12. Some elastostatic problems of the half-spac

Introduction. The equilibrium problems for the half-spa in particular the second and the mixed problem, are of great practical interest: relevant to the theory of foundations and soil-mechanics; pertinent to the study of local effects near the boundary of a body of arbitrary shape; essential prerequisites for the analysis of punch and contact problems.

Boussinesq's problem: the half-space under a concentrated surface load.

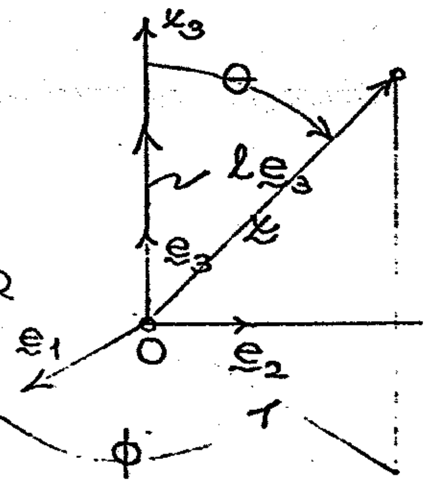
$$R = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in E,\ 0 < x_3 < \infty\} \qquad (12.$$

Note that R is an unbounded regular region (not an exterior region!)

(A) Concentrated normal load

$\underset{\sim}{f} = \underset{\sim}{0}$ on R

(r, θ, ϕ) ... standard spher. coords.



Guided by the properties of the Kelvin-state (Thm. 11.3) we adopt the following <u>direct formulation</u>.

Find $s = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}]$ subject to the requirements:

(a) $s = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}] \in \mathcal{E}(\mu, \nu, \overline{R} - \{\underset{\sim}{0}\})$, $\mu > 0$, $-1 < \nu < 1/2$

(b) $\displaystyle\int_{\Sigma_\rho} \underset{\sim}{s}\, dA = \ell \underset{\sim}{e}_3 \quad \forall\ \rho > 0, \quad \Sigma_\rho = \partial B_\rho(\underset{\sim}{0}) \cap R$

$\underset{\sim}{s}$ ... traction of s on the side of Σ_ρ that faces $\underset{\sim}{0}$

(c) $\underset{\sim}{s} = \underset{\sim}{0}$ on $\partial R - \{\underset{\sim}{0}\}$, i.e. $\sigma_{3i} = 0$ on $\partial R - \{\underset{\sim}{0}\}$

(d) $\underset{\sim}{u}(\underset{\sim}{x}) = O(r^{-1})$, $\underset{\sim}{S}(\underset{\sim}{x}) = O(r^{-2})$ as $r \to 0$

(e) $\underset{\sim}{u}(\underset{\sim}{x}) = o(1)$ as $r \to \infty$

<u>Discussion</u>. Clearly (a), (b), (c), (d) $\Rightarrow$

$$\int_{\Sigma_\rho} \underset{\sim}{x} \wedge \underset{\sim}{s}\, dA = \underset{\sim}{0} \quad \forall\ \rho > 0. \quad \text{(Explain)}.$$

It follows from a unique thm. for conc.-load probs. (Turteltaub & Co.)* that the above requirements determine s <u>uniquely. One can</u> show that the foregoing direct

* See p. 14.

formulation is equivalent to a limit formulation analogous to the one given for Kelvin's problem in Thm. 11.1. See also a subsequent exercise.

We now attempt to construct $\mathcal{S}$ by composition of basic singular solutions. Clearly, $\mathcal{S}$ is axisymmetric about the x_3-axis. The only sing. solutions in Ch. 11 possessing this symmetry and satisfying (a), (d), (e) are scalar multiples of:

$\overset{3}{\mathcal{S}}(\cdot, \mathbf{0})$ ... normalized Kelvin-state correspdg. to $\mathbf{e}_3$ at $\mathbf{0}$

$\hat{\mathcal{S}}^3$ .... half-line of dilatation along L_3

Recall from (11.8), (11.28) that

$$\overset{3}{\mathcal{S}}(\cdot, \mathbf{0}) : \quad \varphi = 0, \quad \psi_1 = \psi_3 = 0, \quad \psi_3 = -\frac{1}{8\pi(1-\nu)r}$$

$$\hat{\mathcal{S}}^3 \quad : \quad \varphi = \log[r(1+\cos\theta)], \; \psi_i = 0$$

Thus we are led to set

$$\mathcal{S}(\mathbf{x}) = \alpha \overset{3}{\mathcal{S}}(\mathbf{x}, \mathbf{0}) + \beta \hat{\mathcal{S}}^3(\mathbf{x}) \quad \forall \; \mathbf{x} \in \bar{\mathcal{R}} - \{\mathbf{0}\} \quad (\alpha, \beta \text{... const.}) \; (1$$

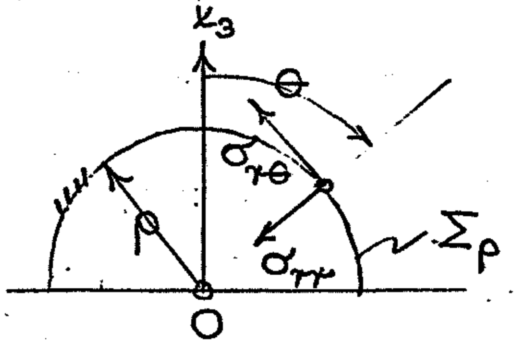
We now seek to determine α and $\beta \ni$ (b), (c) hold. In spherical coordinates (c) becomes

$$\sigma_{\theta r}(r,\pi/2) = \sigma_{\theta\theta}(r,\pi/2) = \sigma_{\theta\phi}(r,\pi/2) = 0 \quad (0<r<\infty)$$

or, from (11.8), (11.28),

$$\frac{\alpha(1-2\nu)}{8\pi(1-\nu)\cancel{r^2}} + \frac{\beta}{\cancel{r^2}} \doteq 0 \qquad (1)$$

Turn to (b).



