sim}{V} = \tfrac{1}{3}(\operatorname{tr}\underset{\sim}{W})\underset{\sim}{1} \\ \text{or} \quad &U_{ij} = W_{ij} - V_{ij}, \quad V_{ij} = \tfrac{1}{3} W_{kk}\delta_{ij} \end{aligned}\right\} \quad (1.40)$$

Proof. Assume first the desired decomposition (*) exists.

(*) and Thm. 1.11 $\Rightarrow$

$$\underset{\sim}{U} = \underset{\sim}{W} - \underset{\sim}{V}, \quad \underset{\sim}{V} = \alpha \underset{\sim}{1}, \quad \operatorname{tr} \underset{\sim}{W} = \operatorname{tr} \underset{\sim}{V} = 3\alpha, \quad \underset{\sim}{V} = \tfrac{1}{3}(\operatorname{tr} \underset{\sim}{W})\underset{\sim}{1}$$

Hence if $\exists\ \underset{\sim}{U}, \underset{\sim}{V}$ satisfying (*), they are unique and given by (1.40). Conversely, $\underset{\sim}{U}, \underset{\sim}{V}$ defined by (1.40) clearly satisfy (*). qed.

Remark. One calls $\underset{\sim}{U}$ and $\underset{\sim}{V}$ above the deviatoric and the isotropic parts of the tensor $\underset{\sim}{W}$, respectively. Mention importance of Thm. 1.12 in plasticity and viscoelasticity theory.

In preparation for yet another decomposition of two-tensors we require the following auxiliary result

Lemma. Let $\underset{\sim}{S}$ be a pos. definite symm. two-tensor. Then $\exists$ a unique pos. definite symm. two-tensor $\underset{\sim}{U}$ $\ni$ $\underset{\sim}{U}^2 = \underset{\sim}{S}$. One calls $\underset{\sim}{U}$ the "square root of $\underset{\sim}{S}$" and writes $\underset{\sim}{U} = \sqrt{\underset{\sim}{S}}$.

Proof. By hyp. and Thms. 1.7, 1.9 $\exists$ a p. frame $X \in$

for $\underset{\sim}{S}$ and

$$[S_{ij}^{X}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \lambda_i > 0,$$

where λ_i are the p. values of $\underset{\sim}{S}$. Now <u>define</u> a two-tensor $\underset{\sim}{U}$ such that

$$[U_{ij}^{X}] = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}, \quad \sqrt{\lambda_i} > 0.$$

Clearly, $\underset{\sim}{U}^2 = \underset{\sim}{S}$ and $\underset{\sim}{U}$ is symm., pos. def. (explai

Re <u>uniqueness</u> of $\underset{\sim}{U} \equiv \sqrt{\underset{\sim}{S}}$ (Stephenson): To see that $\underset{\sim}{U}$ is unique it suffices to show that $\underset{\sim}{U}$ symm. pos. def. $\Rightarrow$ every p. frame of $\underset{\sim}{S} = \underset{\sim}{U}^2$ is also a p. frame of $\underset{\sim}{U}$. Thus sufficient to show that

$$\underset{\sim}{S}\underset{\sim}{n} \equiv \underset{\sim}{U}^2\underset{\sim}{n} = \mu\underset{\sim}{n}, \quad \underset{\sim}{n}^2 = 1, \quad \mu > 0 \qquad (i)$$

implies

$$\underset{\sim}{U}\underset{\sim}{n} = \sqrt{\mu}\,\underset{\sim}{n}. \qquad (ii)$$

To see this note that (i) $\Rightarrow$

$$(\underset{\sim}{U}^2-\mu \mathbf{1})\underset{\sim}{n} = (\underset{\sim}{U}+\sqrt{\mu}\mathbf{1})(\underset{\sim}{U}-\sqrt{\mu}\mathbf{1})\underset{\sim}{n} = \underset{\sim}{0} \quad (1)$$

Set

$$\underset{\sim}{v} = (\underset{\sim}{U}-\sqrt{\mu}\mathbf{1})\underset{\sim}{n}. \quad (2)$$

