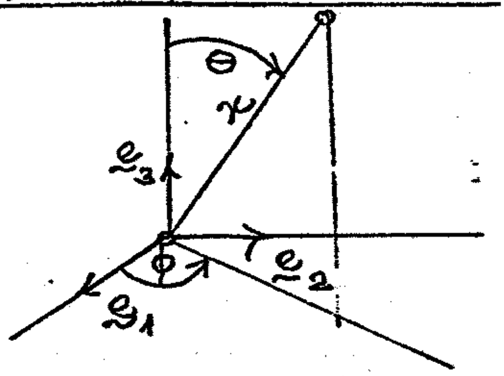
10. Problems of polar and rotational symmetry in elastosta



The problem of polar symmetry

We seek

$$\mathcal{S} = [\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\sigma}] \in \mathfrak{E}(\mu, \nu, \mathbf{f}, E - \{\mathbf{0}\}), \quad \mu > 0, -1 < \nu < 1/2 \quad (10.1$$

subject to the requirement of polar symmetry about $\mathbf{0}$, i.e

$$u_r = u(r), \quad u_\theta = u_\phi = 0 \quad (0 < r < \infty), \qquad (10$$

where (r, θ, ϕ) are standard spherical coords. From (10.2) together with the displt-strain rels. in spherical coords. one has

$$\left.\begin{aligned} &\gamma_{rr} = \frac{du}{dr} \equiv u', \quad \gamma_{\theta\theta} = \gamma_{\phi\phi} = \frac{u}{r}, \quad \gamma_{r\theta} = \gamma_{\theta\phi} = \gamma_{\phi r} = 0, \\ &\vartheta = \nabla \cdot \mathbf{u} = u' + \frac{2u}{r}, \quad \mathbf{w} = \tfrac{1}{2}\nabla \wedge \mathbf{u} = \mathbf{0} \end{aligned}\right\} (1$$

From (10.3) and the stress-strain rels.,

$$\sigma_{rr} = \frac{2\mu}{1-2\nu}\left[(1-\nu)u' + 2\nu\frac{u}{r}\right], \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2\mu}{1-2\nu}\left(\nu u' + \frac{u}{r}\right),$$

$$\sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{\phi r} = 0$$

Substitution from (12.4) into the stress eqs of equil. in spherical coords. leads to

$$u'' + \frac{2u'}{r} - \frac{2u}{r^2} = -\frac{1-2\nu}{2(1-\nu)\mu} f, \quad f = f_r, \ f_\theta = f_\phi = 0 \quad (10.5)$$

Thus f is a central field with $f_r = f(r)$ $0 < r < \infty$.
We assume $f \in C([0,\infty))$.
The displt. equ. of equilibrium in (10.5) is an ord. diff eq. of Euler's type and is reducible to an eq. with const. coeffs. by the transf. $t = \log r$. Alternatively, note that (10.5) may be written as

$$\frac{d}{dr}\left\{\frac{1}{r^2}\frac{d}{dr}\left[r^2 u(r)\right]\right\} = -\frac{1-2\nu}{2(1-\nu)\mu} f(r) \quad (0<r<\infty) \quad (10$$

The complete sol. of (10.6) admits the representation

$$u(r) = c_1 u^{(1)}(r) + c_2 u^{(2)}(r) + F(r) \quad (0<r<\infty)$$

$$u^{(1)}(r) = r, \quad u^{(2)}(r) = r^{-2}, \quad F(r) = -\frac{1-2\nu}{2(1-\nu)\mu r^2}\int_0^r \xi^2 \int_0^\xi f(\eta)\, d\eta\, d$$

c_1, c_2 arb. const.

From here on we assume

$$\underset{\sim}{f} = \underset{\sim}{0} \text{ on } E \quad (\text{no body forces}). \tag{10.8}$$

For convenience set

$$k_1 = \frac{2\mu(1+\nu)c_1}{1-2\nu}, \quad k_2 = -2\mu c_2. \tag{10.9}$$

From (10.2) to (10.9) follows easily

$$\left.\begin{aligned} \mathcal{S}:\ u_r = u(r) = \frac{(1-2\nu)k_1}{2\mu(1+\nu)}\,r - \frac{k_2}{2\mu r^2}, \quad u_\theta = u_\phi = 0 \\ \sigma_{rr} = k_1 + \frac{2k_2}{r^3}, \ \sigma_{\theta\theta} = \sigma_{\phi\phi} = k_1 - \frac{k_2}{r^3}, \ \sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{\phi r} = 0 \end{aligned}\right\} \tag{10.10}$$

This is the complete solution of the elastostatic field eqs. for polar symm. about O in absence of body forces. One confirms readily with the aid of Thm. 8.9 that $\mathcal{S}$ admits the following representation in terms of the PN-potentials.

$$\left.\begin{aligned} \mathcal{S}(\underset{\sim}{x}) = k_1 \overset{(1)}{\mathcal{S}}(\underset{\sim}{x}) + k_2 \overset{(2)}{\mathcal{S}}(\underset{\sim}{x}) \quad \forall\, \underset{\sim}{x} \in E - \{\underset{\sim}{0}\} \\ \overset{(1)}{\mathcal{S}}:\ \varphi = 0,\ \underset{\sim}{\psi} = -\frac{\underset{\sim}{x}}{2(1+\nu)}; \quad \overset{(2)}{\mathcal{S}}:\ \varphi = \frac{1}{r},\ \underset{\sim}{\psi} = \underset{\sim}{0} \end{aligned}\right\} \tag{10.11}$$

~~Henceforth adopt the notation~~

$\cancel{s=[\underset{\sim}{u},\underset{\sim}{E},\underset{\sim}{S}]\in\mathcal{E}(\mu,\nu;R)} \Leftrightarrow \cancel{s\in\mathcal{E}(\mu,\nu,\underset{\sim}{f},R),\ \underset{\sim}{f}=\underset{\sim}{0}\text{ on }R}$

Clearly,

$s^{(1)}\in\mathcal{E}(\mu,\nu;E)$, $\underset{\sim}{S}^{(1)}=\underset{\sim}{1}$ (const. isotropic stress field

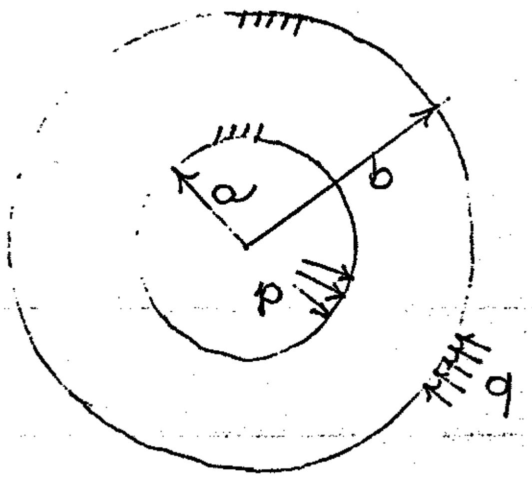
$s^{(2)}\in\mathcal{E}(\mu,\nu;E-\{\underset{\sim}{0}\})$ is a basic singular elastostat

state corresponding to a "center of compression

at $\underset{\sim}{x}=\underset{\sim}{0}$". See later.