In view of the rotational symmetry of ℓ, (b) becomes

$$\int_0^{\pi/2} [\sigma_{r\theta}(\rho,\theta)\sin\theta - \sigma_{rr}(\rho,\theta)\cos\theta]\, 2\pi\rho^2 \sin\theta \, d\theta = \ell$$

or, using (12.2), (11.8), (11.28),

$$\frac{\alpha}{2} + 2\pi\beta \doteq \ell \qquad (2)$$

(1), (2) $\Longrightarrow$

$$\alpha \doteq 4\ell(1-\nu), \quad \beta \doteq -\frac{(1-2\nu)\ell}{2\pi} \qquad (12.3$$

$\alpha c = \frac{\ell}{2\pi}$

From (12.2), (12.3), (11.8), (11.28) finally follows:

$\ell(z)$: $\varphi \doteq -\frac{(1-2\nu)\ell}{2\pi} \log[r(1+\cos\theta)]$, $\psi_1 \doteq \psi_2 \doteq 0$, $\psi_3 \doteq -\frac{\ell}{2\pi r}$,

$$u_r \doteq -\frac{\ell}{4\pi\mu r}[1-2\nu-4(1-\nu)\cos\theta],$$

$$u_\theta \doteq -\frac{\ell}{4\pi\mu r}\left[3-4\nu-\frac{1-2\nu}{1+\cos\theta}\right]\sin\theta, \quad u_\phi \doteq 0,$$

$$\sigma_{rr} \doteq \frac{\ell}{2\pi r^2}[1-2\nu-2(2-\nu)\cos\theta], \quad \sigma_{\theta\theta} \doteq \frac{(1-2\nu)\ell}{2\pi r^2}\frac{\cos^2\theta}{1+\cos\theta},$$

$$\sigma_{\phi\phi} \doteq \frac{(1-2\nu)\ell}{2\pi r^2}\frac{\cos^2\theta+\cos\theta-1}{1+\cos\theta}, \quad \sigma_{r\theta} \doteq \frac{(1-2\nu)\ell}{2\pi r^2}\frac{\cos\theta\sin\theta}{1+\cos\theta}$$

The shear stresses $\sigma_{\phi r}$, $\sigma_{\phi\theta}$ vanish identically (rot. sym

Mention conc. normal load at a point of curved bdy. Singularity has same order but is not the same as abo

See :

Rosenthal & Co., J. Appl. Mech. 19, p. 413 (1952)

Eubanks & Co., Proc., 2nd U.S. Nat. Cong. Appl. Mech. (

Neidhardt & Co., J. Appl. Mech. 23, p. 541 (1956)

(B) Concentrated tangential load

Here (b) is to be replaced by

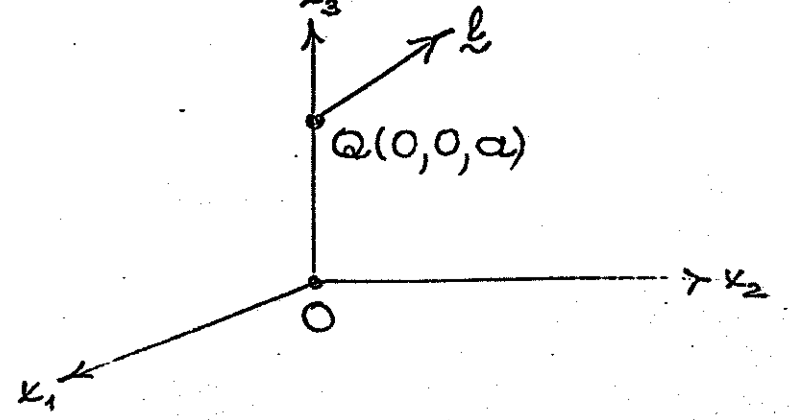
(b′) $$\int_{\Sigma_\rho} \underset{\sim}{s}\, dA = \ell\, \underset{\sim}{e}_1 \quad \forall\ \rho > 0$$

Problem is no longer axisymmetric. Its solution is found to be generated by:

$$\left.\begin{aligned} \varphi &= \frac{(1-2\nu)^2 \ell}{4\pi(1-\nu)} \frac{x_1}{r + x_3}, \\ \psi_1 &= -\frac{\ell}{4\pi(1-\nu)} \frac{1}{r}, \quad \psi_2 = 0, \quad \psi_3 = -\frac{(1-2\nu)\ell}{4\pi(1-\nu)} \frac{x_1}{r(r+x_3)} \end{aligned}\right\} \quad (12$$

~~(11.5) correspond to a linear combination of $\delta^1(\cdot, 0)$ and a half-line of rotation along L_3.~~

Mindlin's problem: concentrated load at an interior point of half-space (generalizes Kelvin's & Boussinesq's problems).