(1), (2) $\Rightarrow$

$$(\underset{\sim}{U}+\sqrt{\mu}\mathbf{1})\underset{\sim}{v} = \underset{\sim}{0}, \quad \underset{\sim}{U}\underset{\sim}{v} = -\sqrt{\mu}\underset{\sim}{v} \quad (3)$$

Suppose $\underset{\sim}{v} \neq \underset{\sim}{0}$. Then (3) $\Rightarrow$

$$\underset{\sim}{v}\cdot\underset{\sim}{U}\underset{\sim}{v} = -\sqrt{\mu}\,\underset{\sim}{v}\cdot\underset{\sim}{v} = -\sqrt{\mu}\,\underset{\sim}{v}^2 < 0,$$

which contradicts the pos. definiteness of $\underset{\sim}{U}$. Hence $\underset{\sim}{v} = \underset{\sim}{0}$ and thus (ii) follows from (2). /

<u>Thm</u>. 1.13 (Polar decomposition). Let $\underset{\sim}{W}$ be a nonsingular two-tensor. Then $\exists$ <u>unique</u> two-tensors $\underset{\sim}{U}, \underset{\sim}{V}, \underset{\sim}{Q}, \underset{\sim}{Q}'$ $\ni$

$$\underset{\sim}{W} = \underset{\sim}{Q}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{Q}'; \quad \underset{\sim}{Q}, \underset{\sim}{Q}' \text{ orthog.}, \ \underset{\sim}{U}, \underset{\sim}{V} \text{ symm. pos. def.}$$

and these two resolutions are given by

$$\underset{\sim}{U} = \sqrt{\underset{\sim}{W}^T\underset{\sim}{W}}, \quad \underset{\sim}{Q} = \underset{\sim}{W}\underset{\sim}{U}^{-1}, \quad \underset{\sim}{V} = \sqrt{\underset{\sim}{W}\underset{\sim}{W}^T}, \quad \underset{\sim}{Q}' = \underset{\sim}{V}^{-1}\underset{\sim}{W}. \quad (1.4$$

Further,

$$\underset{\sim}{Q} = \underset{\sim}{Q}' \ , \ \underset{\sim}{V} = \underset{\sim}{Q}\underset{\sim}{U}\underset{\sim}{Q}^T \ , \ \underset{\sim}{U} = \underset{\sim}{Q}^T\underset{\sim}{V}\underset{\sim}{Q} \ , \qquad (1.42)$$

and $\underset{\sim}{Q}$ is <u>proper</u> orthogonal if $\det \underset{\sim}{W} > 0$.

<u>Proof</u>. Clearly, (*) and $\underset{\sim}{Q} = \underset{\sim}{Q}'$ imply the last two of (1.42). We prove only existence & uniqueness of "right decomp." $\underset{\sim}{W} = \underset{\sim}{Q}\underset{\sim}{U}$. See Ex.6 for completion of proof. Thus assume first $\exists$ two-tensors $\underset{\sim}{Q}, \underset{\sim}{U} \ni$

$$\underset{\sim}{W} = \underset{\sim}{Q}\underset{\sim}{U} \ , \ \underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{1} \ , \ \underset{\sim}{U} = \underset{\sim}{U}^T , \ \underset{\sim}{U} \text{ pos. def.} \qquad (1)$$

Let

$$\underset{\sim}{S} = \underset{\sim}{W}^T\underset{\sim}{W} \quad \text{or} \quad S_{ij} = W_{ki}W_{kj} \qquad (2)$$

(1), (2) $\Rightarrow$

$$\underset{\sim}{S} = \underset{\sim}{U}^T\underset{\sim}{Q}^T\underset{\sim}{Q}\underset{\sim}{U} = \underset{\sim}{U}^T\underset{\sim}{U} = \underset{\sim}{U}^2 , \ \underset{\sim}{S} = \underset{\sim}{S}^T , \ \underset{\sim}{S} \text{ pos. def} \quad (3)^1$$

Hence from Lemma and (2),

$$\underset{\sim}{U} = \sqrt{\underset{\sim}{S}} = \sqrt{\underset{\sim}{W}^T\underset{\sim}{W}} \quad .(4)$$

and from first of (1),

$$\underset{\sim}{Q} = \underset{\sim}{W}\underset{\sim}{U}^{-1} \quad (5)^2$$

Consequently, <u>if</u> the first decomp. exists, it is <u>unique</u> with $\underset{\sim}{Q}$ and $\underset{\sim}{U}$ given by (1.41).

