11. Kelvin's problem. Basic singular solutions in elastostatics

Introduction

~~Remainder of course is devoted to applications of the equil. theory appropriate to homog. and isotropic linear elastic solids. Refer to separate course on linear elasto-dynamics.~~

State Kelvin's problem roughly. Notion of "concentrated load" inherently alien to continuum mechanics — in particular to linear elasticity theory. Explain: freak-show of singularities, violation of underlying approximative assumptions; no conc. loads in a laboratory.

Motivation for admission of "conc. loads":

(a) fiction useful if made meaningful

(b) fundamental role of such singular solutions in the theory of integration appropriate to regular probs. — Green's functions; dislocation theory.

Criticize excuse for traditional vagueness & ambiguity based on fictitious nature of conc. loads or on violation of assumption of infinitesimal deformations.

Sketch essence of Kelvin's own approach to his problem via limit of a sequence of solutions corresponding to sequence of body-force fields that "tend to given conc. load". We now proceed to a precise limit formulation of Kelvin's problem.

Definition 11.1 (Sequence of body force fields tending to a conc. load). Let $\xi \in E$ and ℓ be a vector. Let $\{B^m\}$ be a sequence of balls $\ni$

$$B^m = \{x \mid (x-\xi)^2 < \rho_m^2\} \quad (m=1,2,\cdots), \ \rho_m \to 0 \text{ as } m \to \infty.$$

We say that $\{f^m\}$ is a sequence of body-force fields on E tending to a conc. load ℓ applied at ξ if:

(a) $f^m \in \mathcal{C}^2(E)$ $(m=1,2,3,\cdots)$;

(b) $f^m = 0$ on $E - B^m$ $(m=1,2,3,\cdots)$;

(c) $\lim_{m\to\infty} \int_E f^m dV = \ell$;

(d) the sequence $\{\int_E |f^m| dV\}$ is bounded

Remarks. Any null sequence is acceptable for $\{\rho_m\}$, in particular $\rho_m = 1/m$ $(m=1,2,\cdots)$. Because of (b) one may replace E in (c), (d) by B^m.

Discussion of condition (d)

While (a), (b), (c) are natural, (d) is as yet mysterious. It would seem natural to require, instead of (d),

$$\int_E (\underset{\sim}{x} - \underset{\sim}{\xi}) \wedge \underset{\sim}{f}^m(\underset{\sim}{x})\, dV \to \underset{\sim}{0} \quad \text{as } m \to \infty \quad (d').$$

Now (a), (b), (c), (d) $\Rightarrow$ (d') since

$$\left| \int_E (\underset{\sim}{x} - \underset{\sim}{\xi}) \wedge \underset{\sim}{f}^m(\underset{\sim}{x})\, dV \right| \leq \int_E \rho_m |\underset{\sim}{f}^m|\, dV \leq \rho_m M \to 0 \quad \text{as } m \to \infty.$$

But (a), (b), (c), (d') $\not\Rightarrow$ (d), as may be shown by choosing $\{\underset{\sim}{f}^m\}$ to be a sequence of central fields with center $\underset{\sim}{\xi}$ that meets (a), (b) and hence (c) with $\underset{\sim}{\ell} = \underset{\sim}{0}$, as well as (d'), but violates (d). Such an $\{\underset{\sim}{f}^m\}$ is easily constructed.

Thus (d) is a stronger restriction than (d'). In view of (b) and (c), $\{\underset{\sim}{f}^m\}$ cannot be uniformly bounded on E. Otherwise $\exists$ $K>0$ (independent of $\underset{\sim}{x}$) $\ni$ $|\underset{\sim}{f}^m| < K$ on E $(m = 1, 2, \cdots)$, whence

$$\left| \int_E \underset{\sim}{f}^m\, dV \right| \leq \int_{B^m} |\underset{\sim}{f}^m|\, dV < \frac{4}{3}\pi \rho_m^3 K \to 0 \quad \text{as } m \to \infty$$

so that

$$\lim_{m\to\infty} \int_E \underset{\sim}{f}^m\, dV = \underset{\sim}{0}, \text{ which implies } \underset{\sim}{\ell} = \underset{\sim}{0}$$

Requirement (d) restricts the manner in which the sequence $\{\underset{\sim}{f}^m\}$ becomes unbounded.

<u>Claim</u>: (a), (b), (c) $\Rightarrow$ (d) and thus (d) may be omitted if $\{\underset{\sim}{f}^m\}$ is "uni-directional", i.e. if

$$\underset{\sim}{f}^m(\underset{\sim}{x}) = \underset{\sim}{e}\, f^m(\underset{\sim}{x}) \,,\ f^m(\underset{\sim}{x}) \geq 0 \ \forall\ \underset{\sim}{x} \in E \quad (m=1,2,\cdots) \,,\ |\underset{\sim}{e}| = 1$$

In this case

$$\int_E |\underset{\sim}{f}^m|\, dV = \int_E f^m\, dV \qquad (m=1,2,\cdots) \qquad (*)$$

and (c) $\Rightarrow$ $\underset{\sim}{e} = \dfrac{\underset{\sim}{\ell}}{|\underset{\sim}{\ell}|}$ when $\underset{\sim}{\ell} \neq \underset{\sim}{0}$. Also, here (c) $\Rightarrow$

$$\int_E \underset{\sim}{f}^m\, dV = \underset{\sim}{e} \int_E f^m\, dV \to \underset{\sim}{\ell} \text{ as } m \to \infty, \text{ so that } \left\{\int_E f^m\, dV\right\}$$

is bounded. Therefore (d) follows from (*).

The indispensable need for (d) when $\{\underset{\sim}{f}^m\}$ is not restricted to be uni-directional will become clear later on.

Def. (Sequence of body-force fields tending to a concentrated load). Let $\boldsymbol{\xi} \in E$ and $\boldsymbol{\ell} \neq \mathbf{0}$ be a vector. Let $\{B^m\}$ be a sequence of balls $\ni$

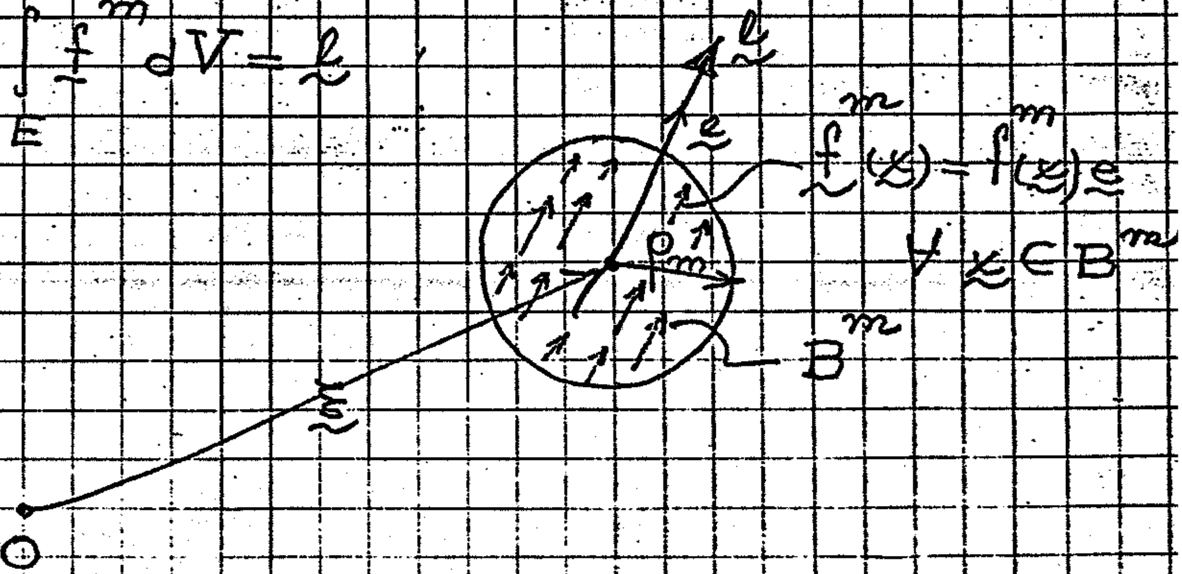
$$B^m = \{\mathbf{x} \mid |\mathbf{x} - \boldsymbol{\xi}| < \rho_m\} \ (m = 1, 2, \ldots); \ \rho_m \to 0 \text{ as } m \to \infty$$