See Mindlin, Physics, 7, p.195 (1936)

Proc., 1st Midwestern Conf. Solid Mech (195

Describe Mindlin's original and subsequent approa

For $\underset{\sim}{\ell} = \ell\, \underset{\sim}{e}_3$ Mindlin's solution is generated by th PN-potentials:

$$\left.\begin{aligned}
&\varphi = \frac{\ell}{8\pi}\left[\frac{a}{(1-\nu)r_1} + \frac{(3-4\nu)a}{(1-\nu)r_2} - 4(1-2\nu)\log(r_2+x_3+a)\right],\\
&\psi_1 = \psi_2 = 0\,,\quad \psi_3 = -\frac{\ell}{8\pi(1-\nu)}\left[\frac{1}{r_1} + \frac{3-4\nu}{r_2} + \frac{2a(x_3+a)}{r_2^3}\right],\\
&r_1 = \sqrt{x_1^2+x_2^2+(x_3-a)^2}\,,\quad r_2 = \sqrt{x_1^2+x_2^2+(x_3+a)^2}
\end{aligned}\right\} (1$$

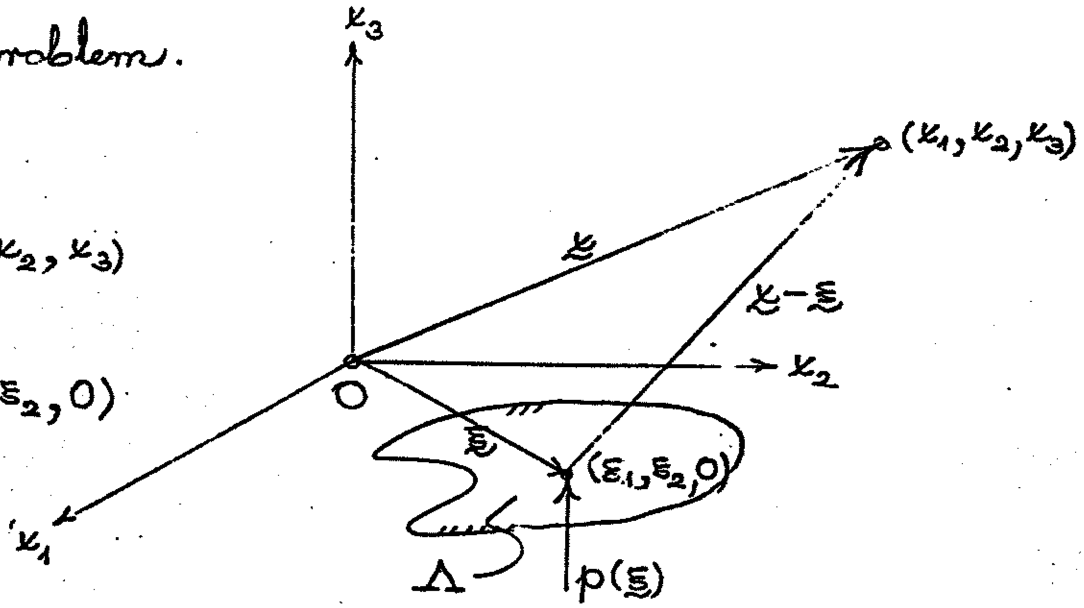
Remark on limit as $a \to 0$.

Half-space under distributed normal loads

Let R again be the upper half-space given by (12.1)

Λ ... bounded, closed subregion of ∂R

Describe problem.



$\underset{\sim}{x} = (x_1, x_2, x_3)$

$\underset{\sim}{\xi} = (\xi_1, \xi_2, 0)$

Suppose:

$$p \in \mathcal{C}^1(\partial R) \;, \quad p(\underset{\sim}{x}) = 0 \;\; \forall \; \underset{\sim}{x} \in \partial R - \Lambda \quad . \tag{12.7}$$

We seek $s = [\underset{\sim}{u}, \underset{\sim}{\tau}, \underset{\sim}{\sigma}]$ subject to:

(a) $s = [\underset{\sim}{u}, \underset{\sim}{\tau}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu; R)$; $\mu > 0$, $-1 < \nu < 1/2$

(b) $\sigma_{33}(\underset{\sim}{x}) \to -p(\underset{\sim}{y})$, $\sigma_{31}(\underset{\sim}{x}) \to 0$, $\sigma_{32}(\underset{\sim}{x}) \to 0$ as $\underset{\sim}{x} \to \underset{\sim}{y}$ $\forall \; \underset{\sim}{y} \in \partial R$

(c) $\underset{\sim}{u}(\underset{\sim}{x}) = o(1)$ as $r \to \infty$ $(\underset{\sim}{x} \in R)$

<u>Remarks</u>. Note that (12.7), (a), (b) $\Rightarrow \sigma_{3i} \in \mathcal{C}(\bar{R})$. But s has not been required to be cont. on $\bar{R}$. Explain. Also note weakness of (c). Conditions (a), (b), (c) uniquely

characterize $\mathbf{s}$ according to a special uniqueness theorem for the half-space: Turteltaub & Co., ARMA, 24, 3 (196

We now construct $\mathbf{s}$ by the method of Green's functions (influence functions). Let $\hat{\mathbf{s}}(\mathbf{x},\boldsymbol{\xi})$ be the solution of Boussinesq's problem for R corresponding to a concentrated load $\boldsymbol{\ell} = \mathbf{e}_3$ at $\boldsymbol{\xi} \in \partial R$. Then by (12.4), Thm

$$\hat{\mathbf{s}}(\mathbf{x},\boldsymbol{\xi}) = \hat{\mathbf{s}}(\mathbf{x}-\boldsymbol{\xi},\mathbf{0}) \quad \forall\, \mathbf{x} \in \bar{R} - \{\boldsymbol{\xi}\}$$

$$\hat{\mathbf{s}}(\mathbf{x},\boldsymbol{\xi}) : \quad \hat{\varphi} = -\frac{1-2\nu}{2\pi}\log(r + x_3), \quad \hat{\psi}_1 = \hat{\psi}_2 = 0, \quad \hat{\psi}_3 = \frac{-1}{2\pi r}$$

$$r = |\mathbf{x}-\boldsymbol{\xi}| = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}$$

Define formally

$$\mathbf{s}(\mathbf{x}) = [\mathbf{u}(\mathbf{x}), \mathbf{T}(\mathbf{x}), \mathbf{S}(\mathbf{x})] = \int_{\Lambda} p(\boldsymbol{\xi})\, \hat{\mathbf{s}}(\mathbf{x},\boldsymbol{\xi})\, dA_{\boldsymbol{\xi}} \quad \forall\, \mathbf{x} \in \bar{R} \qquad (12$$