---

1) The pos. definiteness of $\underset{\sim}{S}$ follows from that of $\underset{\sim}{U}$ and the fact that $\underset{\sim}{S} = \underset{\sim}{U}^2$ (explain)

2) Note that $\underset{\sim}{U}$ is nonsingular since pos. def. (explain

We show next that $\underset{\sim}{Q}, \underset{\sim}{U}$ defined by (4), (5) actually supply the desired decomposition (1).

To see that (4) is meaningful (see Lemma) we must show $\underset{\sim}{S}$ defined by (2) is symm., pos. def. Now (2) $\Rightarrow \underset{\sim}{S} = \underset{\sim}{S}^T$. Also, by (2) and hyp.,

$$\det \underset{\sim}{S} = (\det \underset{\sim}{W})^2 > 0, \; \underset{\sim}{S} \text{ is nonsingular.}$$

Show further $\underset{\sim}{S}$ is pos. def.: choose $\underset{\sim}{x} \neq \underset{\sim}{0}$ & let

$$\underset{\sim}{y} = \underset{\sim}{W}\underset{\sim}{x} \quad \text{or} \quad y_i = W_{ij} x_j$$

Since $\underset{\sim}{x} \neq \underset{\sim}{0}$ & $\det \underset{\sim}{W} \neq 0$, one has $\underset{\sim}{y} \neq \underset{\sim}{0}$. Also, (2), $(1.23)^1 \Rightarrow$

$$\underset{\sim}{x} \cdot \underset{\sim}{S}\underset{\sim}{x} = S_{ij} x_i x_j = \underset{\sim}{x} \cdot (\underset{\sim}{W}^T \underset{\sim}{W}) \underset{\sim}{x} = \underbrace{\underset{\sim}{x} \cdot \underset{\sim}{W}^T}_{\underset{\sim}{n}} \underbrace{(\underset{\sim}{W}\underset{\sim}{x})}_{\underset{\sim}{m}} = \underset{\sim}{W}\underset{\sim}{x} \cdot \underset{\sim}{W}\underset{\sim}{x} = \underset{\sim}{y}^2 >$$

So $\underset{\sim}{S}$ is symm, pos. def. and thus $\underset{\sim}{U}$ symm, pos. def

To see that $\underset{\sim}{Q}$ is orthog., note from (5), (4) & $(1.29)^2$ that

$$\underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{W}\underset{\sim}{U}^{-1}\underset{\sim}{U}^{-T}\underset{\sim}{W}^T = \underset{\sim}{W}(\underset{\sim}{U}^T\underset{\sim}{U})^{-1}\underset{\sim}{W}^T = \underset{\sim}{W}(\underset{\sim}{U}^2)^{-1}\underset{\sim}{W}^T$$

$$= \underset{\sim}{W}(\underset{\sim}{W}^T\underset{\sim}{W})^{-1}\underset{\sim}{W}^T = \underset{\sim}{W}\underset{\sim}{W}^{-1}\underset{\sim}{W}^{-T}\underset{\sim}{W}^T = \underset{\sim}{1}$$

Hence $\underset{\sim}{Q}$ orthog. Also, (5) $\Rightarrow \underset{\sim}{W} = \underset{\sim}{Q}\underset{\sim}{U}$. Hence right decomposition exists, is unique, and $\underset{\sim}{U}, \underset{\sim}{Q}$ are given by first two of (1.41). Also,

$$\det \underset{\sim}{W} = \det \underset{\sim}{Q} \det \underset{\sim}{U}, \; \det \underset{\sim}{U} > 0 \;\; \text{(explain)}$$

Therefore $\det \underset{\sim}{Q} = +1$ ($\underset{\sim}{Q}$ proper orthog.) if $\det \underset{\sim}{W} > 0$

For remainder of proof see Exercise 6.

Exercise 6.