We say that $\{\mathbf{f}^m\}$ is a sequence of body-force fields on E tending to a conc. load $\boldsymbol{\ell}$ applied at $\boldsymbol{\xi}$, provided:

(a) $\mathbf{f}^m = f^m \mathbf{e}$, $\mathbf{e} = \boldsymbol{\ell}/|\boldsymbol{\ell}|$, $f^m \geq 0$ on E, $\mathbf{f}^m \in C^2(E)$ $(m = 1, 2, 3, \ldots)$

(b) $\mathbf{f}^m = \mathbf{0}$ on $E - B^m$ $(m = 1, 2, \ldots)$

(c) $\displaystyle \lim_{m \to \infty} \int_E \mathbf{f}^m \, dV = \boldsymbol{\ell}$



Discussion

i). Any null sequence is acceptable as $\{\rho_m\}$; in particular $\rho_m = 1/m$ $(m=1,2,\cdots)$. In view of (b) one may replace E in (c) by B^m (proper integral).

(ii) (a), (b), (c) $\Rightarrow$ $\{\underset{\sim}{f}^m\}$ cannot be uniformly bounded on E. To see this suppose $\exists$ a constant $K>0 \ni |\underset{\sim}{f}^m| < K$ on E $(m=1,2,\cdots)$. Then,

$$\left|\int_E \underset{\sim}{f}^m dV\right| \le \int_E |f^m| dV = \int_{B^m} f^m dV < \tfrac{4}{3}\pi\rho_m^3 K,$$

whence

$$\int_E \underset{\sim}{f}^m dV \to \underset{\sim}{0} \text{ as } m \to \infty,$$

which contradicts (c) since $\underset{\sim}{\ell} \neq \underset{\sim}{0}$.

(iii) (a), (b), (c) $\Rightarrow$

$$\left\{\int_E |\underset{\sim}{f}^m| dV\right\} \text{ is bounded}$$

To see this note

$$\int_E \underset{\sim}{f}^m \, dV = \underset{\sim}{e} \int_E f^m \, dV = \frac{\underset{\sim}{\ell}}{|\underset{\sim}{\ell}|} \int_E |\underset{\sim}{f}^m| \, dV \to \underset{\sim}{\ell} \text{ as } m \to \infty,$$

so that

$$\int_E |\underset{\sim}{f}^m| \, dV \to |\underset{\sim}{\ell}| \text{ as } m \to \infty,$$

which implies $\exists\, M > 0 \ni$

$$\int_E |\underset{\sim}{f}^m| \, dV < M \quad (m = 1, 2, \dots)$$

Remark on indept. requirement if $\underset{\sim}{f}^m$ not uni-directional

(iv) (a), (b), (c) $\Rightarrow$

$$\int_E (\underset{\sim}{x} - \underset{\sim}{\xi}) \wedge \underset{\sim}{f}^m(\underset{\sim}{x}) \, dV \to \underset{\sim}{0} \text{ as } m \to \infty$$

since, using (iii), one has

$$\left| \int_E (\underset{\sim}{x} - \underset{\sim}{\xi}) \wedge \underset{\sim}{f}^m(\underset{\sim}{x}) \, dV \right| \leq \int_E \rho_m |\underset{\sim}{f}^m| \, dV < \rho_m M \to 0$$

as $m \to \infty$

We turn next to a theorem that supplies a limit-def., as well as an explicit representation, of the solution to Kelvin's problem.

<u>Thm.</u> 11.1 (Limit definition of the solution to Kelvin's prob.) Let $\xi \in E$ and let $\mathbf{b}$ be a vector. Let $\{\mathbf{f}^m\}$ be a sequence of body-force fields on E tending to a concentrated load $\mathbf{b}$ at ξ. Suppose $\mu > 0, -1 < \nu < 1/2$. Then:

(a) for each m $(m = 1, 2, \cdots)$ $\exists$ a unique state $s^m \ni$

$$s^m = [\mathbf{u}^m, \mathbf{E}^m, \mathbf{S}^m] \in \mathcal{E}(\mu, \nu, \mathbf{f}^m; E), \quad \mathbf{u}^m(\mathbf{x}) = o(1) \text{ as } |\mathbf{x}| \to \infty;$$

(b) $\{s^m\}$ converges to a state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ on $E - \{\xi\}$, the convergence being <u>uniform</u> on every closed bd subset of $E - \{\xi\}$;

(c) the limit-state s is independent of the particular choice of $\{\rho_m\}$ and $\{\mathbf{f}^m\}$,

$$s = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \in \mathcal{E}(\mu, \nu; E - \{\xi\})$$

and — in the sense of Thm. 8.9 — s is generated by the Papkovich-Neuber stress-functions

$$\varphi(\mathbf{x}) = \frac{\boldsymbol{\xi}\cdot\mathbf{b}}{8\pi(1-\nu)|\mathbf{x}-\boldsymbol{\xi}|}\ ,\ \boldsymbol{\psi}(\mathbf{x}) = -\frac{\mathbf{b}}{8\pi(1-\nu)|\mathbf{x}-\boldsymbol{\xi}|}\ \ \forall\ \mathbf{x}\in E-\{\boldsymbol{\xi}\} \quad (11.$$

We call s the Kelvin-state corresponding to a concentrat load $\mathbf{b}$ at $\boldsymbol{\xi}$. (and to the elastic constants μ,ν).

<u>Proof.</u> Define $\{\varphi^m\}$, $\{\boldsymbol{\psi}^m\}$ on E by setting $\forall\ \mathbf{x}\in E$,

$$\varphi^m(\mathbf{x}) = \frac{1}{8\pi(1-\nu)}\int_E \frac{\mathbf{y}\cdot\mathbf{f}^m(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV_{\mathbf{y}}\ ,\ \boldsymbol{\psi}^m(\mathbf{x}) = \frac{-1}{8\pi(1-\nu)}\int_E \frac{\mathbf{f}^m(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV_{\mathbf{y}} \quad (1)$$

(1), Thm. 8.6, and (a), (b) of Def. 11.1 $\Rightarrow$

$$\varphi^m\in\mathcal{C}^3(E),\ \boldsymbol{\psi}^m\in\mathcal{C}^3(E),\ \nabla^2\varphi^m = \frac{-\mathbf{x}\cdot\mathbf{f}^m}{2(1-\nu)}\ ,\ \nabla^2\boldsymbol{\psi}^m = \frac{\mathbf{f}^m}{2(1-\nu)} \text{ on } E \quad (2$$

Define

$$\left.\begin{array}{l}\mathbf{u}^m = \frac{1}{2\mu}[\nabla(\varphi^m+\mathbf{x}\cdot\boldsymbol{\psi}^m) - 4(1-\nu)\boldsymbol{\psi}^m] \text{ on } E\\ \mathbf{E}^m = \text{sym}\nabla\mathbf{u}^m,\ \mathbf{S}^m = 2\mu\left[\frac{\nu}{1-2\nu}\mathbf{1}\,\text{tr}\,\mathbf{E}^m + \mathbf{E}^m\right] \text{ on } E\end{array}\right\}(3$$

(2), (3), Thm. 8.9, Thm. 6.7, (1) $\Rightarrow$

$$s^m = [\mathbf{u}^m, \mathbf{E}^m, \mathbf{S}^m] \in \mathcal{E}(\mu,\nu,\mathbf{f}^m; E)\ ,\ \mathbf{u}^m(\mathbf{x}) = o(1) \text{ as } |\mathbf{x}|\to\infty \quad (4$$

Further, Thm. 6.5 (uniqueness thm. for exterior regular regions) assures that (4) characterize $\mathcal{S}^m$ uniquely. This establishes conclusion (a).