(12.8), (12.9), Thm. 8.9 $\Rightarrow$

$$\mathbf{s}(\mathbf{x}) : \quad \varphi(\mathbf{x}) = \int_{\Lambda} p(\boldsymbol{\xi})\, \hat{\varphi}(\mathbf{x},\boldsymbol{\xi})\, dA_{\boldsymbol{\xi}}, \quad \boldsymbol{\psi}(\mathbf{x}) = \int_{\Lambda} p(\boldsymbol{\xi})\, \hat{\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\xi})\, dA_{\boldsymbol{\xi}} \qquad (12$$

$$\forall\, \mathbf{x} \in R$$

(12.8), (12.10) $\Rightarrow$

$$\left.\begin{aligned}
&\varphi = -\frac{1-2\nu}{2\pi}\,\chi \;,\quad \psi_1 = \psi_2 = 0\;,\quad \psi_3 = -\frac{1}{2\pi}\,U \quad \text{on } R \\
&\chi(\underset{\sim}{x}) = \int_\Lambda p(\underset{\sim}{\xi}) \log(r + x_3)\, dA_{\underset{\sim}{\xi}}\;,\quad U(\underset{\sim}{x}) = \int_\Lambda \frac{p(\underset{\sim}{\xi})}{r}\, dA_{\underset{\sim}{\xi}} \\
&r = r(\underset{\sim}{x},\underset{\sim}{\xi}) = |\underset{\sim}{x} - \underset{\sim}{\xi}| = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}
\end{aligned}\right]\text{(1.}$$

U ... Newtonian pot. of a uniform disk of mass density p occupying Λ

χ ... "Boussinesq's three-dimensional logarithmic pot." of the same disk

One verifies trivially (explain) that

$$U \in \mathcal{C}^\infty(R),\ \chi \in \mathcal{C}^\infty(R),\ \nabla^2 U = 0,\ \nabla^2\chi = 0,\ U = \frac{\partial \chi}{\partial x_3} \text{ on } R \quad \text{(12}$$

The displacements and stresses of $\mathcal{S}$ now follow from (12.11) and Thm. 8.9. If Greek subscripts have the range (1,2) one obtains for $\mathcal{S}$ on R the following results.

$$
\mathcal{S}: \quad
\left.
\begin{aligned}
u_\alpha &= -\frac{1}{4\pi\mu}\left[x_3 U_{,\alpha} + (1-2\nu)\chi_{,\alpha}\right], \\
u_3 &= \frac{1}{4\pi\mu}\left[2(1-\nu)U - x_3 U_{,3}\right], \\
\sigma_{\alpha\alpha} &= \frac{1}{2\pi}\left[2\nu U_{,3} - x_3 U_{,\alpha\alpha} - (1-2\nu)\chi_{,\alpha\alpha}\right] \quad (\text{no sum}), \\
\sigma_{33} &= \frac{1}{2\pi}\left(U_{,3} - x_3 U_{,33}\right), \\
\sigma_{3\alpha} &= -\frac{1}{2\pi} x_3 U_{,3\alpha}, \quad \sigma_{12} = -\frac{1}{2\pi}\left[(1-2\nu)\chi_{,12} - x_3 U_{,12}\right]
\end{aligned}
\right\} \text{(1}
$$

We confirm now that $\mathcal{S}$ given by (12.13) satisfies (a),(b)

Re (a). Immediate from (12.12) and Thm 8.9.

Re (c). Obvious from (12.11), (12.13)

Re (b). Consider first $\mathbf{y} \in \partial R - \Lambda$. From (12.11),

$$
U(\mathbf{x}) = O(1),\ U_{,i}(\mathbf{x}) = O(1),\ U_{,ij}(\mathbf{x}) = O(1) \text{ as } \mathbf{x} \to \mathbf{y}\ \forall\ \mathbf{y} \in \partial R
$$

Also, by (12.11),

$$
U_{,3}(\mathbf{x}) = -x_3 \int_\Lambda \frac{p(\boldsymbol{\xi})}{r^3}\, dA_{\boldsymbol{\xi}} \to 0 \quad \text{as } \mathbf{x} \to \mathbf{y}\ \forall\ \mathbf{y} \in \partial R - \Lambda
$$

Hence, from (12.13),

$$
\sigma_{3i}(\mathbf{x}) \to 0 \quad \text{as } \mathbf{x} \to \mathbf{y}\ \forall\ \mathbf{y} \in \partial R - \Lambda
$$

Further, from the theory of Newtonian potentials of surface distributions (see Kellogg, chapter 5) one has here:

$$\left.\begin{array}{l} ?\quad U_{,3i}(\underset{\sim}{x}) = O(1) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \;\forall\; \underset{\sim}{y} \in \partial R \\ \checkmark\quad U_{,3}(\underset{\sim}{x}) \to -2\pi p(\underset{\sim}{y}) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \;\forall\; \underset{\sim}{y} \in \partial R \quad (\underset{\sim}{x} \in R) \end{array}\right\} (*)$$

(*), (12.13) $\Rightarrow$ (b).

Exercise 36. Obtain the solution (12.4) for a half-space under a conc. normal surface load through a limit process analogous to that underlying Thm. 11.1.