<u>The spherical shell under uniform normal loads</u>

(Lamé's problem)



$R=\{\underset{\sim}{x}\,|\,a<|\underset{\sim}{x}|<b\}$, $0<a<b$

p, q ... constants

no body forces

<u>Formulation</u>: Find

$$s=[\underset{\sim}{u},\underset{\sim}{E},\underset{\sim}{S}]\in\mathcal{E}(\mu,\nu;\overline{R}) \qquad (\mu>0, -1<\nu<\tfrac{1}{2})$$

subject to the boundary conditions

$$\left.\begin{aligned} \sigma_{rr} &= -p\,, \quad \sigma_{r\theta} = \sigma_{r\phi} = 0 \quad \text{at } r = a \\ \sigma_{rr} &= -q\,, \quad \sigma_{r\theta} = \sigma_{r\phi} = 0 \quad \text{at } r = b \end{aligned}\right\} \quad (10$$

Clearly, S is given by (10.10) provided k_1, k_2 satisfy

$$k_1 + \frac{2k_2}{a^3} = -p \quad , \quad k_1 + \frac{2k_2}{b^3} = -q$$

from which

$$k_1 = \frac{pa^3 - qb^3}{b^3 - a^3} \quad , \quad k_2 = \frac{(q-p)a^3 b^3}{2(b^3 - a^3)} \qquad (10.$$

(10.11), (10.13) $\Longrightarrow$

$$\left.\begin{aligned} \sigma_{rr} &= \frac{pa^3(r^3 - b^3) + qb^3(a^3 - r^3)}{r^3(b^3 - a^3)}\,, \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \frac{pa^3(2r^3 + b^3) - qb^3(2r^3 + a^3)}{2r^3(b^3 - a^3)} \end{aligned}\right\} \quad (10$$

Check on bdy. condns.
Note independence of stress field of Poisson's ratio. Rema
on uniqueness of solution.

<u>Special case</u>: $p = 0$, $q > 0$ (external pressure only)

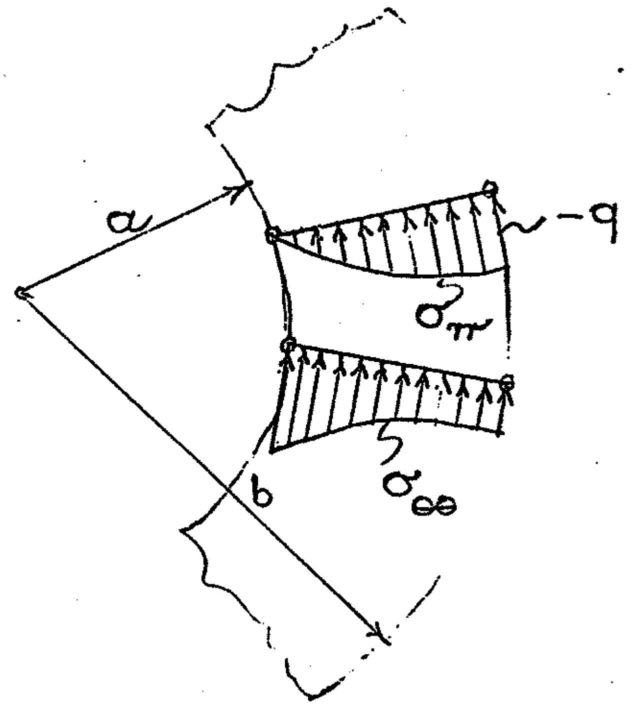
Here (10.14) reduces to

$$\sigma_{rr} = \frac{qb^3}{b^3-a^3}\left[\left(\frac{a}{r}\right)^3 - 1\right]; \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{-qb^3}{b^3-a^3}\left[\frac{1}{2}\left(\frac{a}{r}\right)^3 + 1\right] \quad (10.15)$$

$\sigma_{rr} < 0$ on $(a,b]$, σ_{rr} motone decreasing on $[a,b]$

$\sigma_{\theta\theta} < 0$ on $[a,b]$, $\sigma_{\theta\theta}$ monotone increasing on $[a,b]$

$$\sigma_{\theta\theta}\Big|_{r=a} = -\frac{3q}{2[1-(a/b)^3]} = \min_{[a,b]} \sigma_{\theta\theta} \quad (10.16)$$



Solid sphere. Consider $R = \{\underset{\sim}{x} \mid 0 \leq |\underset{\sim}{x}| < b\}$,

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, \bar{R}), \quad \mu > 0, -1 < \nu < 1/2$$

$$\sigma_{rr} = -q, \; \sigma_{r\theta} = \sigma_{r\phi} = 0 \text{ at } r = b$$

$\mathcal{S}$ is evidently given by (10.10) with $k_1 = -q$, $k_2 = 0$

In particular,

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = -q\,, \quad \sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{\phi r} = 0 \qquad (10.1$$

(uniform hydrostatic stress field).

Consider the solution of spherical-shell prob. with b and q fixed. From (10.15),

$$\lim_{a\to 0} \sigma_{rr}(r;a) = -q\,, \quad \lim_{a\to 0} \sigma_{\theta\theta}(r;a) = -q \quad \forall\, r \in (a,b)$$

On the other hand, from (10.15),

$$\lim_{a\to 0} \sigma_{rr}(a;a) = 0\,, \quad \lim_{a\to 0} \sigma_{\theta\theta}(a;a) = -\frac{3}{2}q \quad !$$

Discuss stress concentration.

To clarify matters hold b, q fixed and set

$$\varepsilon = \frac{a}{b} \quad (0 \le a < b)\,, \quad \rho = \frac{r-a}{b-a} \quad (a \le r \le b)$$

Thus,

$$0 \le \varepsilon < 1\,, \quad 0 \le \rho \le 1\,, \quad \varepsilon = 0 \Rightarrow \rho = r/b \ \text{(solid sphe}$$

Then from (10.15), after an elementary computation,

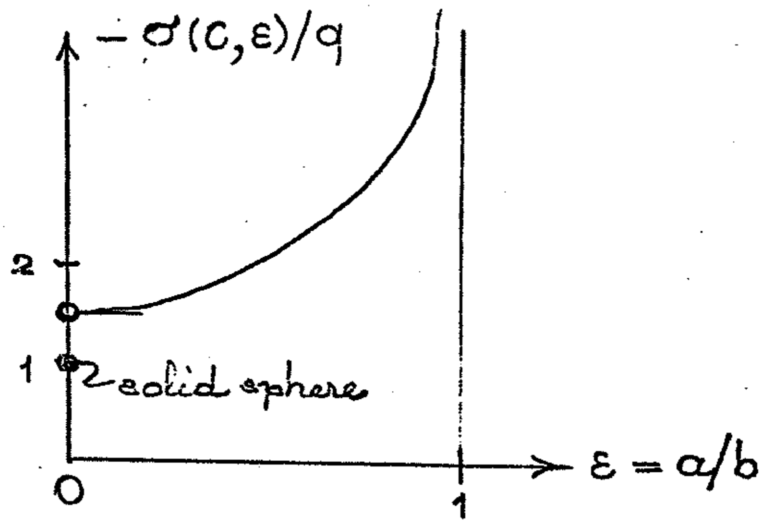
$$\sigma_{\theta\theta} = \sigma(\rho,\varepsilon) = q\,\frac{2\rho^3(1-\varepsilon)^3 + 6\rho^2\varepsilon(1-\varepsilon)^2 + 6\rho\varepsilon^2(1-\varepsilon) + 3\varepsilon^3}{2(\varepsilon^3-1)[\varepsilon+\rho(1-\varepsilon)]^3}$$