Next, (11.1) $\Rightarrow$

$$\varphi \in C^\infty(E-\{\xi\}),\ \underline{\psi} \in C^\infty(E-\{\xi\}),\ \nabla^2\varphi = 0,\ \nabla^2\underline{\psi} = \underline{0} \text{ on } E-\{\xi\} \quad (5)$$

Hence, by Thm. 8.9, if $\mathcal{S}$ is the state generated by $\varphi, \underline{\psi}$,

$$\mathcal{S} = [\underline{u}, \underline{E}, \underline{S}] \in \mathcal{E}(\mu, \nu; E-\{\xi\}) \quad \text{(no body forces)}$$

Choose a compact set $R \subset E-\{\xi\}$. To complete the proof we need to show that

$$\underline{u}^m \to \underline{u},\ \underline{E}^m \to \underline{E},\ \underline{S}^m \to \underline{S} \text{ uniformly on } R \quad (6)$$

To this end, according to Thm. 8.9, it suffices to show

$$\left.\begin{aligned} &\nabla\varphi^m \to \nabla\varphi,\ \nabla\nabla\varphi^m \to \nabla\nabla\varphi \text{ uniformly on } R \\ &\underline{\psi}^m \to \underline{\psi},\ \nabla\underline{\psi}^m \to \nabla\underline{\psi},\ \nabla\nabla\underline{\psi}^m \to \nabla\nabla\underline{\psi} \text{ uniformly on } R \end{aligned}\right] \quad (7)$$

Since each of (7) is deducible by strictly similar means, we show merely that

$$\underline{\psi}^m \to \underline{\psi} \text{ as } m \to \infty, \text{ uniformly on } R \quad (8)$$

(1), (11.1), (b) of Def. 11.1 $\Rightarrow$

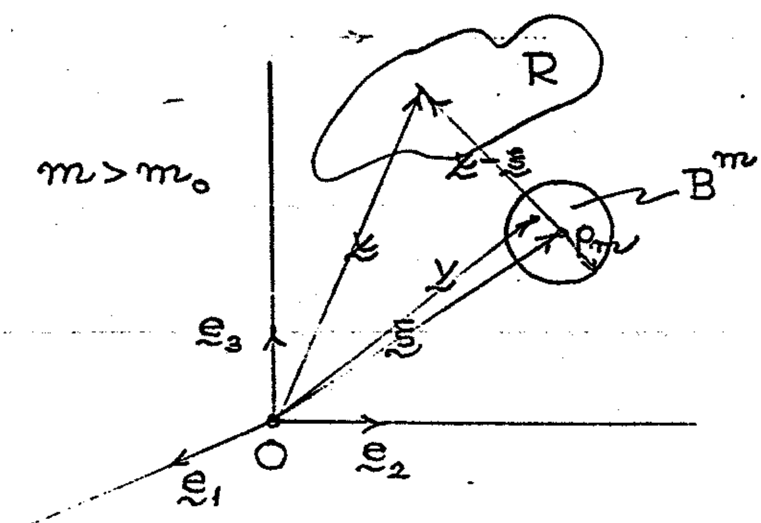
$$-8\pi(1-\nu)[\underset{\sim}{\psi}^m(\underset{\sim}{x})-\underset{\sim}{\psi}(\underset{\sim}{x})] = \int_{B^m} \frac{\underset{\sim}{f}^m(\underset{\sim}{y})}{|\underset{\sim}{x}-\underset{\sim}{y}|}\,dV_y - \frac{\underset{\sim}{b}}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}$$

$$= \underbrace{\int_{B^m} \underset{\sim}{f}^m(\underset{\sim}{y})\left[\frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|}-\frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\right]dV_y}_{\underset{\sim}{I}_1^m(\underset{\sim}{x})} + \underbrace{\frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\left[\int_{B^m}\underset{\sim}{f}^m(\underset{\sim}{y})\,dV-\underset{\sim}{b}\right]}_{\underset{\sim}{I}_2^m(\underset{\sim}{x})} \;\forall\; \underset{\sim}{x}\in R \qquad (9)$$

Now $\dfrac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|} \le \dfrac{1}{M}\;\forall\;\underset{\sim}{x}\in R$, $M = \min\limits_{\underset{\sim}{x}\in R}|\underset{\sim}{x}-\underset{\sim}{\xi}| > 0$; whence

by hyp. and (c) of Def. 11.1, $\underset{\sim}{I}_2 \to 0$ uniformly on R.

Next, since $\rho_m \to 0$ as $m\to\infty$, $\exists\, m_0 \ni m > m_0 \Rightarrow R \cap \bar{B}^m = \emptyset$



Thus $\forall\; m > m_0$ the integral $\underset{\sim}{I}_1(\underset{\sim}{x})$ is proper and

$$\left|\frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|}-\frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\right| \le \frac{|\underset{\sim}{y}-\underset{\sim}{\xi}|}{|\underset{\sim}{x}-\underset{\sim}{y}||\underset{\sim}{x}-\underset{\sim}{\xi}|} \le K\rho_m \;\forall\; (\underset{\sim}{x},\underset{\sim}{y}) \in R\times\overline{B}^m$$

$$K=\sup \frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}||\underset{\sim}{x}-\underset{\sim}{\xi}|} \text{ for } (\underset{\sim}{x},\underset{\sim}{y})\in R\times\overline{B}^m, \text{ so that}$$

$$|\overset{m}{\underset{\sim}{I}}_1(\underset{\sim}{x})| \le K\rho_m \int_{B^m} |\underset{\sim}{f}^m|\,dV \;\forall\; \underset{\sim}{x}\in R \;(m>m_0) \qquad (10)$$

But (10), hyp. ~~and~~ (d) of Def. 11.1 $\Rightarrow \overset{m}{\underset{\sim}{I}}_1 \to 0$ unif'ly on $R \Rightarrow$ (8,

qed.

<u>Thm</u> 11.2. Theorem 11.1 is false if (d) in Def. 11.1 is replaced by (d′).

See Eubanks & Co., JRMA, <u>4</u> (1955). Describe counter-examp based on $\underset{\sim}{\ell}=\underset{\sim}{0}$.