Discontinuous distributed normal load (generalized pro

Suppose (12.7) is replaced by

$$p \in \mathcal{C}^1(\mathring{\Lambda}),\ p \text{ integrable on } \Lambda,\ p = 0 \text{ on } \partial R - \Lambda \qquad (12.1$$

In the original formulation replace (b) by

(b′) $\quad \sigma_{33}(\underset{\sim}{x}) \to -p(\underset{\sim}{y})\,,\ \sigma_{3\alpha}(\underset{\sim}{x}) \to 0$ as $\underset{\sim}{x} \to \underset{\sim}{y}\ \forall\ \underset{\sim}{y} \in \partial R - \partial\Lambda$

Then $\mathscr{S}$ given by (12.11), (12.13) satisfies (a), (b′), (c).
We adopt $\mathscr{S}$ as the solution of the generalized problem
Remark on limit-definition of sol. to modified problem

Discussion

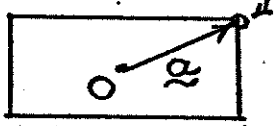
(12.11), (12.13) reduces sol. of generalized prob. to determination of U, χ for given Λ and given p on Λ. Clo[sed] representations are available for several important special cases. Mention transform approach to axisymm[etric] problem.

Love, Trans., Roy. Soc. London, Ser. A, 228, p. 377 (1929)

Λ circular, $p = \text{const}$ on Λ (elliptic integrals)

Λ rectangular, $p = \text{const}$ on Λ

$$\tau_{12} = -\frac{p(1-2\nu)}{2\pi}\log(|\underset{\sim}{x} - \ldots$$
$$+ O(1) \text{ as } \ldots$$

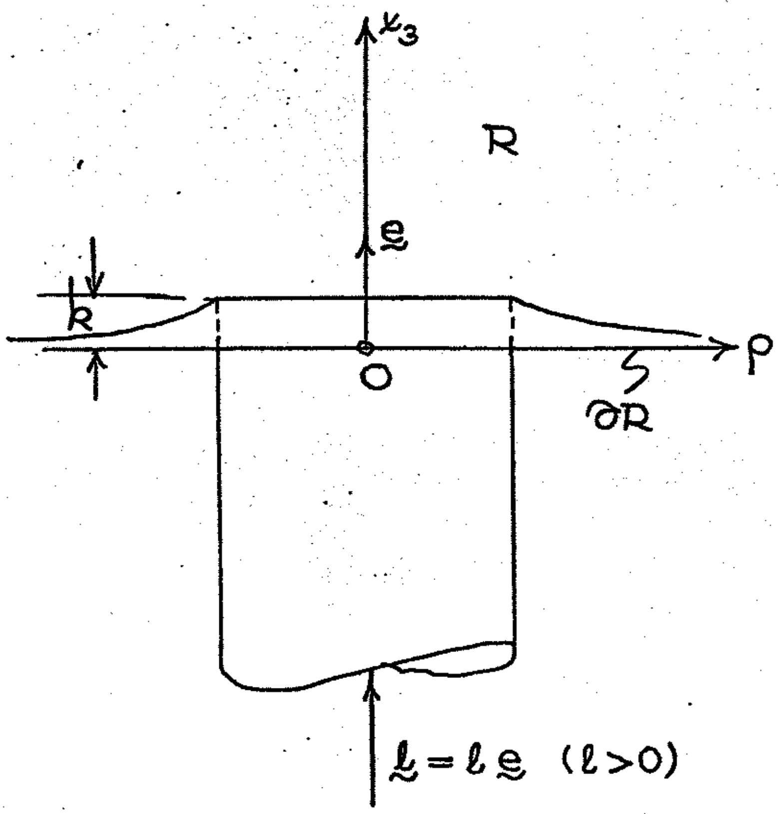


Huber: Λ circular, p hemi-spheroidal
Remark on half-space under tang. distributed loads
Exercise 37. Let $\Lambda = \{(x_1, x_2) \mid 0 \le x_1^2 + x_2^2 \le a^2\}$, $p = p_0$ on Λ
Show that Boussinesq's sol. (12.11), (12.13) predicts

$$u_3(\underset{\sim}{0}) = \frac{2(1-\nu^2)p_0 a}{\eta}, \quad \sigma_{33}(\underset{\sim}{e}_3 z) = p_0\left[\frac{z^3}{(a^2+z^2)^{3/2}} - 1\right]$$

The problem of the smooth rigid circular punch

$$\left.\begin{aligned} R &= \{\underset{\sim}{x} \mid \underset{\sim}{x} \in E,\ 0 < x_3 < \infty\} \\ \Lambda &= \{\underset{\sim}{x} \mid \underset{\sim}{x} \in \partial R,\ x_1^2 + x_2^2 \le a^2\} \end{aligned}\right\} \quad (12.1$$



Find $\mathcal{S}$ subject to:

(a) $\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\gamma}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{0}; R)$

(b) $u_3(\underset{\sim}{x}) \to k$ as $\underset{\sim}{x} \to \underset{\sim}{y}\ \forall\ \underset{\sim}{y} \in \mathring{\Lambda}$ $(k > 0$, constant$)$

$\sigma_{33}(\underset{\sim}{x}) \to 0$ as $\underset{\sim}{x} \to \underset{\sim}{y}\ \forall\ \underset{\sim}{y} \in \partial R - \Lambda$

$\sigma_{3\alpha}(\underset{\sim}{x}) \to 0$ as $\underset{\sim}{x} \to \underset{\sim}{y}\ \forall\ \underset{\sim}{y} \in \partial R - \partial\Lambda$

(c) $\underset{\sim}{u}(\underset{\sim}{x}) = o(1)$ as $r \to \infty$

Note that this is an example of a mixed-mixed pb

Method of solution: assume δ is given by (12.11), (12.13) and seek to determine p so as to satisfy the displacement boundary-cond. in (b).