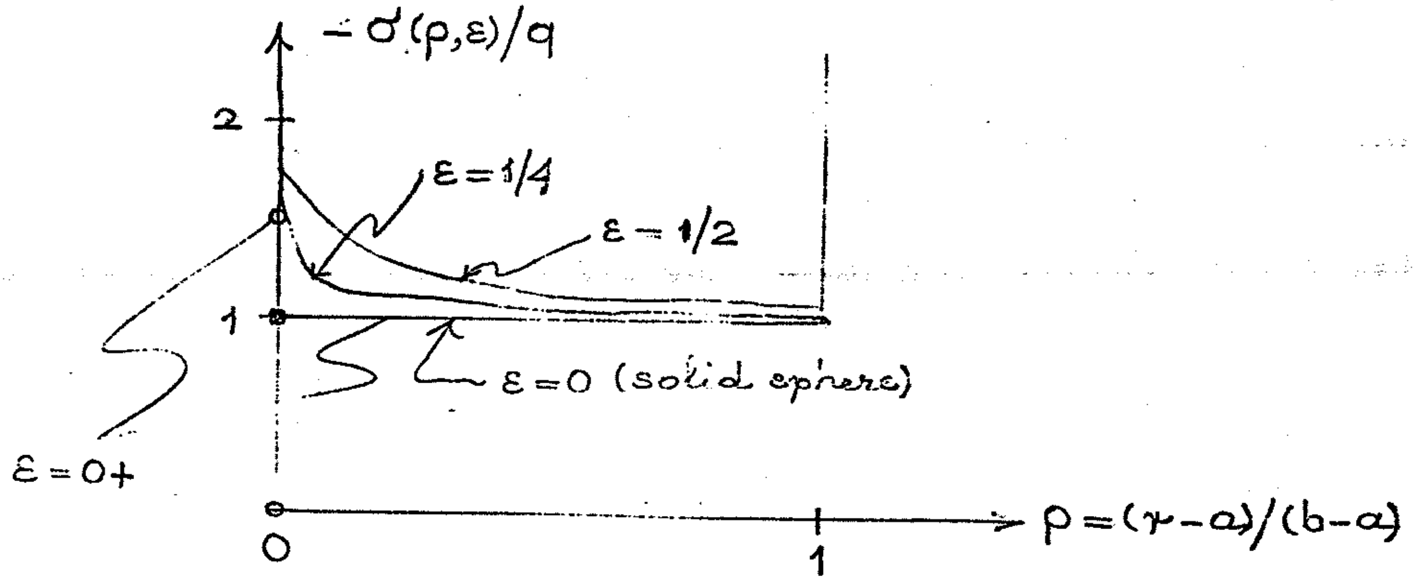
$$(0<\varepsilon<1,\ 0\le\rho\le 1$$

$$\sigma(\rho,0) = -q \quad (0\le\rho\le 1) \quad \text{(solid sphere)},\ \rho = r/b$$

$$\sigma(\rho,0+) = -q\ \forall\ \rho\in(0,1)\ ,\quad \sigma(0,\varepsilon) = -\frac{3q}{2(1-\varepsilon^3)}\ \forall\ \varepsilon\in($$

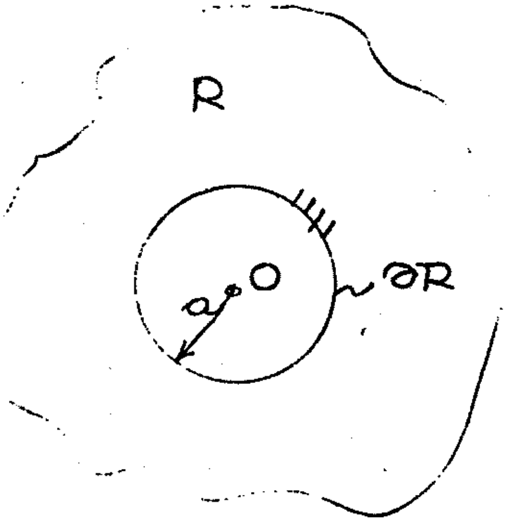
$$\sigma(0,0+) = -\frac{3q}{2}$$





Note non-uniform convergence (discont. limit functio

Spherical cavity in infinite medium under isotropic stress at infinity



$$R = \{\underset{\sim}{x} \mid a < |\underset{\sim}{x}| < \infty\} \quad (a > 0)$$

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu; \bar{R})$$

$$\underset{\sim}{s} = \underset{\sim}{0} \text{ on } \partial R, \quad \underset{\sim}{\sigma} = -q\underset{\sim}{1} + o(1) \text{ as } r \to \infty$$

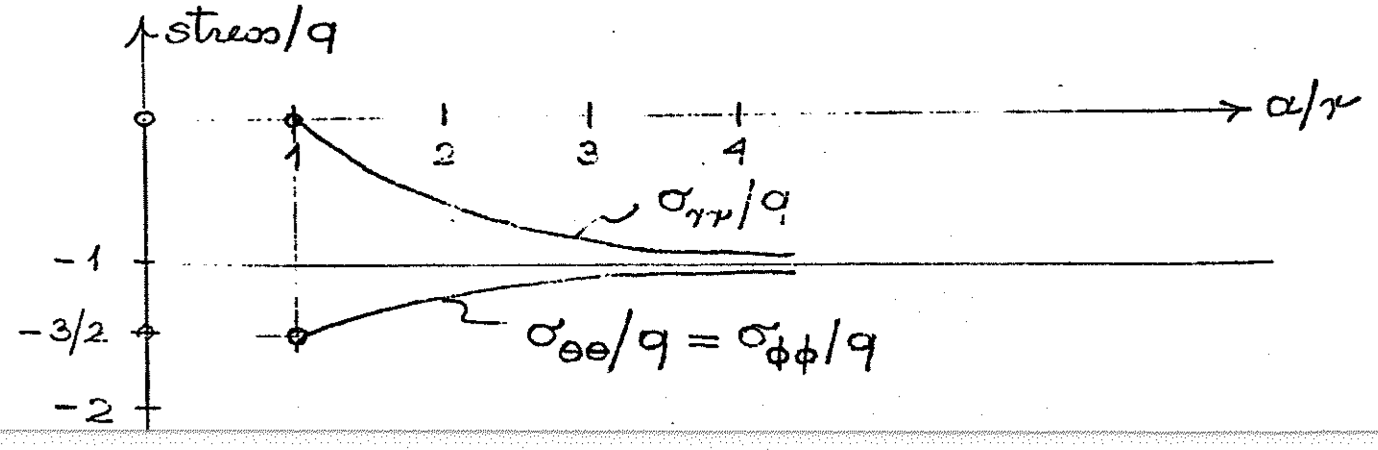
In view of polar symmetry of problem, use (10.10) subject to

$$\sigma_{rr}(a) = 0, \quad \sigma_{rr}(\infty) = \sigma_{\theta\theta}(\infty) = \sigma_{\phi\phi}(\infty) = -q$$

This leads to $k_1 = -q$, $k_2 = \dfrac{a^3 q}{2}$ and thus gives

$$\sigma_{rr} = q\left[\left(\frac{a}{r}\right)^3 - 1\right], \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -q\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]$$

in agreement with the result of letting $b \to \infty$ in (10.15) with a held fixed.



Note independence of stress concentration of elastic constants and cavity size.

Mention problem of perfectly bonded spherical elastic inclusion in the infinite medium. Sketch solution. See Exercise 33 for limiting case of rigid inclusion. Dependence of stress conc. on shape of ca[vity] or inclusion.

Mention analogue of sphere and pressurized cavity problem in elastodynamics (focussing).

Remarks. Lamé's problem is a special case of the second elastostatic bdy-value prob. for region between two concentric spheres.: This prob. is under complete control, as is the corresponding displt. prob.. Theory of these probs. is intimately connected with theory of spher. harmonics. In preparation for certain axisymm. probs. for such a region, we now recall relevant elements of theory of spherical harmonics.