(a) Complete the proof of the polar decomp. thm. by first proving the existence and uniqueness of the left polar decomposition $\underset{\sim}{W} = \underset{\sim}{V}\,\underset{\sim}{Q}'$, thus arriving at $\underset{\sim}{V} = \sqrt{\underset{\sim}{W}\,\underset{\sim}{W}^T}$, $\underset{\sim}{Q}' = \underset{\sim}{V}^{-1}\underset{\sim}{W}$. Show next that $\underset{\sim}{U}$ and $\underset{\sim}{Q}\,\underset{\sim}{U}\,\underset{\sim}{Q}^T$ have same p. values. Use this fact and the uniqueness of the left decomposition to infer that $\underset{\sim}{V} = \underset{\sim}{Q}\,\underset{\sim}{U}\,\underset{\sim}{Q}^T$, $\underset{\sim}{Q}' = \underset{\sim}{Q}$.

(b) Note from (a) that $\underset{\sim}{U}, \underset{\sim}{V}$ have the same p. values. Assume $\det \underset{\sim}{W} > 0$ and let $X = \{0; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ be p. for $\underset{\sim}{U}$. Let $\underset{\sim}{e}_i' = \underset{\sim}{Q}\,\underset{\sim}{e}_i$. Show that $X' = \{0; \underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3'\} \in \mathcal{F}$ and is a p. frame for $\underset{\sim}{V}$.

Tensor fields, differentiation of tensor fields

Let $S \subset E_3$. A tensor field of order N on S is a an N-th order tensor-valued function of position defined on S (e.g. scalar, vector, two-tensor fields)

Smoothness of tensor fields. Notation.

R... region in E_3 (R open, closed, or neither)

Let $\underset{\sim}{v}$ be an N-th order tensor field on R.

(i) We say $\underset{\sim}{v}$ is continuous on R & write $\underset{\sim}{v} \in \mathcal{C}(R)$ if the functions $v^{\mathfrak{X}}_{ij\ldots k}$ (N subs) are cont. on R $\forall \mathfrak{X} \in \mathfrak{F}$.

(ii) We say $\underset{\sim}{v}$ is M times continuously differentiable on R (M… pos. integer) and write $\underset{\sim}{v} \in \mathcal{C}^M(R)$ if $\underset{\sim}{v} \in \mathcal{C}(R)$, the partial derivatives

$$\frac{\partial^m v^{\mathfrak{X}}_{ij\ldots k}}{\partial x_p \partial x_q \ldots \partial x_r} \equiv v^{\mathfrak{X}}_{\underbrace{ij\ldots k}_{N},\underbrace{pq\ldots r}_{m}} \quad (m = 1, 2, \cdots M),$$

(where x_p are the cartesian coordinates in $\mathfrak{X}$) exist on $\mathring{R}$ $\forall\ \mathfrak{X} \in \mathfrak{F}$ and there coincide with functions cont. on R.

<u>Remarks</u>. Recall that $\mathring{R}$ is the interior of R, $\mathring{R} = R$ if R is a domain. Note that cont. and differentiability need to be verified only for single $\mathfrak{X}$.

<u>Thm</u>. 1.14. Let R be a domain and $\underset{\sim}{v} \in \mathcal{C}^1(R)$ a tensor field of order N. Define

$$w^{\mathfrak{X}}_{\underbrace{ij,\ldots ks}_{N+1 \text{ subs.}}}(\underset{\sim}{x}) = \frac{\partial}{\partial x_s}\{v^{\mathfrak{X}}_{\underbrace{ij\ldots k}_{N \text{ subs.}}}(\underset{\sim}{x})\} \quad \forall\, \underset{\sim}{x} \in R, \quad \forall \mathfrak{X} \in \mathfrak{F}.$$

where x_s are the cartesian coords. in $\mathfrak{X}$ of points in R. Then $w^{\mathfrak{X}}_{ij\ldots ks}$ are the components in $\mathfrak{X}$ of a tensor field $\underset{\sim}{w}$ of order N+1 and $\underset{\sim}{w} \in \mathcal{C}(R)$.