Recall from (12.13),

$$u_3(\underset{\sim}{x}) = \frac{1}{4\pi\mu}\left[2(1-\nu)U(\underset{\sim}{x}) - x_3 U_{,3}(\underset{\sim}{x})\right] \quad \forall\ \underset{\sim}{x} \in R \qquad (1)$$

From potential theory (see previous discussion),

$$U \in \mathcal{C}(\bar{R}) \,, \quad U_{,3}(\underset{\sim}{x}) = O(1) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \ \forall\ \underset{\sim}{y} \in \mathring{\Lambda} \qquad (2)$$

Hence* (a), (b), (c) are met provided

$$\frac{1-\nu}{2\pi\mu} U(\underset{\sim}{x}) = k \quad \forall\ \underset{\sim}{x} \in \mathring{\Lambda} . \qquad (3)$$

or, by (12.11),

$$\left.\begin{aligned} U(\underset{\sim}{x}) &= \int_{\Lambda} \frac{p(\underset{\sim}{\xi})}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\, d\Lambda_{\underset{\sim}{\xi}} = \frac{2\pi\mu k}{1-\nu} \quad \forall\ \underset{\sim}{x} \in \mathring{\Lambda} \\ |\underset{\sim}{x}-\underset{\sim}{\xi}| &= \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2} \end{aligned}\right\} \quad (1\!\!\!\!$$

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* provided p is sufficiently well behaved.

Note that (12.16) is a singular integral equation for the unknown pressure distribution. Interpret the pro of solving (12.16) as an inverse pb. in pot. theory.

The solution of (12.16) is found to be

$$\left.\begin{aligned} p(\underline{x}) = q(\rho) = \frac{2k\mu}{\pi(1-\nu)}\frac{1}{\sqrt{a^2-\rho^2}} \quad (0 \le \rho < a) \ \forall\, \underline{x} \in \mathring{\Lambda}, \\ \rho = \sqrt{x_1^2 + x_2^2}. \end{aligned}\right\} \quad (12.$$

Exercise 38. Show that p given by (12.17) satisfies (12.16)

To relate p to the total punch load ℓ, observe that

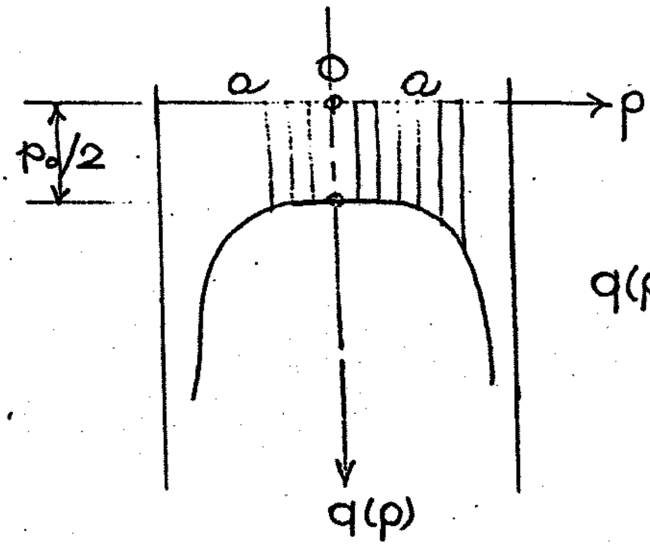
$$\ell = \int_\Lambda p\, dA = \int_0^{2\pi}\int_0^a q(\rho)\,\rho\, d\rho\, d\phi = \frac{4k\mu}{1-\nu}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}}, \text{ so that}$$

$$\ell = \frac{4k\mu a}{1-\nu} \quad , \quad k = \frac{\ell(1-\nu)}{4\mu a} \qquad (12.1$$

(12.17),(12.18) $\Rightarrow$

$$\left.\begin{aligned} q(\rho) = \frac{p_0 a}{2\sqrt{a^2-\rho^2}} \quad , \quad p_0 = \frac{\ell}{\pi a^2} \ \text{(average pressure)} \\ q(0) = p(\underline{0}) = \frac{p_0}{2} \end{aligned}\right\} \quad (12.$$