Review of axisymmetric spherical harmonics

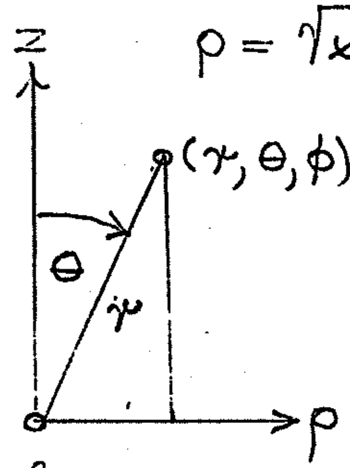
References: Hobson, Spherical & ellipsoidal harm
Lebedev, Special functions & their applica
AM 113 Lecture notes.

Define spherical coords. (r, θ, ϕ) through

$$\left.\begin{array}{c} x_1 = r\sin\theta\cos\phi,\; x_2 = r\sin\theta\sin\phi,\; x_3 = r\cos\phi \\ 0 \le r < \infty,\; 0 \le \theta \le \pi,\; 0 \le \phi < 2\pi \end{array}\right\} (10$$

φ $(\rho, \phi; z)$ are standard circular cyl. coords.,

$$\rho = \sqrt{x_1^2 + x_2^2} = r \sin\theta, \; z = x_3 = r\cos\theta \quad (10.19)$$



For convenience set

$$p = \cos\theta, \; q = \sqrt{1-p^2} = \sin\theta \quad (10.20)$$

Clearly,

$$F(\rho, z) = F(rq, rp) = H(r, p) = G(r, \theta) \quad (10.21)$$

By (10.20) & ∇^2 in spherical coords., $\nabla^2 G = 0 \Leftrightarrow$

$$\frac{\partial^2 H}{\partial r^2} + \frac{2}{r}\frac{\partial H}{\partial r} + \frac{1-p^2}{r^2}\frac{\partial^2 H}{\partial p^2} - \frac{2p}{r^2}\frac{\partial H}{\partial p} = 0 \quad (10.22)$$

Separate variables, setting

$$H(r, p) = U(r)V(p) \quad (10.23).$$

This yields:

$$r^2 U'' + 2rU' - n(n+1)U = 0 \quad \text{(Euler type)},$$

$$(1-p^2)V'' - 2pV' + n(n+1)V = 0 \quad \text{(Legendre's eq.)},$$

where $k = n(n+1)$ is the separation param. Thus

$$H(r, p) = [r^n \text{ or } r^{-n-1}][P_n(p) \text{ or } Q_n(p)] \quad (10.2$$

P_n, Q_n ... Legendre fcs. of first and second kind.

Restrict n to non-negative integers. Then,

$$P_n(p) = \frac{1}{2^n n!} \frac{d^n}{dp^n} (p^2-1)^n \quad \text{(Rodrigues)}$$

$$Q_n(p) = \frac{1}{2^n n!} \frac{d^n}{dp^n} \left[(p^2-1)^n \log \frac{1+p}{1-p} \right] - \frac{1}{2} P_n(p) \log \frac{1+p}{1-p}$$

$(n = 0, 1, 2, \dots)$

$P_n(p)$ ... Legendre polynomial of degree n

$P_n(p) = P_n(-p)$ (only even powers) if n even

$P_n(p) = -P_n(-p)$ (only odd powers) if n odd

$$P_n(1) = 1, \; P_n(-1) = (-1)^n$$

$$r^{-n-1} P_n\left(\tfrac{z}{r}\right) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right), \; r = \sqrt{\rho^2+z^2}$$

$$P_0(p) = 1, \; P_1(p) = p, \; P_2(p) = \tfrac{1}{2}(3p^2-1), \; P_3(p) = \tfrac{1}{2}(5p^3-3p)$$

Note that $Q_n(p)$ is singular at $p = \pm 1$ $(\theta = 0, \pi)$. Hence if $H(r,p)$ in (10.25) involves $Q_n(p)$, it is singular along z-axis (x_3-axis).

(10.25) and above yield the following two sequences of "axisymm. spherical harmonics":

$$\left.\begin{aligned} H_n(r,p) &= r^n P_n(p) \quad (n=0,1,2,\cdots) \\ H_{-n}(r,p) &= r^{-n} P_{n-1}(p) \quad (n=1,2,3,\cdots) \end{aligned}\right\} \quad (10.28)$$

$H_n(r,p)$ ... <u>interior</u> harmonic of degree n

$H_{-n}(r,p)$ ... <u>exterior</u> harmonic of degree n

Motivate terminology. Define

$$P_{-n-1}(p) = P_n(p) \quad (n=0,1,2,\cdots) \quad (10.29)$$

Then $H_n = r^n P_n(p)$ $(n=0,\pm1,\pm2,\cdots)$ and is interior for $n \geq 0$, exterior for $n<0$.

<u>Recursion relations</u>:

$$\left.\begin{aligned} (2n+1)pP_n(p) &= (n+1)P_{n+1}(p) + nP_{n-1}(p), \\ (1-p^2)P_n'(p) &= nP_{n-1}(p) - npP_n(p) \\ &\qquad (n=0,\pm1,\pm2,\cdots) \end{aligned}\right\} (10.$$

<u>Orthogonality</u>: If m,n are non-neg. integers,

$$\int_{-1}^{1} P_n(p)P_m(p)\,dp = \begin{cases} 0 & (m\neq n) \\ 2/(2n+1) & (m=n) \end{cases} \quad (10.$$

$$\int_{-1}^{1} (1-p^2)P_n'(p)P_m'(p)\,dp = \begin{cases} 0 & (m\neq n) \\ 2(n+1)!/(2n+1)(n-1)! & (m= \end{cases} \quad (10$$

Completeness, Fourier-Legendre expansions

Let f and g be piecewise continuously differentiable on $[-1,1]$ with $g(1)=g(-1)=0$. Then $\forall\ p\in[-1,1]$ at which f is continuous,

$$f(p)=\sum_{0}^{\infty}\xi_n P_n(p)\quad,\quad \xi_n=\frac{2n+1}{2}\int_{-1}^{1}f(p)P_n(p)\,dp \tag{10.33}$$

and $\forall\ p\in[-1,1]$ at which g is continuous,

$$g(p)=q\sum_{1}^{\infty}\eta_n P_n'(p)\quad,\quad \eta_n=\frac{(2n+1)(n-1)!}{2(n+1)!}\int_{-1}^{1}q\,g(p)P_n'(p)\,dp, \tag{10.34}$$

where $q=\sqrt{1-p^2}$.

Remarks. See Hobson for less stringent conditions on f and g. Note that $q=0$ for $p=\pm1$, whence if the series in (10.34) converges to $g(p)$ at $p=\pm1$, we must have $g(\pm1)=0$. Observe that the formulas for ξ_n, η_n follow formally from above expansion if one assumes uniform convergence on $[-1,1]$ and uses (10.31), (10.32). Elaborate.

Completeness of axisymmetric spherical harmonics

Let

$$R = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in E,\ r_1 < |\underset{\sim}{x}| < r_2\},\ 0 \le r_1 < r_2 \le \infty$$

and suppose $G \in \mathcal{C}^2(R)$, $\nabla^2 G = 0$ on R with

$$G(\underset{\sim}{x}) = H(r,p)\ \forall\ (r,p) \in (r_1,r_2) \times [-1,1].$$

Then $\exists\ \{c_n\}_{n=-\infty}^{n=\infty}$ such that

$$H(r,p) = \sum_{-\infty}^{\infty} c_n H_n(r,p) = \sum_{-\infty}^{\infty} c_n r^n P_n(p)\ \forall\ (r,p) \in (r_1,r_2) \times [$$

(10.