Proof. Consider $\underline{X} = \{O; \underline{e}_1, \underline{e}_2, \underline{e}_3\} \in \mathcal{F}$, $\underline{X}' = \{O; \underline{e}'_1, \underline{e}'_2, \underline{e}'_3\} \in$ with $[\underline{A}] : \underline{X} \to \underline{X}'$. If $\underline{x}$ is the position vector of points in R one has because of (1.17),

$$\underline{x} = \underline{e}_i x_i = \underline{e}'_i x'_i \ , \quad x_\ell = A_{n\ell} x'_n \qquad (1)$$

By hyp. and (1.18),

$$w^{\underline{X}'}_{pq\ldots rs} = \frac{\partial}{\partial x'_s} v^{\underline{X}'}_{pq\ldots r} = \frac{\partial}{\partial x'_s}\left(A_{pi} A_{qj} \ldots A_{rk} v^{\underline{X}}_{ij\ldots k}\right) \quad (2)$$

From (1), (2) and the chain rule follows

$$w^{\underline{X}'}_{pq\ldots rs} = A_{pi} A_{qj} \ldots A_{rk} \underbrace{\frac{\partial}{\partial x_\ell} v^{\underline{X}}_{ij\ldots k}}_{w^{\underline{X}}_{ij\ldots k\ell}} \frac{\partial x_\ell}{\partial x'_s} \quad \text{and} \quad \frac{\partial x_\ell}{\partial x'_s} = A_{s\ell}$$

Therefore,

$$w^{\underline{X}'}_{pq\ldots rs} = A_{pi} A_{qj} \ldots A_{rk} A_{s\ell}\, w^{\underline{X}}_{ij\ldots k\ell} \quad /$$

Relation to vector calculus

Let D be a domain. Let φ and $\underline{v}$ be, respectively, a suitably smooth scalar and vector field on D.. Define the following fields on D:

$$\left.\begin{aligned}
&\underset{\sim}{w} = \operatorname{grad}\varphi \equiv \nabla\varphi \iff w_i = \varphi_{,i}\\
&\psi = \operatorname{div}\underset{\sim}{v} \equiv \nabla\cdot\underset{\sim}{v} \iff \psi = v_{i,i}\\
&\underset{\sim}{w} = \operatorname{curl}\underset{\sim}{v} \equiv \nabla\wedge\underset{\sim}{v} \iff w_i = \varepsilon_{ijk} v_{k,j}\\
&\psi = \nabla^2\varphi \iff \psi = \varphi_{,ii}\\
&\underset{\sim}{w} = \nabla^2\underset{\sim}{v} \iff w_i = v_{i,jj}
\end{aligned}\right\} \quad (1.43)$$

Note on the basis of Thms. 1.14, 1.2, 1.3 that ψ is a scalar field and $\underset{\sim}{w}$ a vector field on D. Explain.

If $\varphi \in \mathcal{C}^2(D)$, $\psi \in \mathcal{C}^2(D)$ are scalar fields and $\underset{\sim}{v} \in \mathcal{C}^2(D)$ is a vector field, one has on D:

$$\left.\begin{aligned}
&\nabla\wedge\nabla\varphi = \underset{\sim}{0}, \quad \nabla\cdot(\nabla\wedge\underset{\sim}{v}) = 0, \quad \nabla\cdot\nabla\varphi = \nabla^2\varphi\\
&\nabla\wedge(\nabla\wedge\underset{\sim}{v}) = \nabla(\nabla\cdot\underset{\sim}{v}) - \nabla^2\underset{\sim}{v}\\
&\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi \qquad \checkmark\\
&\nabla\cdot(\varphi\underset{\sim}{v}) = \varphi(\nabla\cdot\underset{\sim}{v}) + \underset{\sim}{v}\cdot(\nabla\varphi) \qquad \checkmark\\
&\nabla\wedge(\varphi\underset{\sim}{v}) = \varphi(\nabla\wedge\underset{\sim}{v}) + (\nabla\varphi)\wedge\underset{\sim}{v} \qquad \checkmark\\
&\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2(\nabla\varphi)\cdot(\nabla\psi)
\end{aligned}\right\} \quad (1.44)$$

See Exercise 7. Note: The three identities marked ✓ are true if $\psi, \varphi \in \mathcal{C}^1(D)$, $\underset{\sim}{v} \in \mathcal{C}^1(D)$.