<u>Def.</u> 11.2. Denote the Kelvin state corresponding to $\underset{\sim}{\ell}$ at $\underset{\sim}{\xi}$ by

$$\mathcal{S}(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\ell}) \;\forall\; \underset{\sim}{x}\in E-\{\underset{\sim}{\xi}\} \qquad (11.2)$$

Let $X=\{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$. The triplet of normalized Kelvin states appropriate to X and $\underset{\sim}{\xi}$ is defined by

$$\mathcal{S}^k(\underset{\sim}{x},\underset{\sim}{\xi}) = \mathcal{S}(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{e}_k) \;\forall\; \underset{\sim}{x}\in E-\{\underset{\sim}{\xi}\} \qquad (11.3)$$

Clearly,

$$\mathfrak{f}(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\ell}) = \mathfrak{f}^k(\underset{\sim}{x},\underset{\sim}{\xi})\,\ell_k \quad \forall\ \underset{\sim}{x}\in E-\{\underset{\sim}{\xi}\} \qquad (11.4)^*$$

Further, from Thm. 11.1, Thm. 8.9, Def. 11.2 and direct computation one finds the "translation identity"

$$\mathfrak{f}^k(\underset{\sim}{x},\underset{\sim}{\xi}+\underset{\sim}{a}) = \mathfrak{f}^k(\underset{\sim}{x}-\underset{\sim}{a},\underset{\sim}{\xi}) \quad \forall\ \underset{\sim}{x}\in E-\{\underset{\sim}{\xi}+\underset{\sim}{a}\} \qquad (11.5)$$

for every constant vector $\underset{\sim}{a}$. In particular, taking $\underset{\sim}{\xi}=\underset{\sim}{0}$,

$$\mathfrak{f}^k(\underset{\sim}{x},\underset{\sim}{a}) = \mathfrak{f}^k(\underset{\sim}{x}-\underset{\sim}{a},\underset{\sim}{0}) \quad \forall\ \underset{\sim}{x}\in E-\{\underset{\sim}{a}\} \qquad (11.6)$$

We now record the PN-functions, as well as the cartesian components of displacement and stress appropriate to $\mathfrak{f}^k(\underset{\sim}{x},\underset{\sim}{0})$, which follow easily from Thm. 11.1, Thm 8.9 and Def. 11.2.

$$\left.\begin{aligned}
&\mathfrak{f}^k(\underset{\sim}{x},\underset{\sim}{0}):\quad \varphi^k = 0,\quad \psi^k_i = -c\,\frac{\delta_{ki}}{r}\ ,\quad r=|\underset{\sim}{x}|\ ,\quad c=\frac{1}{8\pi(1-\nu)},\\
&\qquad u^k_i(\underset{\sim}{x},\underset{\sim}{0}) = \frac{c}{2\mu r}\left[\frac{x_k x_i}{r^2}+(3-4\nu)\delta_{ki}\right],\\
&\sigma^k_{ij}(\underset{\sim}{x},\underset{\sim}{0}) = -\frac{c}{r^3}\left[\frac{3x_i x_j x_k}{r^2}+(1-2\nu)(\delta_{ki}x_j+\delta_{kj}x_i-\delta_{ij}x_k\right].
\end{aligned}\right\}\ (11.7)$$

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* Recall that <u>all</u> Latin indices have the range (1,2,3), summation over repeated indices being implied.

Let (r,θ,ϕ) be standard spherical coordinates (see Chapter 9). Then $s^k(x,0)$ $(k=3)$, referred to such coordinates, has the following form.

$$\left.\begin{aligned}
&s^3(x,0): \quad \varphi = 0\,,\ \psi_1=\psi_2=0\,,\ \psi_3=-\frac{c}{r}\,,\ c=\frac{1}{8\pi(1-\nu)}\,,\\
&u_r=\frac{2c(1-\nu)}{\mu}\,\frac{\cos\theta}{r}\,,\ u_\theta=-\frac{c(3-4\nu)}{2\mu}\,\frac{\sin\theta}{r}\,,\ u_\phi=0\,,\\
&\sigma_{rr}=-2c(2-\nu)\frac{\cos\theta}{r^2}\,,\ \sigma_{\theta\theta}=\sigma_{\phi\phi}=c(1-2\nu)\frac{\cos\theta}{r^2}\,,\\
&\qquad\sigma_{r\theta}=c(1-2\nu)\frac{\sin\theta}{r^2}\,,\ \sigma_{\phi r}=\sigma_{\phi\theta}=0.
\end{aligned}\right\}\ (11.8)$$

<u>Note</u>: φ and ψ_i in (11.8) are immediate from (11.7); observe axisymmetry of $s^3(x,0)$. The rest of (11.8) are most conveniently deduced with the aid of Boussinesq's solution in spherical coordinates, i.e. (9.43), (9.44).

<u>Thm.</u> 11.3 (Properties of the Kelvin-state). The Kelvin state $s(x)\equiv s(x,\xi,\ell)$ has the properties:

(a) $s=[u,\gamma,\sigma]\in\mathcal{E}(\mu,\nu;E-\{\xi\})$;

(b) $\int_{\partial B_\rho(\xi)} s \, dA = \ell , \quad \int_{\partial B_\rho(\xi)} (x - \xi) \wedge s \, dA = 0 \quad \forall \ \rho > 0,$

where s is the traction vector of $\mathcal{S}$ on the <u>inner</u> side of $\partial B_\rho(\xi)$.

(c) $u(x) = O(|x - \xi|^{-1}) , \quad \sigma(x) = O(|x - \xi|^{-2}) \quad$ as $x \to \xi$

(d) $u(x) = O(r^{-1}) , \quad \sigma(x) = O(r^{-2}) \quad$ as $r \to \infty$.

<u>Proof</u>.

<u>Re (a)</u>: see (c) of Thm. 11.1

<u>Re (b)</u>: by direct computation based on (11.4), (11.6), (11.7)

<u>Re (c), (d)</u>: by inspection of (11.4), (11.6), (11.7).

<u>Discussion</u>. The "traditional" formulation of Kelvin's problem is in terms of (a), (b) of Thm. 11.3 together with

$$u(x) = o(1) , \quad \sigma(x) = o(1) \quad \text{as } r \to \infty \qquad (*)$$

One can show[§] that (a) and the first of $(*) \Rightarrow$ (d). But (a), (b), (d) fail to characterize $\mathcal{S}$ uniquely! To see this consider

[§] See Gurtin & Co., ARMA, 8, 2 (1961).

$$\mathfrak{s}'(\mathbf{x}) = \mathfrak{s}(\mathbf{x}) + \alpha\,\mathfrak{s}^*(\mathbf{x}) \quad \forall\ \mathbf{x} \in E - \{\boldsymbol{\xi}\}\ ,\ \alpha = \text{const.}$$

where $\mathfrak{s}^*$ is generated by the harmonic Papkovich potentials

$$\varphi^*(\mathbf{x}) = \frac{1}{|\mathbf{x} - \boldsymbol{\xi}|}\ ,\ \boldsymbol{\psi}^*(\mathbf{x}) = \mathbf{0}\ \forall\ \mathbf{x} \in E - \{\boldsymbol{\xi}\}.$$

Omit (c) below!

One confirms easily that $\mathfrak{s}'$ satisfies (a), (b), (c), (d) of Thm. 11.3 for <u>every</u> choice of the real constant α. Observe that this non-uniqueness does not contradict Thm. 6.4 (uniqueness thm of elastostatics), which does not apply to the formulation of Kelvin's pb. under consideration. Note that $\mathfrak{s}^*$, the physical significance of which will emerge later on, has a higher-order self-equilibrated singularity at $\boldsymbol{\xi}$ and thus $\mathfrak{s}'$ fails to obey (c) in Thm. 11.3 unless $\alpha = 0$. Explain.

Mention analogous uniqueness issue for concentrated surface loads, eg. problem of sphere under equal and diametrically opposite conc. surface loads. See Rosenthal & Co., Journal of Appl. Mech., <u>19</u>, 4, 1952. Discuss.