Cf. Laurent expansion

The second axisymm. bdy-value pb. for the exterior of sph

(All around infinite medium with a spherical cavity)

$$R = \{\underset{\sim}{x} \mid a < |\underset{\sim}{x}| < \infty\}\quad (a>0)$$

Find

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{e}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu,\nu;\overline{R}) \quad ,\ \mu>0, -1<\nu<1/2 \quad (*)$$

subject to the following requirements:

Boundary conds.

$$\left.\begin{array}{c} f\in C^1(I),\ g\in C^1(I),\ I=[-1,1],\ g(-1)=g(1)=0 \\ \sigma_{rr}(a,p)=f(p),\ \sigma_{r\theta}(a,p)=g(p),\ \sigma_{r\phi}(a,p)=0 \\ p=\cos\theta\ ,\ q=\sqrt{1-p^2}=\sin\theta,\ -1\le p\le 1 \end{array}\right\}\ (10$$

Regularity at infinity

$$\underline{s}(\underline{x})=o(1)\ \text{as}\ x\to\infty \qquad (**)$$

Note that the above problem is covered by the uniqueness theorem for exterior regions (Thm. 6.5). Remark that $\sigma_{r\theta}(a,\pm 1)=0$ is implied by axisymmetry and regularity of $\mathcal{S}$. Expanding f and g in Fourier-Legendre series, one has

$$f(p)=\sum_0^\infty \xi_n P_n(p)\ ,\ \ g(p)=q\sum_0^\infty \eta_n P_n'(p)\quad (-1\le p\le 1),\quad (10.$$

with ξ_n, η_n given by (10.33), (10.34). Now seek to construct the axisymmetric state $\mathcal{S}$ by means of Boussinesq sol in spherical coordinates. For this purpose conside the sequences $\{\mathcal{S}_n\}$, $\{\bar{\mathcal{S}}_n\}$ defined as follows.

Axisymmetric elastostatic states generated by spherical harmonics*

$$\boxed{S_n: \ \varphi = r^{-n-1} P_n(p), \ \psi = 0 \quad (n = 0, \pm 1, \pm 2, \cdots)} \quad (10.$$

Substitution into Boussinesq's axisymmetric solution referred to spherical coordinates, upon use of use of Legendre's equation and the recursion formulas, yields:

$$\left.\begin{aligned}
&2\mu u_r \doteq -\frac{(n+1)P_n}{r^{n+2}}, \quad 2\mu u_\theta \doteq -\frac{qP_n'}{r^{n+2}}; \\
&\sigma_{rr} \doteq \frac{(n+1)(n+2)}{r^{n+3}} P_n, \\
&\sigma_{\theta\theta} \doteq \frac{1}{r^{n+3}} \left[P_{n+1}' - (n+1)(n+2) P_n \right], \\
&\sigma_{\phi\phi} \doteq -\frac{P_{n+1}'}{r^{n+3}}, \quad \sigma_{r\theta} \doteq (n+2)\frac{qP_n'}{r^{n+3}}. \\
&P_n = P_n(p), \ p = \cos\theta, \ q = \sin\theta
\end{aligned}\right\} \quad (10.$$

* No body forces.

$$\boxed{\bar{s}_n : \quad \varphi = 0 , \quad \psi = r^{-n-1} P_n(p) \quad (n = 0, \pm 1, \pm 2, \cdots)} \quad (10.$$

$$2\mu u_r = -\frac{(n+4-4\nu)}{(2n+1) r^{n+1}} \left[(n+1) P_{n+1} + n P_{n-1} \right] ,$$

$$2\mu u_\theta = -\frac{q}{(2n+1) r^{n+1}} \left[(n-3+4\nu) P'_{n+1} \right.$$

$$\left. + (n+4-4\nu) P'_{n-1} \right] ;$$

$$\sigma_{rr} = \frac{n+1}{(2n+1) r^{n+2}} \left\{ \left[(n+1)(n+4) - 2\nu \right] P_{n+1} \right.$$

$$\left. + n(n+4-4\nu) P_{n-1} \right\} ,$$

$$\sigma_{\theta\theta} = -\frac{1}{(2n+1) r^{n+2}} \left[(n+1)(n^2 - n + 1 - 2\nu) P_{n+1} \right.$$

$$\left. + n(n+1)(n+4-4\nu) P_{n-1} - (2n+1) P'_n \right]$$

$$\sigma_{\phi\phi} = -\frac{1}{r^{n+2}} \left[(n+1)(1-2\nu) P_{n+1} + P'_n \right]$$

$$\sigma_{r\theta} = \frac{q}{(2n+1) r^{n+2}} \left[(n+1)(n+4-4\nu) P'_{n-1} \right.$$

$$\left. + (n^2 + 2n - 1 + 2\nu) P'_{n+1} \right] ,$$

$$P_n = P_n(p) , \quad p = \cos\theta , \quad q = \sin\theta$$

Clearly, $\mathcal{S}_n$ and $\bar{\mathcal{S}}_n$ satisfy (*) and conform to (**) as well, provided $n \geq 0$.(§)

Note that while $\mathcal{S}_n$ matches the structure of the boundary conditions (10.36), (10.37), such is not true for $\bar{\mathcal{S}}_n$. For this reason we are led to abandon $\{\bar{\mathcal{S}}_n\}$ in favor of the sequence of axisymmetric solutions $\{\hat{\mathcal{S}}_n\}$ defined by

$$\hat{\mathcal{S}}_n = (2n+1)\bar{\mathcal{S}}_n - (n+4-4\nu)\mathcal{S}_{n-1} \quad (n=0,1,2,\cdots). \text{ (10.}$$

The spherical components of displacement and stress for $\hat{\mathcal{S}}_n$ follow from (10.42), (10.3 and (10.41). The results thus obtained are those listed in (10.43).

---

§ Observe from (10.39) that $\mathcal{S}_{-1}$ is the null-state since $P_{-1}(p) = P_0(p) = 1$.

$$\hat{s}_n: \quad 2\mu u_r = -\frac{(n+1)(n+4-4\nu)}{r^{n+1}} P_{n+1},$$

$$2\mu u_\theta = -\frac{q(n-3+4\nu)}{r^{n+1}} P'_{n+1};$$

$$\sigma_{rr} = (n+1)[(n+1)(n+4)-2\nu]\frac{P_{n+1}(p)}{r^{n+2}},$$

$$\sigma_{r\theta} = (n^2+2n-1+2\nu)\,q\,\frac{P'_{n+1}(p)}{r^{n+2}},$$

$$\sigma_{\theta\theta} = \frac{-1}{r^{n+2}}[(n+1)(n^2-n+1-2\nu)P_{n+1}(p) - (n-3+4\nu)P'_n(p)],$$

$$\sigma_{\phi\phi} = \frac{-1}{r^{n+2}}[(1-2\nu)(n+1)(2n+1)P_{n+1}(p) + (n-3+4\nu)P'_n(p)]. \quad (10.4$$

We now assume s in the form:

$$s = \sum_0^\infty [a_n s_n + b_n \hat{s}_n] \quad \text{on } \overline{R} \quad (10.4$$

The application of the bdy condc. (10.36), (10.38) yields two simult. linear alg. eqs. for a_n, b_{n-1}, whose sol. is easily found to be:

$$a_n \doteq -\frac{[(n^2-2+2\nu)\xi_n - n(n^2+3n-2\nu)\eta_n]\,a^{n+3}}{2(n+2)[n^2+n+1-(2n+1)\nu]}$$

$$b_n \doteq \frac{[\xi_{n+1}-(n+2)\eta_{n+1}]\,a^{n+2}}{2[n^2+3n+3-(2n+3)\nu]} \quad (10.45)$$

$(n = 0, 1, 2, \dots)$

Eqs. (10.44), (10.45, $\overbrace{\text{\&}~\cancel{(10.40)}}^{(10.33),(10.34),(10.39)}$, (10.43) constitute the formal solution of problem posed in (*); (**), (10.36); (1[…]

Discuss verification of sol. In this connection and for explicit sol. of corresponding prob. for solid sphere and spherical shell, see Sadowsky, Eubanks, & Co., Proc. First U.S. Nat. Cong. Appl. Mech, 1952. Mention general (non-axisymm.) problem.