<u>Discussion</u>. Describe pressure distribution

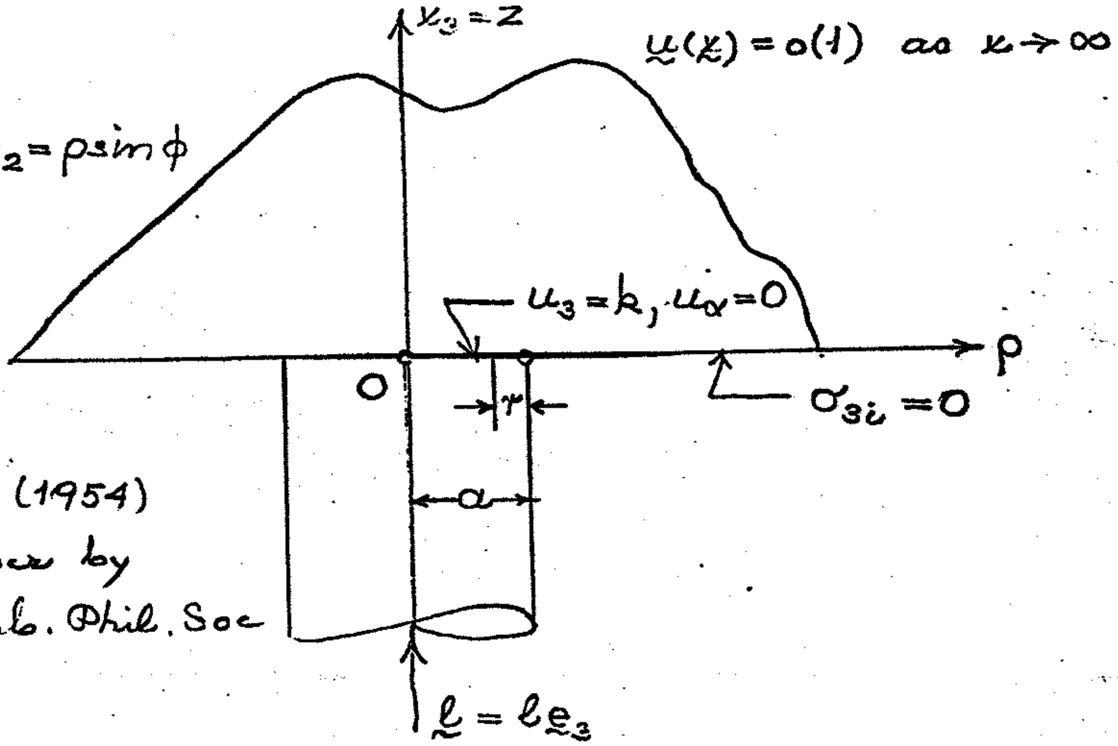


$q(\rho) \to \infty$ as $\rho \to a$

Compare with Winkler-foundation. Remark on complet of solution (Huber). Boussinesq solved (12.16) by a trick. Mention constructive solution via transforms (see Gre & Zerna, p. 172 et seq., Sneddon, p. 455 et seq.). Sned deals with arb. smooth axisymm. rigid punch via Love's stress function, Hankel transform, dual integra eqs. (in particular spherical & conical punch). Refe to generalization to asymmetric punch load, arb Λ, multiple punches, etc. See the books of Galin, Shtaerman, Lurie. Mention rough circular punch (singular mixed prob.)

Re bonded-punch problem

$x_1 = \rho\cos\phi$, $x_2 = \rho\sin\phi$
$x_3 = z$



Mossakovski (1954)
See survey paper by
Keer, Proc. Camb. Phil. Soc
<u>63</u> (1967)

$$\sigma_{zz}(\rho, 0) = -q(\rho) \sim \frac{-\ell(1+\kappa)}{2\pi\sqrt{2\kappa a}}\, r^{-1/2} \cos[\beta \log(r/2a)]$$

$$\sigma_{z\rho}(\rho, 0) \sim \frac{\ell(1+\kappa)}{2\pi\sqrt{2\kappa a}}\, r^{-1/2} \sin[\beta \log(r/2a)] \quad \text{as } r \to$$

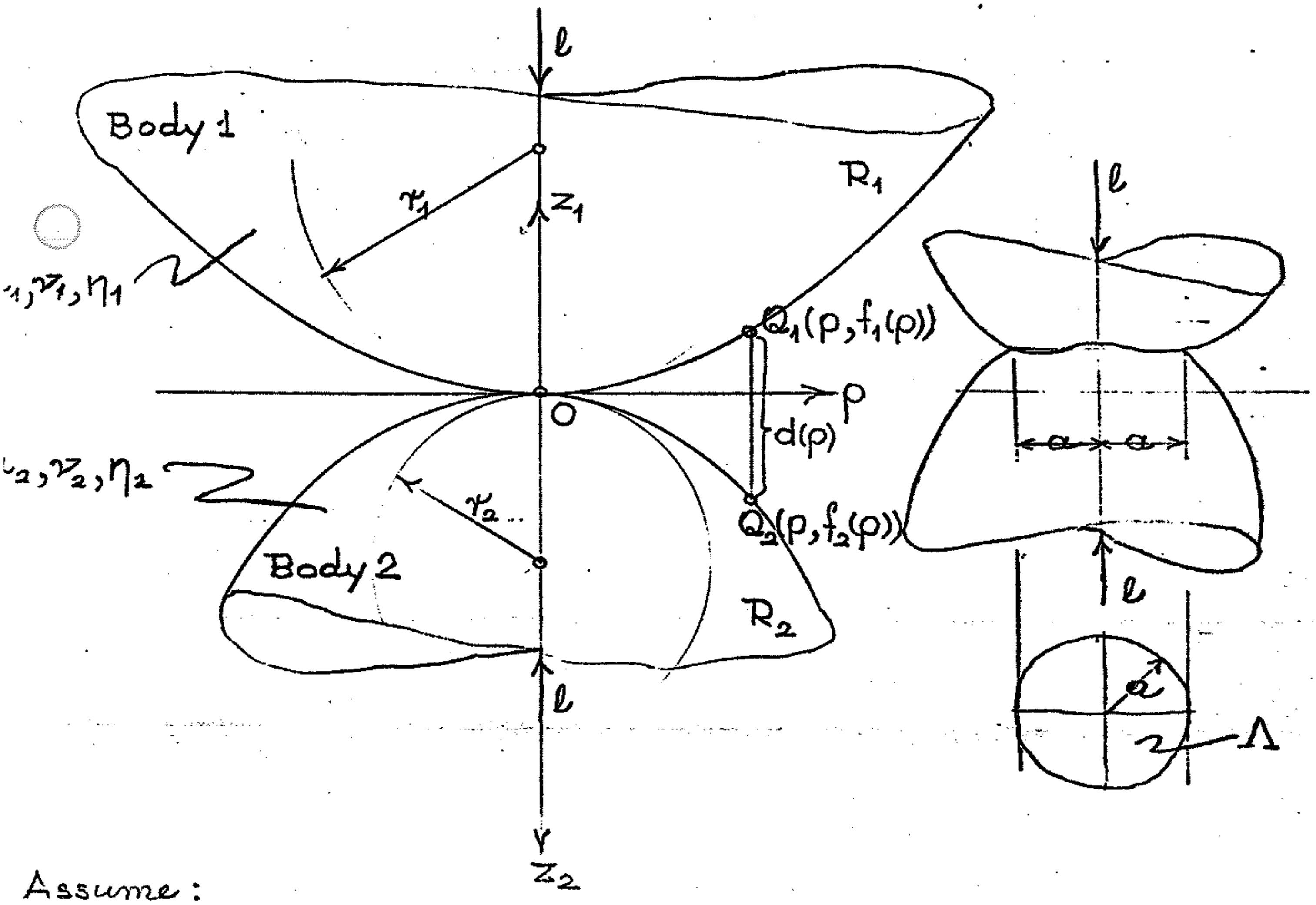
where $r = a - \rho$ $(\rho < a)$, $\kappa = 3 - 4\nu$, $\beta = \frac{1}{2\pi}\log\kappa$

Discuss oscillatory character of singularities if $\nu \neq \frac{1}{2}$. Ex
difficulty with problem of "rough" punch. Mention
two-dimensional analogue.