Question: Do properties (a), (b), (c), (d) in Thm. 11.3 determine $\mathcal{S}$ uniquely? Yes! Indeed, somewhat less is sufficient for uniqueness, as is apparent from the next theorem, which we cite without proof.

Thm. 11.4. The Kelvin state $\mathcal{S}(\underset{\sim}{x}) \equiv \mathcal{S}(\underset{\sim}{x}; \underset{\sim}{\xi}, \underset{\sim}{\ell})$, but for an arbitrary additive inf. rigid displacement, is uniquely determined by:

(a) $\mathcal{S} \doteq [\underset{\sim}{u}, \underset{\sim}{\tau}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, E - \{\underset{\sim}{\xi}\})$

(b) $\displaystyle\int_{\partial B_\rho(\underset{\sim}{\xi})} \underset{\sim}{s} \, dA = \underset{\sim}{\ell}$

(c) $\underset{\sim}{\sigma}(\underset{\sim}{x}) = O(|\underset{\sim}{x} - \underset{\sim}{\xi}|^{-2})$ as $\underset{\sim}{x} \to \underset{\sim}{\xi}$

(d) $\underset{\sim}{\sigma}(\underset{\sim}{x}) = o(1)$ as $x \to \infty$

See Turteltaub & Co., ARMA, 29, 3 (1968) for a general uniqueness theorem appropriate to the direct formulation of concentrated load problems in elastostatics. See also Gurtin's Handbuch article.

Higher-order basic singular solutions

Thm. 11.5. Let R be an open region and

$$\mathscr{s} = [\underset{\sim}{u}, \underset{\sim}{\gamma}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu; R).$$

Suppose $\hat{\mathscr{s}}^k = [\hat{\underset{\sim}{u}}^k, \hat{\underset{\sim}{\gamma}}^k, \hat{\underset{\sim}{\sigma}}^k] = \mathscr{s}_{,k}$ on R, i.e.,

$$\hat{u}_i^k = u_{i,k}, \quad \hat{\gamma}_{ij}^k = \gamma_{ij,k}, \quad \hat{\sigma}_{ij}^k = \sigma_{ij,k} \quad \text{on } R.$$

Then,

$$\hat{\mathscr{s}}^k \in \mathcal{E}(\mu, \nu; R)$$

Further, if $\mathscr{s}$ is generated by the (harmonic) PN-potentia[ls] φ and $\underset{\sim}{\psi}$, then $\mathscr{s}^k$ is generated by

$$\hat{\varphi}^k = \varphi_{,k} + \psi_k, \quad \hat{\psi}_i^k = \psi_{i,k} \quad \text{on } R. \qquad (11.9)$$

Re proof. By hyp. and Thm. 6.11, $\mathscr{s} \in \mathcal{C}^\infty(R)$. Hence $\hat{\mathscr{s}}^k \in \mathcal{C}^\infty(R)$. Sketch argument to show $\hat{\mathscr{s}}^k$ satisfies fund. field eqs. on R. (11.9) follows with the aid of Thm. 8.9.

Exercise 34. Prove Thm. 11.5.

Thm. 11.5 may be used to generate new singular sols. by successive space-differentiations of Kelvin's solution.

<u>Def.</u> 11.3. Let $\mathfrak{X} = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and let $\mathfrak{s}^{k}(\mathbf{x}, \boldsymbol{\xi})$ be the normalized Kelvin-states of Def. 11.2. We call the nine states

$$\mathfrak{s}^{kl}(\mathbf{x}, \boldsymbol{\xi}) = [\mathbf{u}^{kl}(\mathbf{x}, \boldsymbol{\xi}), \mathbf{E}^{kl}(\mathbf{x}, \boldsymbol{\xi}), \mathbf{S}^{kl}(\mathbf{x}, \boldsymbol{\xi})] \ \forall \ \mathbf{x} \in E - \{\boldsymbol{\xi}\}$$

defined by

$$\mathfrak{s}^{kl}(\mathbf{x}, \boldsymbol{\xi}) = \frac{\partial}{\partial x_l} \mathfrak{s}^{k}(\mathbf{x}, \boldsymbol{\xi}) \ \forall \ \mathbf{x} \in E - \{\boldsymbol{\xi}\} \tag{11.10}$$

states corresponding to force-doublets applied at $\boldsymbol{\xi}$ (and to the elastic constants μ, ν).

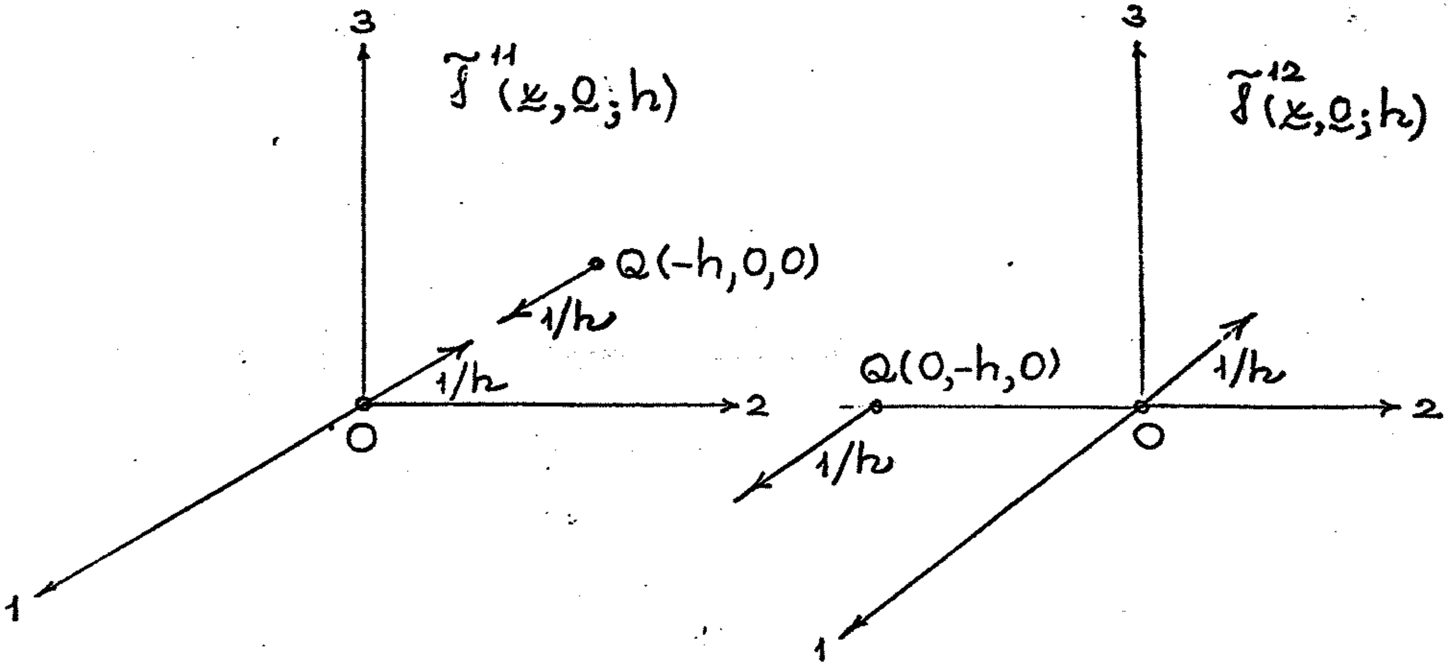
<u>Physical interpretation of the doublet states</u>

(11.10) $\Rightarrow$

$$\mathfrak{s}^{kl}(\mathbf{x}, \boldsymbol{\xi}) = \lim_{h \to 0} \left\{ \frac{1}{h} [\mathfrak{s}^{k}(\mathbf{x} + h\mathbf{e}_l, \boldsymbol{\xi}) - \mathfrak{s}^{k}(\mathbf{x}, \boldsymbol{\xi})] \right\}$$

Hence, by (11.5),

$$\left.\begin{aligned} \delta^{kl}(\boldsymbol{x},\boldsymbol{\xi}) &= \lim_{h\to 0}\tilde{\delta}^{kl}(\boldsymbol{x},\boldsymbol{\xi};h)\\ \tilde{\delta}^{kl}(\boldsymbol{x},\boldsymbol{\xi};h) &= \frac{1}{h}\left[\delta^{k}(\boldsymbol{x},\boldsymbol{\xi}-h\,\boldsymbol{e}_l)-\delta^{k}(\boldsymbol{x},\boldsymbol{\xi})\right] \quad (h>0)\end{aligned}\right\} \quad (11.11)$$



$\delta^{kk}(\boldsymbol{x},\boldsymbol{\xi})$ (no sum) ... force-doublet without moment, applied at $\boldsymbol{\xi}$ parallel to $\boldsymbol{e}_k$-axis.