Stress concentration around a spherical cavity in a arbitrary, uniform stress field

$$R = \{\underset{\sim}{x} \mid a < |\underset{\sim}{x}| < \infty\}$$

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu; \bar{R}) \quad , \mu > 0, -1 < \nu < 1/2 \quad (*)$$

Boundary conditions

$$\underset{\sim}{s} = \underset{\sim}{0} \text{ on } \partial R \text{ or}$$

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0 \text{ at } r = a \qquad (10.46)$$

Regularity conds. at ∞.

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \overset{\infty}{\underset{\sim}{\sigma}} + o(1) \text{ as } \cancel{x} \to \infty \qquad (\overset{\infty}{\underset{\sim}{\sigma}} \ldots \text{const.}) \qquad (1[…]$$

In view of the polar symmetry of sphere and because of the princ. of superposition, we may — without loss of generality — replace (10.47) by

$$\sigma_{ij}(x) = \delta_{3i}\delta_{3j}\sigma + o(1) \text{ as } r\to\infty \qquad (10.48$$

Explain reduction of original prob. Note axisymmetry of reduced prob. Now, assume sol. in the form

$$\mathcal{S} = \overset{\circ}{\mathcal{S}} + \overset{*}{\mathcal{S}} \text{ on } \overline{R}, \qquad (10.4$$

where

$$\left.\begin{aligned} \overset{\circ}{\mathcal{S}} &= [\overset{\circ}{u}, \overset{\circ}{\varepsilon}, \overset{\circ}{\sigma}] \in \mathcal{E}(\mu,\nu;\overline{R}) \\ \overset{\circ}{\sigma}_{ij} &= \delta_{3i}\delta_{3j}\sigma \text{ on } \overline{R} \end{aligned}\right\} (10$$

One confirms easily that $\overset{\circ}{\mathcal{S}}$ may be generated as bel

$$\left.\begin{aligned} \overset{\circ}{\mathcal{S}}: \quad \overset{\circ}{\varphi} &= \frac{\nu\sigma}{2(1+\nu)}(2z^2-\rho^2) = \frac{\nu\sigma r^2}{2(1+\nu)}(3p^2-1) = \frac{\nu\sigma}{1+\nu}r^2P_2(p) \\ \overset{\circ}{\psi} &= -\frac{\sigma z}{2(1+\nu)} = -\frac{\sigma r p}{2(1+\nu)} = -\frac{\sigma}{2(1+\nu)}rP_1(p) \end{aligned}\right] (10$$

(10.51), $\overset{n=-3}{(10.39)}$, $\overset{n=-2}{(10.41)}$ $\Rightarrow$

[§] $\overset{\circ}{\mathcal{S}}$ is not to be confused with the center of compression.

$$\left.\begin{array}{l}\overset{\circ}{u}_r \overset{\checkmark}{=} \frac{\sigma r}{2\mu}\left(p^2 - \frac{\nu}{1+\nu}\right), \quad \overset{\circ}{u}_\theta \overset{\checkmark}{=} -\frac{\sigma r}{2\mu} qp ; \\ \overset{\circ}{\sigma}_{rr} \overset{\checkmark}{=} \sigma p^2, \quad \overset{\circ}{\sigma}_{r\theta} \overset{\checkmark}{=} -\sigma pq, \quad \overset{\circ}{\sigma}_{\theta\theta} \overset{\checkmark}{=} \sigma q^2, \quad \overset{\circ}{\sigma}_{\phi\phi} \overset{\checkmark}{=} 0. \end{array}\right\} (10.52)$$

Remark on alternative derivation of (10.52) via (10.50)§:

In view of (10.49), (10.52), (10.50), (10.48), and (10.46), $\overset{*}{s}$ is axisymmetric and obeys:

$$\left.\begin{array}{l}\overset{*}{s} = [\overset{*}{u}, \overset{*}{\varepsilon}, \overset{*}{\sigma}] \in \mathcal{E}(\mu, \nu, \bar{R}) \\ \overset{*}{\sigma}_{rr}(a,p) = -\sigma p^2, \quad \overset{*}{\sigma}_{r\theta}(a,p) = \sigma pq \quad (-1 \le p \le 1) \\ \overset{*}{\sigma}(x) = o(1) \text{ as } x \to \infty \end{array}\right\} (10.5$$

Thus $\overset{*}{s}$ is a special case of the sol. to the axisymm. cavity problem deduced earlier. Hence $\overset{*}{s}$ is given by s of (10.44), (10.45), (10.39), (10.42), provided ξ_n, η_n are determined consistent with the bdy conds. in (10.53). Hence, recalling (10.37), (10.27) one has now

$$\left.\begin{array}{l}\xi_0 \overset{\checkmark}{=} -\sigma/3, \quad \xi_1 \overset{\checkmark}{=} 0, \quad \xi_2 \overset{\checkmark}{=} -2\sigma/3, \quad \xi_n \overset{\checkmark}{=} 0 \quad (n>2), \\ \eta_0 \overset{\checkmark}{=} \eta_1 = 0 ; \quad \eta_2 \overset{\checkmark}{=} \sigma/3, \quad \eta_n \overset{\checkmark}{=} 0 \quad (n>2). \end{array}\right\} (10.$$

From (10.54), (10.49), (10.45), (10.44) one finds for the

✓ § Use $\hat{\sigma}_{ij} = A_{ip} A_{jq} \sigma_{pq}$.

complete sol. of the present problem:

$$\mathscr{S} \overset{\vee}{=} \overset{\circ}{\mathscr{S}} - \frac{a^3\sigma}{6}\mathscr{S}_0 + \frac{a^5\sigma}{7-5\nu}\mathscr{S}_2 - \frac{5a^3\sigma}{6(7-5\nu)}\hat{\mathscr{S}}_1 \text{ on } \bar{R} \qquad (10.55)$$

The stress field of $\mathscr{S}$ in spherical coords. is immediate from (10.55), (10.52), (10.39), (10.43). Remark on phys. interpretation of $\mathscr{S}$ in terms of basic singular sols: $\mathscr{S}_0$ ... center of comp. at O; $\hat{\mathscr{S}}_1$ ... linear comb. of center of comp. & force doublet without moment par. x_3-axis at O $\mathscr{S}_2$ ... higher-order self-equil. singularity at O. Remark on ad-hoc derivation of (10.55).