Gradient of a tensor field

Let $\underset{\sim}{T} \in C^1(D)$ be a tensor field of order $N \geq 1$. Define

$$\underset{\sim}{W} = \nabla \underset{\sim}{T} \Leftrightarrow W_{ij\ldots kl} = T_{\underbrace{ij\ldots k}_{N \text{ subs.}},l} \qquad (1.45)$$

Thus $\underset{\sim}{W}$ is a tensor field of order $N+1$. In particular, if $\underset{\sim}{v} \in C^1(D)$ is a vector field, then $\nabla \underset{\sim}{v}$ is a 2-tensor field.

Divergence, curl, and Laplacian of a two-tensor field

If $\underset{\sim}{W}$ is a suitably smooth two-tensor field on D, define the following fields on D:

$$\left.\begin{aligned}
\underset{\sim}{v} &= \operatorname{div} \underset{\sim}{W} \equiv \nabla \cdot \underset{\sim}{W} \Leftrightarrow v_i = W_{ij,j} \\
\underset{\sim}{U} &= \operatorname{curl} \underset{\sim}{W} \equiv \nabla \wedge \underset{\sim}{W} \Leftrightarrow U_{ij} = \varepsilon_{ipq} W_{jq,p} \\
\underset{\sim}{U} &= \nabla^2 \underset{\sim}{W} \Leftrightarrow U_{ij} = W_{ij,kk}
\end{aligned}\right\} \quad (1.46)$$

Note that $\underset{\sim}{v}$ is a vector field and $\underset{\sim}{U}$ a two-tensor field.

If $\underset{\sim}{v} \in C^2(D)$ is a vector field and $\underset{\sim}{W} \in C^2(D)$ a two-tensor field, one has on D:

$$\left.\begin{aligned}
&\nabla\wedge\nabla\underset{\sim}{v}=\underset{\sim}{0}\ ,\ \nabla\cdot(\nabla\wedge\underset{\sim}{W})^T=\underset{\sim}{0}\\
&\nabla\wedge(\nabla\underset{\sim}{v})^T=\nabla(\nabla\wedge\underset{\sim}{v})\ ,\ \nabla\cdot(\nabla\wedge\underset{\sim}{W})=\nabla\wedge(\nabla\cdot\underset{\sim}{W}^T)\\
&\operatorname{tr}\nabla\wedge\underset{\sim}{W}=0 \text{ if } \underset{\sim}{W}=\underset{\sim}{W}^T,\ \{\nabla\wedge(\nabla\wedge\underset{\sim}{W})\}^T=\nabla\wedge(\nabla\wedge\underset{\sim}{W}^T)\\
&\nabla\cdot(\underset{\sim}{W}^T\underset{\sim}{v})=\underset{\sim}{v}\cdot(\nabla\cdot\underset{\sim}{W})+\underset{\sim}{W}\cdot\nabla\underset{\sim}{v}\\
&\nabla\wedge(\underset{\sim}{W}^T\underset{\sim}{a})=(\nabla\wedge\underset{\sim}{W})\underset{\sim}{a} \text{ if } \underset{\sim}{a} \text{ is a const. vector}
\end{aligned}\right\}(1.4$$

<u>Exercise</u> 7: Use indicial notation to prove (1.44), (1.4

<u>Thm</u>. 1.15 (Divergence theorem[1]). Let R be a bounded <u>regular</u> region. Let $\underset{\sim}{v}\in\mathcal{C}(\overline{R})\cap\mathcal{C}^1(R)$ be a vector field and $\nabla\cdot\underset{\sim}{v}\in\mathcal{C}(\overline{R})$. Then,

$$\int_{\partial R}\underset{\sim}{v}\cdot\underset{\sim}{n}\,dA=\int_R\nabla\cdot\underset{\sim}{v}\,dV \text{ or } \int_{\partial R}v_j n_j\,dA=\int_R v_{j,j}\,dV,\quad (*)$$

where $\underset{\sim}{n}$ is the outer unit normal vector of ∂R. Further, the indicial version of (*) is valid if v_i is merely a triplet of real-valued functions of the required smoothness.

<u>Remarks</u>

Recall def. of "bounded regular region". Give examples. Note that hyp. $\nabla\cdot\underset{\sim}{v}\in\mathcal{C}(\overline{R})$ is superfluous if $\underset{\sim}{v}\in\mathcal{C}^1(\overline{R})$;

[1] Green, Gauss, Ostrogradskii (?)

otherwise volume integrals may be improper and need not exist. Motivate need for weak smoothness assumptions (uniqueness thms.)