$\delta^{kl}(\boldsymbol{x},\boldsymbol{\xi})$ $(k \neq l)$ ... force-doublet with moment about the $\boldsymbol{e}_m$-axis $(m\neq k, m \neq l)$ applied at $\boldsymbol{\xi}$ parallel to $\boldsymbol{e}_k$-axis.

(11.10), (11.6) $\Longrightarrow$

$$\delta^{kl}(\boldsymbol{x},\boldsymbol{\xi}) = \delta^{kl}(\boldsymbol{x}-\boldsymbol{\xi},\boldsymbol{0}) \quad \forall \quad \boldsymbol{x}\in E-\{\boldsymbol{\xi}\} \qquad (11.12)$$

From (11.10), (11.7), and Thm. 11.5 one obtains the following formulas for the displacement and stress components of $\mathfrak{s}^{k\ell}(\underline{x},\underline{0})$.

$$\left.\begin{aligned}
&\mathfrak{s}^{k\ell}(\underline{x},\underline{0}) : \quad \varphi^{k\ell} = -\frac{c\,\delta_{k\ell}}{r}\,, \quad \psi_i^{k\ell} = \frac{c\,\delta_{ki}x_\ell}{r^3}\,, \quad c = \frac{1}{8\pi(1-\nu)}\,,\\
&u_i^{k\ell} = -\frac{c}{2\mu r^3}\left[\frac{3x_k x_\ell x_i}{r^2} + (3-4\nu)\,\delta_{ki}x_\ell - \delta_{k\ell}x_i - \delta_{\ell i}x_k\right],\\
&\sigma_{ij}^{k\ell} = \frac{c}{r^3}\Big[\frac{15x_k x_\ell x_i x_j}{r^4} + \frac{3(1-2\nu)}{r^2}(\delta_{ki}x_\ell x_j + \delta_{kj}x_\ell x_i - \delta_{ij}x_k x_\ell)\\
&\qquad -\frac{3}{r^2}(\delta_{\ell i}x_k x_j + \delta_{\ell j}x_k x_i + \delta_{k\ell}x_i x_j)\\
&\qquad -(1-2\nu)(\delta_{ki}\delta_{\ell j} + \delta_{kj}\delta_{\ell i} - \delta_{ij}\delta_{k\ell})\Big].
\end{aligned}\right\} \quad (11.1$$

We turn now to an analogue for doublet-states of Thm. 11.3.

<u>Thm.</u> 11.6. (Properties of doublet-states). The doublet-states $\mathfrak{s}^{k\ell}(\underline{x}) \equiv \mathfrak{s}^{k\ell}(\underline{x},\underline{0})$ have the properties:

(a) $\quad \mathfrak{s}^{k\ell} = [\underline{u}^{k\ell}, \underline{\sigma}^{k\ell}, \underline{b}^{k\ell}] \in \mathcal{E}(\mu,\nu;\, E-\{\underline{0}\})$

(b) $\int_{\partial B_\rho(\mathbf{0})} \mathbf{s}^{kl}\, dA = \mathbf{0}$, $\int_{\partial B_\rho(\mathbf{0})} \mathbf{x} \wedge \mathbf{s}^{kl}\, dA = \varepsilon_{klm}\, \mathbf{e}_m \quad \forall\ \rho > 0$,

where $\mathbf{s}^{kl}$ is the traction vector of $\mathcal{S}^{kl}$ on the inner side of $\partial B_\rho(\mathbf{0})$;

(c) $\mathbf{u}^{kl}(\mathbf{x}) = O(r^{-2})$, $\boldsymbol{\sigma}^{kl}(\mathbf{x}) = O(r^{-3})$ as $r = |\mathbf{x}| \to 0$;

(d) $\mathbf{u}^{kl}(\mathbf{x}) = O(r^{-2})$, $\boldsymbol{\sigma}^{kl}(\mathbf{x}) = O(r^{-3})$ as $r \to \infty$.

<u>Proof</u>:

<u>Re (a)</u>: immediate from Def. 11.3, Thm 11.3, Thm 11.5

<u>Re (b)</u>: by computation based on (11.13) {note plausibility of (b) in view of (11.11)}

<u>Re (c), (d)</u>: by inspection of (11.13).

<u>Discussion</u>. Compare orders in (c), (d) with those in (c), (d) of Thm. 11.3. Note that $\mathcal{S}^{kk}$ has a self-equil. singularity at $\mathbf{0}$; hence (a), (b), (c), (d) cannot characterize $\mathcal{S}^{kl}$ uniquely! Explain. Observe that $\mathcal{S}^{kl} \neq \mathcal{S}^{lk}$ on $E - \{\mathbf{0}\}$; contrast to mechanics of rigid bodies.

It is useful to single out two particular linear combinations of doublet-states.

<u>Center of compression (dilatation) & center of rotation</u>

<u>Def</u>. 11.4. (Center of compression). We call

$$\delta^{\circ}(\underset{\sim}{x},\xi) = [\underset{\sim}{u}^{\circ}(\underset{\sim}{x},\xi), \underset{\sim}{\gamma}^{\circ}(\underset{\sim}{x},\xi), \underset{\sim}{\sigma}^{\circ}(\underset{\sim}{x},\xi)] \ \forall \ \underset{\sim}{x} \in E - \{\xi\}$$

defined by

$$\delta^{\circ}(\underset{\sim}{x},\xi) = \frac{4\pi(1-\nu)}{1-2\nu}\, \delta^{kk}(\underset{\sim}{x},\xi) \ \forall \ \underset{\sim}{x} \in E - \{\xi\} \quad (11.14)$$

the state corresponding to a center of compression at ξ

(11.14), (11.12) $\Rightarrow$

$$\delta^{\circ}(\underset{\sim}{x},\xi) = \delta^{\circ}(\underset{\sim}{x}-\xi, \underset{\sim}{0}) \ \forall \ \underset{\sim}{x} \in E - \{\xi\} \quad (11.15)$$