Results for stress in the equatorial plane

Here $\theta = \frac{\pi}{2}$, $p = 0$, $q = 1$. Assume $\sigma > 0$. Set $\xi = r/a$.

$$\sigma_{\theta\theta}(r,0) = \frac{\sigma}{2(7-5\nu)}\left[2(7-5\nu) + \frac{4-5\nu}{\xi^2} + \frac{9}{\xi^5}\right] \quad (1 \le \xi < \infty)$$

$$\sigma_{\phi\phi}(r,0) = \frac{\sigma}{2(7-5\nu)\xi^3}\left(-6 + 15\nu + \frac{3}{\xi^2}\right) \quad (1 \le \xi < \infty)$$

$$\sigma_{rr}(r,0) = \frac{6\sigma}{(7-5\nu)\xi^3}\left(1 - \frac{1}{\xi^2}\right) \quad (1 \le \xi < \infty)$$

In particular,

$$\sigma_{\theta\theta}(a,0) = \max_{a \le r < \infty} \sigma_{\theta\theta}(r,0) \overset{\checkmark}{=} \frac{(27-15\nu)\sigma}{2(7-5\nu)} \quad (\text{indept. of } a\,!)$$

For $\nu = 0.3$; $\sigma_{\theta\theta}(a,0) = \frac{45}{22}\sigma$

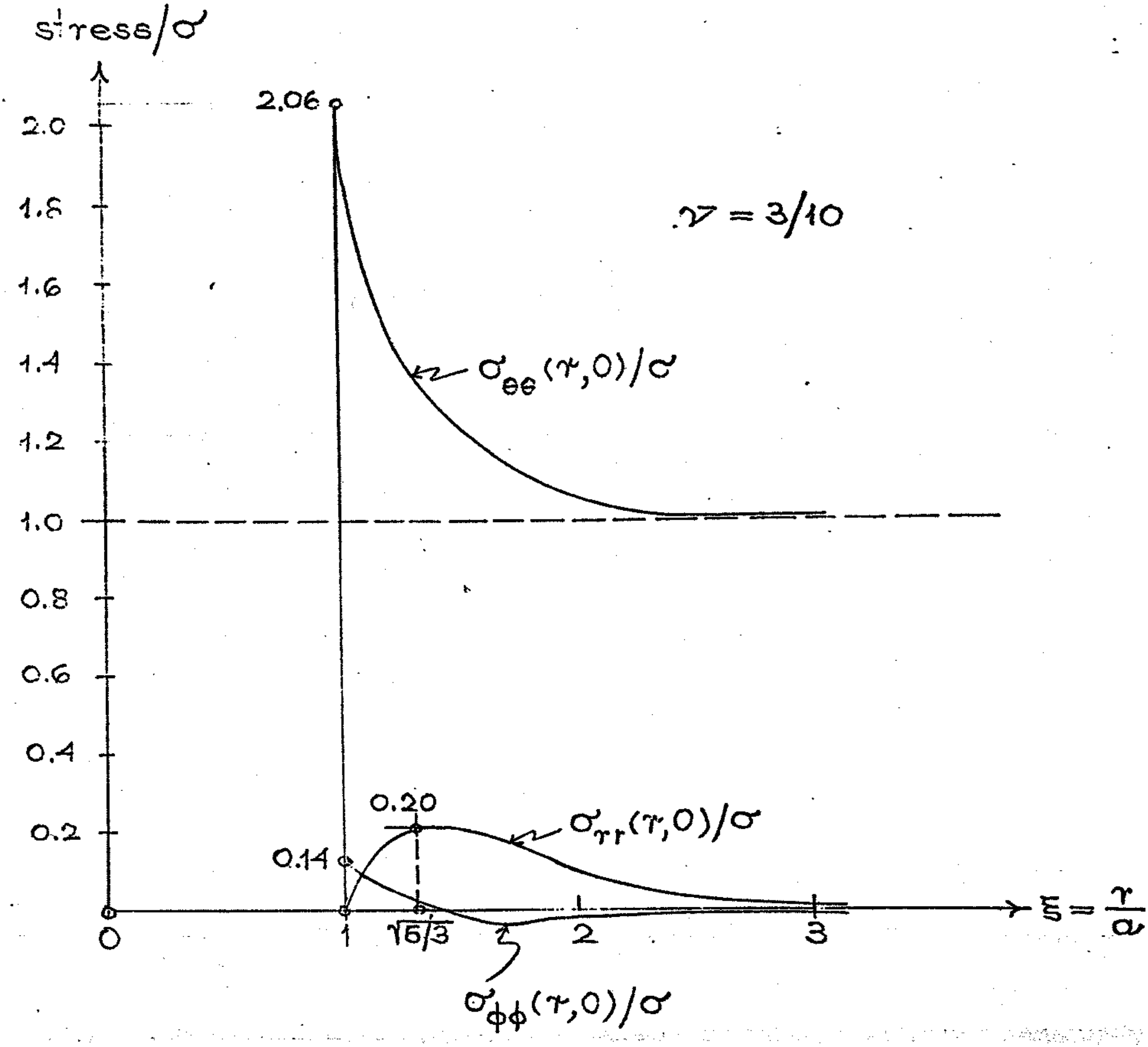
$$\sigma_{\phi\phi}(a,0) \overset{\checkmark}{=} -\frac{3(1-5\nu)\sigma}{2(7-5\nu)}, \quad \sigma_{rr}(a,0) = 0 \quad (\text{cf. bdy cond.})$$

<u>Observe</u>: $\sigma_{\theta\theta}(r,0) > 0 \ (a \le r < \infty)$, $\sigma_{rr}(r,0) > 0 \ (a < r < \infty$

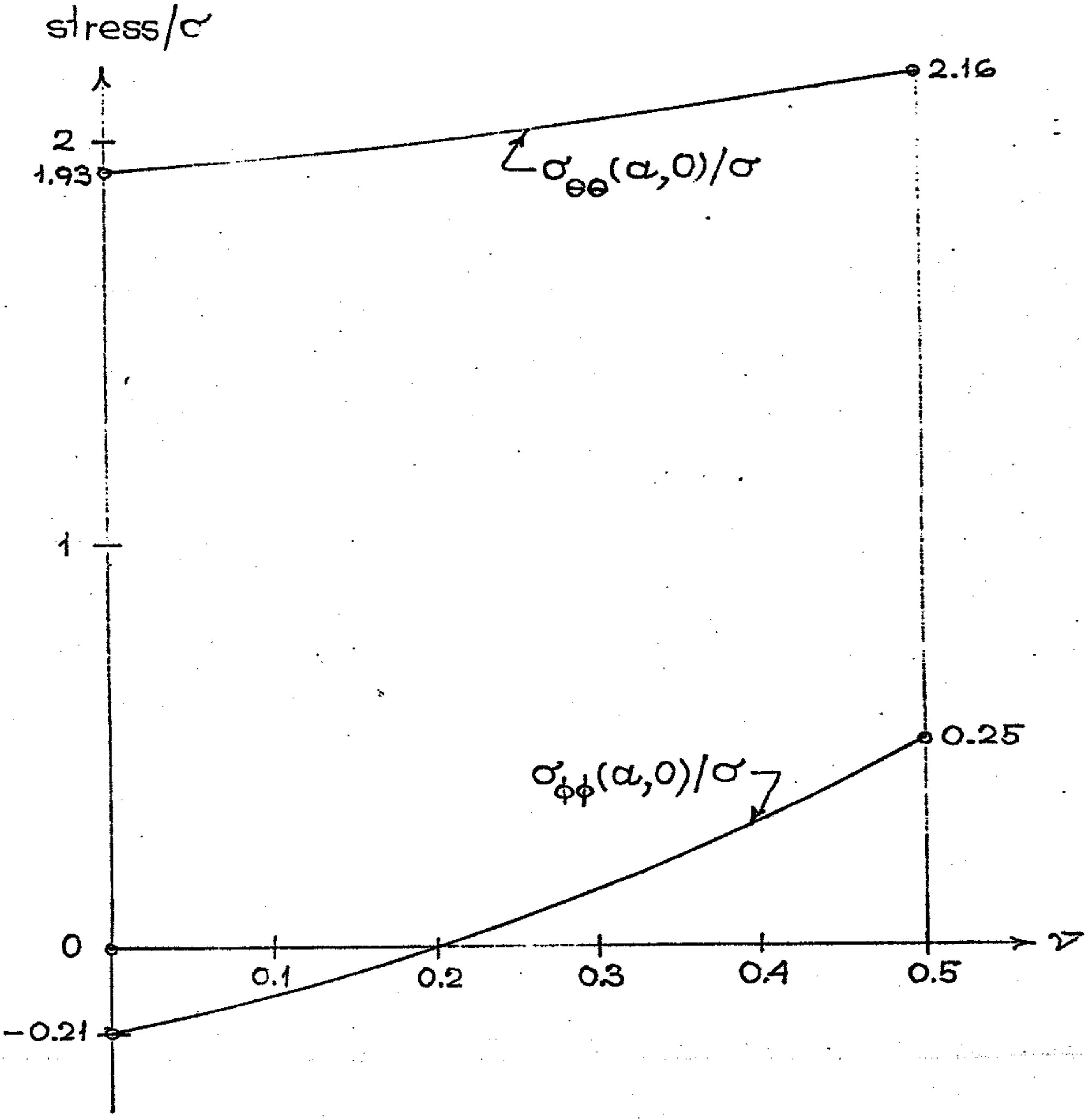
$$\sigma_{\phi\phi}(a,0) < 0 \ (0 \le \nu < 1/5), \quad \sigma_{\phi\phi}(a,0) > 0 \ (1/5 < \nu \le 1/2)$$