For proof of much less general versions of Thm. 1.1[5] (R convex) see calculus books (eg. Courant). The thm. as stated here follows from strongest form of divergence thm. proved by Kellogg*.

Corollary (Tensorial analogue of divergence thm.) Let R be a bounded regular region. Let $\underset{\sim}{W} \in \mathcal{C}(\bar{R}) \cap \mathcal{C}^1(R$ be a two-tensor field and $\nabla \cdot \underset{\sim}{W} \in \mathcal{C}(\bar{R})$. Then,

$$\int_{\partial R} \underset{\sim}{W}\, \underset{\sim}{n}\, dA = \int_R \nabla \cdot \underset{\sim}{W}\, dV \quad \text{or} \quad \int_{\partial R} W_{ij} n_j \, dA = \int_R W_{ij,j}\, dV,$$

where $\underset{\sim}{n}$ is the outer unit normal vector of ∂R.

Proof. Apply indicial version of Thm. 1.15 to the triplet of real-valued functions W_{ij} (i fixed).

---

* Kellogg's def. restricts ∂R to a single "closed regular surface" (unnecessary restriction, readily removed)

Remark on extension of Thm. 1.15 and its Corollary to unbounded regions

Thm. 1.16 (Stokes). Let D be a domain, $\underset{\sim}{v}$ a vector field on D, and $\overset{\circ}{\underset{\sim}{x}}$ a point in D.

(a) If $\phi \in C^{N+1}(D)$ $(N \geq 1)$ is a scalar field $\ni$

$$\underset{\sim}{v} = \nabla \phi \text{ on } D, \quad (i)$$

then

$$\underset{\sim}{v} \in C^{N}(D), \; \nabla \wedge \underset{\sim}{v} = \underset{\sim}{0} \text{ or } v_{i,j} = v_{j,i} = 0 \text{ on } D \quad (ii)$$

and the line integral

$$\int_{\overset{\circ}{\underset{\sim}{x}}}^{\underset{\sim}{x}} \underset{\sim}{v}(\underset{\sim}{\xi}) \cdot d\underset{\sim}{\xi} \equiv \int_{\overset{\circ}{\underset{\sim}{x}}}^{\underset{\sim}{x}} v_i(\underset{\sim}{\xi})\, d\xi_i \quad (iii)$$

is path-independent $\forall\, \underset{\sim}{x} \in D$ and $\forall$ regular path $C \subset D$ joining $\overset{\circ}{\underset{\sim}{x}}$ to $\underset{\sim}{x}$.

(b) Conversely, if D is <u>simply connected</u> and $\underset{\sim}{v}$ satisfies (ii), then the line integral (iii) is path-independent and $\exists$ a scalar field $\phi \in C^{N+1}(D)$, unique up to an arbitrary additive constant, $\ni$ (i) holds; further such a ϕ is supplied by

$$\phi(\underset{\sim}{x}) = \int_{\underset{\sim}{x}^{0}}^{\underset{\sim}{x}} \underline{v}(\underset{\sim}{\xi}) \cdot d\underset{\sim}{\xi} \quad \forall \; \underset{\sim}{x} \in D .$$

Remarks

Recall def. of simple connectivity. Note that

$$v_{i,j} = v_{j,i} \text{ on } D \Leftrightarrow \varepsilon_{ijk} v_{k,j} = 0 \text{ on } D$$

For proofs of Stokes' theorem see calculus books (e.g. Courant II, Apostol). Here D need be neither bounded nor a regular region. Further, the conclusion in this theorem remains valid also if v_1, v_2, v_3 are merely three scalar fields of the requisite smoothness.

Thm. 1.17 (A tensorial analogue of Stokes' theorem[1]).

Assume D is a simply connected domain.