(11.14), (11.13) $\Rightarrow$

$$\left.\begin{aligned} \delta^{\circ}(\underset{\sim}{x},\underset{\sim}{0}):\quad & \varphi^{\circ} = \frac{-3}{2(1-2\nu)r}, \quad \psi^{\circ}_i = \frac{1}{2(1-2\nu)}\frac{x_i}{r^3}, \\ & u^{\circ}_i = -\frac{x_i}{2\mu r^3} \quad \text{or} \quad \underset{\sim}{u}^{\circ} = \frac{1}{2\mu}\nabla\left(\frac{1}{r}\right), \\ & \sigma^{\circ}_{ij} = \frac{1}{r^3}\left(\frac{3x_i x_j}{r^2} - \delta_{ij}\right), \\ & \vartheta^{\circ} = \nabla\cdot\underset{\sim}{u}^{\circ} = 0, \quad \underset{\sim}{w}^{\circ} = \frac{1}{2}\nabla\wedge\underset{\sim}{u}^{\circ} = \underset{\sim}{0}. \end{aligned}\right\} \quad (11.16)$$

Also from (11.16), Thm. 8.9, and (9.44) we obtain for $\mathcal{S}^{\circ}(\underset{\sim}{x},\underset{\sim}{0})$ in spherical coordinates (r,θ,ϕ):

$$\left.\begin{aligned} &\mathcal{S}^{\circ}(\underset{\sim}{x},\underset{\sim}{0}): \quad \varphi = \frac{1}{r}, \quad \underset{\sim}{\psi} = \underset{\sim}{0} \\ &\qquad u_r = -\frac{1}{2\mu r^2}, \quad u_\theta = u_\phi = 0, \\ &\sigma_{rr} = \frac{2}{r^3}, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -\frac{1}{r^3}, \quad \sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{\phi r} = 0. \end{aligned}\right\} \quad (11.1$$

Elaborate on derivation of (11.17).

<u>Discussion</u>. Note non-uniqueness of PN-potentials generating $\mathcal{S}^{\circ}(\cdot,\underset{\sim}{0})$. (11.17) $\Rightarrow$ polar symmetry of $\mathcal{S}^{\circ}(\cdot,\underset{\sim}{0})$. Explain. State properties of $\mathcal{S}^{\circ}(\cdot,\underset{\sim}{0})$ implied by Def. 11.4 and Thm. 11.6. In particular, $\mathcal{S}^{\circ}(\cdot,\underset{\sim}{0})$ has a self-equil. singularity at $\underset{\sim}{0}$. Return to non-uniqueness of conventional formulation of Kelvin's problem and interpret $\mathcal{S}^*$ on p. 40 as $\mathcal{S}^{\circ}(\cdot,\underset{\sim}{\xi})$.

<u>Def</u>. 11.5 (Center of rotation) We call

$$\bar{\mathcal{S}}^{k}(\underset{\sim}{x},\underset{\sim}{\xi}) = [\bar{\underset{\sim}{u}}^{k}(\underset{\sim}{x},\underset{\sim}{\xi}), \bar{\underset{\sim}{E}}^{k}(\underset{\sim}{x},\underset{\sim}{\xi}), \bar{\underset{\sim}{\sigma}}^{k}(\underset{\sim}{x},\underset{\sim}{\xi})] \;\forall\; \underset{\sim}{x} \in E - \{\underset{\sim}{\xi}\}$$

defined by

$$\bar{\tau}^k(\underset{\sim}{x},\underset{\sim}{\xi}) = \frac{1}{2}\varepsilon_{klm}\,\delta^{lm}(\underset{\sim}{x},\underset{\sim}{\xi}) \;\forall\; \underset{\sim}{x}\in E-\{\underset{\sim}{\xi}\} \tag{11.18}$$

the state corresponding to a center of rotation at $\underset{\sim}{\xi}$, parallel to the $\underset{\sim}{e}_k$-axis.

Explicit formulas for $\bar{\underset{\sim}{u}}^k(\underset{\sim}{x},\underset{\sim}{0})$ and $\bar{\underset{\sim}{\sigma}}^k(\underset{\sim}{x},\underset{\sim}{0})$ are easily obtained from (11.18), (11.13). The relevant properties of $\bar{\delta}^k(\underset{\sim}{x},\underset{\sim}{0})$ follow from (11.18), Thm. 11.6. In particular, by means of (1.11), one obtains

$$\int_{\partial B_\rho(\underset{\sim}{0})} \bar{\underset{\sim}{s}}^k(\underset{\sim}{x},\underset{\sim}{0})\,dA = \underset{\sim}{0}, \quad \int_{\partial B_\rho(\underset{\sim}{0})} \underset{\sim}{x}\wedge\bar{\underset{\sim}{s}}^k(\underset{\sim}{x},\underset{\sim}{0})\,dA = \underset{\sim}{e}_k \;\forall\; \rho>0. \tag{11.19}$$

Remarks.

The singular solutions discussed so far play a pivotal role in the theory of Green's functions appropriate to classical elastostatics, which goes back to Betti, Volterra, Lauricella. This theory furnishes integral representations for the the sol. to the basic boundaryvalue problems of elastostatics. See also the papers by Eubanks & Co. and Turteltaub & Co. cited earlier. For a generalization of Kelvin's sol. to <u>anisotropic media</u> see Fredholm, Acta Mathematica, <u>23</u>, 1 (1900).

Mention possibility of generating progressively <u>higher-order singular solutions</u> through successive space-differentiations of Kelvin's solution*.

Refer to <u>elastodynamic analogues</u> of the singular solutions we have deduced. Mention Stoke's problem (see Love). For a treatment of Green's function in linear elastodynamics (of homog., isotropic solids) see Wheeler & Co., ARMA, <u>31</u>, 1 (1968).

One may generate new solutions of the elastostatic field eqs. by continuous composition (infinite superposition) of elastic states with point-singularities. In preparation for an important example of this kind, we

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* All singular states obtained by more than one space-differentiation of $\mathcal{S}^k(\cdot,\xi)$ have self-equil. singularities. This follows directly from their behavior at infinity. Explain.

turn now to a theorem that is an integration-analogue of Thm. 11.5.

Thm. 11.7. Let R be an open region and let

$$\mathfrak{s}(\cdot,\tau) = [\underset{\sim}{u}(\cdot,\tau), \underset{\sim}{E}(\cdot,\tau), \underset{\sim}{S}(\cdot,\tau)] \in \mathcal{E}(\mu,\nu;R) \quad (a \le \tau \le b)$$

be a one-parameter family of elastic states $\ni$

$$\mathfrak{s}_{,i} \in \mathcal{C}(R \times I), \; I = [a,b].$$

Define $\hat{\mathfrak{s}} = [\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{E}}, \hat{\underset{\sim}{S}}]$ on R through

$$\hat{\mathfrak{s}}(\underset{\sim}{x}) = \int_a^b \mathfrak{s}(\underset{\sim}{x},\tau)\,d\tau \quad \forall\; \underset{\sim}{x} \in R, \text{ i.e.}$$

$$\hat{\underset{\sim}{u}}(\underset{\sim}{x}) = \int_a^b \underset{\sim}{u}(\underset{\sim}{x},\tau)\,d\tau, \quad \hat{\underset{\sim}{E}}(\underset{\sim}{x}) = \int_a^b \underset{\sim}{E}(\underset{\sim}{x},\tau)\,d\tau, \quad \hat{\underset{\sim}{S}}(\underset{\sim}{x}) = \int_a^b \underset{\sim}{S}(\underset{\sim}{x},\tau)\,d\tau.$$

Then,

$$\hat{\mathfrak{s}} = [\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{E}}, \hat{\underset{\sim}{S}}] \in \mathcal{E}(\mu,\nu;R).$$

Re proof.