Discuss diagrams on next two pages. Compare with plane analogue. Note tri-axiality of state of stress near cavity.

<u>Exercise</u> 3.4. Deduce the stress distribution in an all-around infinite elastic body with a rigid spherical inclusion, if the medium is in a state of uniform uni-axial tension at infinity. Discuss the ensuing stres concentration.



STRESSES IN THE EQUATCRIAL PLANE $\theta = \pi/2$.

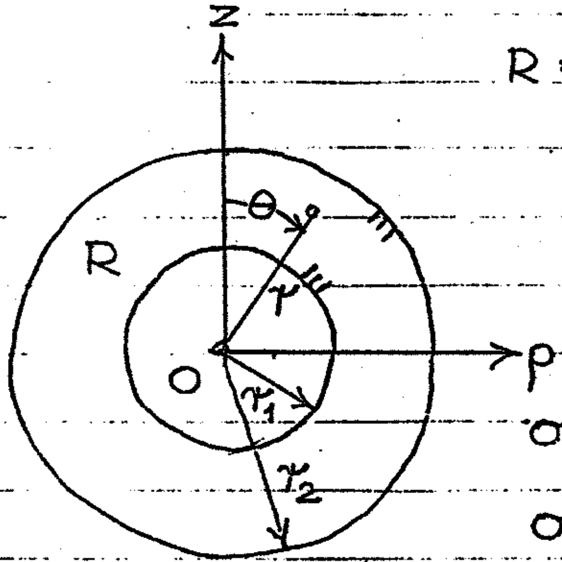


DEPENDENCE OF STRESSES ON POISSON'S RATIO

Remarks. Mention related cavity and inclusion problems (spheroidal, ellipsoidal). Two spherical cavities, stress concentration interference. Notch problems. Refer to Neuber's book. Distribute survey article*.

* Applied Mechanics Reviews, 11 (1958), 1.

Remarks on the axisymmetric equilibrium problem for a spherical shell under given surface loads



Meridional section

$$R = \{\underset{\sim}{x} \mid r_1 < |\underset{\sim}{x}| < r_2\}, \quad 0 < r_1 < r_2$$

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu; \bar{R})$$

(no body forces)

Boundary conditions:

$$\sigma_{rr}(r_\alpha, \theta, \phi) = \overset{*}{f}_\alpha(\theta) \quad (0 \le \theta \le \pi, 0 \le \phi <$$

$$\sigma_{r\theta}(r_\alpha, \theta, \phi) = \overset{*}{g}_\alpha(\theta) \quad '' \quad ''$$

$$\sigma_{r\phi}(r_\alpha, \theta, \phi) = 0 \quad '' \quad ''$$

$$(\alpha = 1, 2)$$

Set $\quad p = \cos\theta, \; q = \sin\theta = \sqrt{1-p^2} \quad (-1 \le p \le 1)$

Use Boussinesq's axisymm. sol. in spherical coords (see notes):

$$\underset{\sim}{u} = \frac{1}{2\mu}[\nabla(\varphi + \underset{\sim}{x}\cdot\underset{\sim}{\psi}) - 4(1-\nu)\underset{\sim}{\psi}] \quad \text{on } R$$

$$\varphi = \varphi(r, p), \; \psi_3 = \psi(r, p), \; \psi_1 = \psi_2 = 0 \quad \text{on } R$$

$$\nabla^2\varphi = 0, \; \nabla^2\psi = 0 \quad \text{on } R$$

Then $\sigma_{rr} = \sigma_{rr}(r, p), \; \sigma_{\theta\theta} = \sigma_{\theta\theta}(r, p), \; \sigma_{\phi\phi} = \sigma_{\phi\phi}(r, p)$

$$\sigma_{r\theta} = \sigma_{r\theta}(r, p), \; \sigma_{\theta\phi} = \sigma_{\phi r} = 0 \; (!)$$

Separation of variables in Laplace's eq. referred to spherical coords. leads to following sequence of ~~sequence~~ of axisymm. harmonics that are regular on R:

$$H(r,p) \equiv U(r)V(p) = H_n(r,p) = r^n P_n(p) \quad (n = 0, \pm 1, \pm 2, \cdots)$$

$P_n(p) = P_{-n-1}(p)$ ... Legendre polynomial of degree n $(n = 0, 1, 2, \cdots)$

$n \geq 0 \Rightarrow H_n(r,p)$ are interior harmonics

$n < 0 \Rightarrow H_n(r,p)$ are exterior harmonics

Assume solution of problem in following form:

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} \{ a_n \mathcal{S}_n(r,p) + b_n \bar{\mathcal{S}}_n(r,p) \} \quad (a_n, b_n \ldots \text{consts})$$

where

$$\mathcal{S}_n : \varphi = H_n(r,p),\ \psi = 0;\quad \bar{\mathcal{S}}_n : \varphi = 0,\ \psi = H_n(r,p)$$

Compute spherical compts. of displacement & stress appropriate to $\mathcal{S}_n, \bar{\mathcal{S}}_n$. Simplify with the aid of the recursion relations & Legendre's eq.

Expand given loads in Fourier-Legendre series. Thus,

$$\sigma_{rr}(r_\alpha, p) = \overset{*}{f}_\alpha(\theta) = f_\alpha(p) = \sum_{n=0}^{\infty} \xi_n^{(\alpha)} P_n(p) \quad (-1 \leq p \leq$$

$$\sigma_{r\theta}(r_\alpha, p) = \overset{*}{g}_\alpha(\theta) = g_\alpha(p) = q \sum_{n=1}^{\infty} \eta_n^{(\alpha)} P_n'(p) \quad (-1 \leq p \leq$$

$$(\alpha = 1, 2)$$

These bdy. condo. lead to a systems of linear algebr equations for the unknown coefficients a_n, b_n $(n = 0, \pm 1, \pm 2, \dots)$ which admits solution in closed form

Mention required convergence proof for formal series solution obtained by above procedure.

Comment on modifications needed in correspdg. axisymm. prob. for the solid sphere & for the spherical cavity.

For details see paper in Proc. First U.S. Nat. Congress of Applied Mech., Chicago, 1951.