(a) Let $\underset{\sim}{T} \in \mathcal{C}^N(D)$ $(N \geq 1)$ be a two-tensor field $\ni$

$$\nabla \wedge \underset{\sim}{T} = \underset{\sim}{0} \text{ on } D. \quad (i)$$

Then $\exists$ a vector field $\underset{\sim}{v} \in \mathcal{C}^{N+1}(D) \ni$

$$\underset{\sim}{T} = \nabla \underset{\sim}{v} \text{ on } D. \quad (ii)$$

(b) Let $\underset{\sim}{T}$ satisfy the hyp. in (a); in addition, let

$$\operatorname{tr} \underset{\sim}{T} = 0 \text{ on } D \quad (iii)$$

Then $\exists$ a two-tensor field $\underset{\sim}{W} \in \mathcal{C}^{N+1}(D) \ni$

$$\underset{\sim}{W}^T = -\underset{\sim}{W}, \quad \underset{\sim}{T} = \nabla \wedge \underset{\sim}{W} \text{ on } D. \quad (iv)$$

Proof.

Re (a). Recall from (1.47) (Exercise 7) that

$$\nabla \wedge (\underset{\sim}{T}^T \underset{\sim}{a}) = (\nabla \wedge \underset{\sim}{T})\underset{\sim}{a} \text{ on } D \text{ if } \underset{\sim}{a} = \text{const. vector}. \quad (1,$$

Choose $\mathfrak{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ and let

$$\underset{\sim}{t}_k = \underset{\sim}{T}^T \underset{\sim}{e}_k \text{ on } D. \quad (2)$$

[1] Actually what follows is an analogue of part (b) in Thm. 1.16.

(1), (2), (i) $\Rightarrow$

$$\nabla \wedge \underset{\sim}{t}_k = (\nabla \wedge \underset{\sim}{T})\underset{\sim}{e}_k = \underset{\sim}{0} \text{ on } D$$

Hence by (b) in Thm 1.16 (Stokes), $\exists\, \phi_k \in C^{N+1}(D) \ni$

$$\underset{\sim}{t}_k = \underset{\sim}{T}^T \underset{\sim}{e}_k = \nabla \phi_k \text{ on } D$$

$$\text{or } \overset{\mathrm{X}}{T_{ji}}\, \delta_{kj} = \phi_{k,i} \text{ or } \overset{\mathrm{X}}{T_{ki}} = \phi_{k,i} \text{ on } D.$$

Define $\underset{\sim}{v} = \underset{\sim}{e}_k \phi_k$. Then $\underset{\sim}{v} \in C^{N+1}(D)$ & $\overset{\mathrm{X}}{T_{ki}} = \overset{\mathrm{X}}{v_{k,i}}$.

Thus $\underset{\sim}{T} = \nabla \underset{\sim}{v}$ on D, so that (ii) holds.

Re (b). Let $\underset{\sim}{v}$ be the vector field established in (

Then (ii), (iii) $\Rightarrow$

$$\operatorname{tr} \underset{\sim}{T} = v_{k,k} = \nabla \cdot \underset{\sim}{v} = 0 \text{ on } D. \quad (3)$$

Define a two-tensor field $\underset{\sim}{W} \in C^{N+1}(D)$ through

$$W_{jq} = \varepsilon_{jqk} v_k \text{ on } D. \quad (4)$$

Then $\underset{\sim}{W}^T = -\underset{\sim}{W}$ and (1.46)[1], (4), (3) $\Rightarrow$

---

[1] Def. of curl of a two-tensor field.

$$(\nabla \wedge \underset{\sim}{W})_{ij} = \varepsilon_{ipq} W_{jq,p} = \varepsilon_{ipq} \varepsilon_{jqk} v_{k,p}$$

$$= -\varepsilon_{qip} \varepsilon_{qjk} v_{k,p} = -(\delta_{ij}\delta_{pk} - \delta_{ik}\delta_{pj}) v_{k,p}$$

$$= -\underbrace{\delta_{ij} v_{k,k}}_{\text{zero by (3)}} + v_{i,j} = v_{i,j}$$

whence $\nabla \wedge \underset{\sim}{W} = \nabla \underset{\sim}{v}$, so that

$$\nabla \wedge \underset{\sim}{W} = \nabla \underset{\sim}{v} = I \text{ on } D.$$

2. Kinematics of motions and deformations of a continuous medium

Objective: Study of purely geom. aspects of motion of an arb. continuous medium.

Introduction. A motion of a "body" (continuous medium) relative to a given $E_3 = E$ is specified if we know the position in the reference space of all material points (particles) of body at every instant during the motion.