Clearly, $\hat{\mathfrak{s}}$ has the required smoothness on R. To see that

$\hat{\mathfrak{s}}$ conforms to the field eqs. consider, for example,

$$\sigma_{ij,j}(\mathbf{x},\tau) = 0 \ \forall \ (\mathbf{x},\tau) \in R \times I \Rightarrow \int_a^b \sigma_{ij,j}(\mathbf{x},\tau)\,d\tau = 0 \ \forall \ \mathbf{x} \in R$$

$$\Rightarrow \frac{\partial}{\partial x_j} \int_a^b \sigma_{ij}(\mathbf{x},\tau)\,d\tau \ \forall \ \mathbf{x} \in R \Rightarrow \hat{\sigma}_{ij,j} = 0 \ \text{ on } R.$$

Remark. The conclusion in Thm. 11.7 evidently remains true if I is unbounded provided the improper integrals defining $\hat{\mathfrak{s}}$ are suitably convergent.

The Half-line of dilatation

Def. 11.6 (Half-line of dilatation). We call

$$\hat{\mathfrak{s}}^k = [\hat{\mathbf{u}}^k, \hat{\mathbf{E}}^k, \hat{\mathbf{S}}^k] \ \text{ on } E - L_k \ , \ L_k = \{\mathbf{x} \mid x_k = -r\} \ , \ r = |\mathbf{x}| \ ^{\S}$$

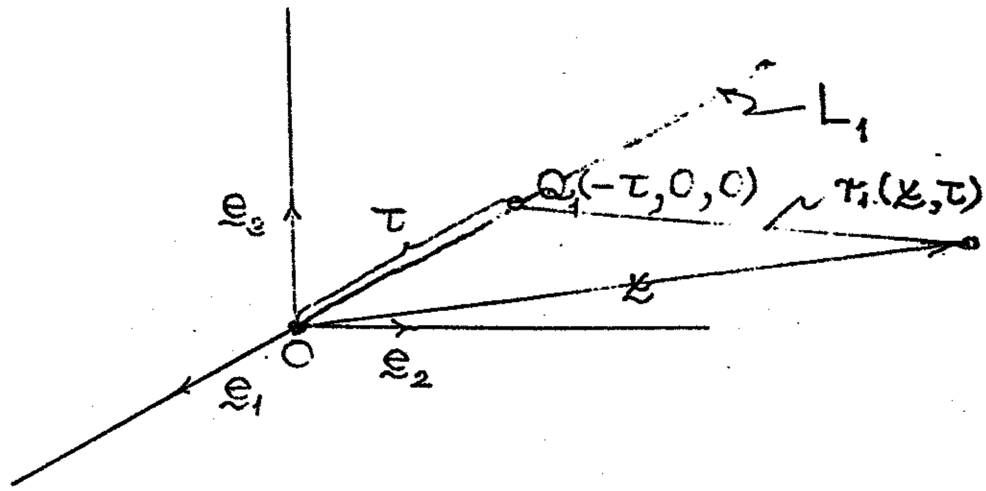
defined by

$$\hat{\mathfrak{s}}^k(\mathbf{x}) = -\int_0^\infty \mathring{\mathfrak{s}}(\mathbf{x}, -\tau \mathbf{e}_k)\,d\tau \ \ \forall \ \mathbf{x} \in E - L_k , \qquad (11.20)$$

the state corresponding to a half-line of dilatation on L

§ $L_k = \{\mathbf{x} \mid x_i = 0 \ (i \neq k), -\infty < x_k \le 0\}$ (non-pos. x_k-axi

Note physical meaning of $\hat{\mathfrak{s}}^k$: e.g. consider $k=1$.



(11.20), (11.15) $\Rightarrow$

$$\hat{\mathfrak{s}}^k(\mathbf{x}) = -\int_0^\infty \mathfrak{s}^\circ(\mathbf{x} + \tau \mathbf{e}_k, \mathbf{0})\, d\tau \quad \forall\ \mathbf{x} \in E - L_k \qquad (11.21)$$

(11.21), (11.16) $\Rightarrow$

$$\left.\begin{aligned} \hat{\mathbf{u}}^k(\mathbf{x}) &= -\frac{1}{2\mu}\int_0^\infty \nabla\left[\frac{1}{r_k(\mathbf{x},\tau)}\right] d\tau \quad \forall\ \mathbf{x} \in E - L_k \\ r_k(\mathbf{x},\tau) &= |\mathbf{x} + \mathbf{e}_k \tau| = \sqrt{(x_i + \delta_{ki}\tau)(x_i + \delta_{ki}\tau)} \\ &= \sqrt{x_i x_i + 2\delta_{ki} x_i \tau + \tau^2} \end{aligned}\right\} (11.22)$$

Carrying out the integration in (11.22) one finds

$$2\mu\, \hat{\mathbf{u}}^k(\mathbf{x}) = \nabla \log(r + x_k) \quad \forall\ \mathbf{x} \in E - L_k \qquad (11.23)$$

Consider

$$\hat{\varphi}^k(\mathbf{x}) \equiv \log(r + x_k) \qquad (11.24)$$

Direct computation confirms

$$\nabla^2 \hat{\varphi}^k \overset{\checkmark}{=} 0 \quad , \quad \hat{\varphi}^k_{,k} \overset{\checkmark}{=} \frac{1}{r} \text{ (no sum) on } E-L_k \qquad (11.25)$$

As suggested by (11.23), (11.24), (11.25) and Thm. 8,9, one finds that

$$\hat{\mathfrak{s}}^k: \quad \hat{\varphi}^k = \log(r+x_k) \ , \ \underset{\sim}{\psi}^k = \underset{\sim}{0} \qquad (11.26)$$

$$\hat{\mathfrak{s}}^k = [\underset{\sim}{u}^k, \underset{\sim}{\varepsilon}^k, \underset{\sim}{\sigma}^k] \in \mathcal{E}(\mu, \nu; E-L_k) \qquad (11.27)$$

From (11.26), (9.44) one obtains the following results for $\hat{\mathfrak{s}}^3$ in spherical coordinates (r, θ, ϕ).

$$\left.\begin{aligned} &\hat{\mathfrak{s}}^3(\underset{\sim}{x}): \quad \varphi = \log[r(1+\cos\theta)] \ , \ \psi_i = 0 \ , \\ &u_r = \frac{1}{2\mu r} \ , \ u_\theta = -\frac{\sin\theta}{2\mu r(1+\cos\theta)} \ , \ u_\phi = 0 \ , \\ &\sigma_{rr} = -\frac{1}{r^2} \ , \ \sigma_{\theta\theta} = \frac{\cos\theta}{r^2(1+\cos\theta)} \ , \ \sigma_{\phi\phi} = \frac{1}{r^2(1+\cos\theta)} \ , \\ &\sigma_{r\theta} = \frac{\sin\theta}{r^2(1+\cos\theta)} \ , \ \sigma_{\phi r} = \sigma_{\phi\theta} = 0 . \end{aligned}\right\} (11.2$$

Exercise 35. Deduce (11.23), (11.28) and confirm (11.25)

From (11.26), (11.25) follows

$$\nabla \cdot \hat{\underset{\sim}{u}}^k = \nabla \wedge \hat{\underset{\sim}{u}}^k = \underset{\sim}{0} \text{ on } E - L_k .$$

Note singularity of $\hat{\delta}^3$ on L_3. Clearly,

$$\left.\begin{aligned} &\hat{\underset{\sim}{u}}^k(\underset{\sim}{x}) = O(r^{-1}),\ \hat{\underset{\sim}{\sigma}}^k(\underset{\sim}{x}) = O(r^{-2}) \\ &\text{as } r \to 0 \text{ and as } r \to \infty \end{aligned}\right\} (11.29)$$

Compare (11.29) with corresponding behavior of $\delta^k(\underset{\sim}{x}, \underset{\sim}{0})$

(Cf. (c), (d) of Thm. 11.3).