The Hertz-problem : contact of elastic bodies

ref. Timo-Goodier, 3rd Ed., art. 140., Love, art 138.

Objective : study of local deformations and stresses in the vicinity of the contact region of two elastic bodies with possibly distinct elastic properties. We shall deal only with axisymmetric case in which the two bodies are locally solids of revolution. No friction.



Assume :

$$f_n \in C^3(I)\ , \ I = [0, \rho_0]\ , \ f_n(0) = f_n'(0) = 0 \quad (n = 1,2) \qquad (11.20)$$

From Taylor expansion,

$$f_n(\rho) = f_n(0) + \rho f_n'(0) + \frac{\rho^2}{2} f_n''(0) + O(\rho^3) = \frac{\rho^2}{2} f_n''(0) + O(\rho^3) \text{ as}$$

Curvature of meridians at contact point

$$\varkappa_n(0) = \frac{1}{r_n} = \frac{f_n''(0)}{\{1+[f_n'(0)]^2\}^{3/2}} = f_n''(0)$$

Hence, using (12.20),

$$f_n(\rho) = \frac{\rho^2}{2r_n} + O(\rho^3) \quad \text{as } \rho \to 0$$

We take

$$f_n(\rho) = \frac{\rho^2}{2r_n} \quad (0 \le \rho \le \rho_0), \; (n=1,2) \qquad (12.2$$

(12.21) $\Rightarrow$

$$d(\rho) = f_1(\rho) + f_2(\rho) = \beta\rho^2 \quad (0 \le \rho \le \rho_0), \; \beta = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \frac{r_1 + r_2}{2r_1 r_2} \quad (1$$

$d(\rho)$ ... vertical distance between meridions before deforma

$\underset{\sim}{u}^{(n)} = [u_\rho^{(n)}, u_\phi^{(n)}, u_z^{(n)}]$ $(n=1,2)$ ... displacement fields

Because of rotational symm.,

$$\underset{\sim}{u}^{(n)} = \underset{\sim}{u}^{(n)}(\rho, z_n), \quad u_\phi^{(n)} = 0 \quad \text{on } \overline{R}_n$$

Let

$$\Delta(\rho) = d(\rho) + \overset{(1)}{w_z}(\rho, z_1) + \overset{(2)}{w_z}(\rho, z_2)\,,\quad z_n = f_n(\rho) \quad (0 \le \rho \le \rho$$

$\Delta(\rho) = 0 \quad (0 \le \rho \le a)$ ... contact condition

$\Delta(\rho) > 0 \quad (a < \rho \le \rho_0)$ ... separation condition

Thus, from (12.22),

$$\overset{(1)}{w_z}(\rho, z_1) + \overset{(2)}{w_z}(\rho, z_2) = -\beta\rho^2\,,\quad z_n = f_n(\rho) \quad (0 \le \rho \le a) \quad (12.2$$

$$\overset{(1)}{w_z}(\rho, z_1) + \overset{(2)}{w_z}(\rho, z_2) > -\beta\rho^2\,,\quad z_n = f_n(\rho) \quad (a < \rho \le \rho_0) \quad (12.$$

From here on Hertz replaces the two bodies by elastic half-spaces occupying

$$R_n = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in E,\ 0 < z_n < \infty\} \quad (n = 1,2) \qquad (12.$$

With this assumption the original problem is reducil to the following simultaneous mixed-mixed prob for the two semi-infinite elastic solids.

Find a and $\overset{(n)}{\mathcal{S}} = [\overset{(n)}{\underset{\sim}{u}}, \overset{(n)}{\underset{\sim}{\varepsilon}}, \overset{(n)}{\underset{\sim}{\sigma}}]$ $(n=1,2)$ subject to

(a) $\mathcal{S}^{(n)} = [\underset{\sim}{u}^{(n)}, \underset{\sim}{\varepsilon}^{(n)}, \underset{\sim}{\sigma}^{(n)}] \in \mathcal{E}(\mu_n, \nu_n; \overline{\mathcal{R}}_n)$, $\mu_n > 0, -1 < \nu_n < 1$

$$\underset{\sim}{u}^{(n)} = [u_\rho^{(n)}(\rho, z_n), 0, u_z^{(n)}(\rho, z_n)]$$

(b) $u_z^{(1)}(\rho,0) + u_z^{(2)}(\rho,0) = -\beta\rho^2 \quad (0 \le \rho \le a) \qquad (*)$

$u_z^{(1)}(\rho,0) + u_z^{(2)}(\rho,0) > -\beta\rho^2 \quad (a < \rho < \infty) \qquad (**)$

$\sigma_{zz}^{(1)}(\rho,0) = \sigma_{zz}^{(2)}(\rho,0) \quad (0 \le \rho \le a)$

$\sigma_{zz}^{(n)}(\rho,0) = 0 \quad (a < \rho < \infty), \; \sigma_{z\rho}^{(n)}(\rho,0) = 0 \quad (0 \le \rho < \infty)$

(c) $u_\rho^{(n)}(\rho, z_n) = o(1), \; \underset{\sim}{\sigma}^{(n)}(\rho, z_n) = o(1)$ as $\rho^2 + z_n^2 \to \infty$

$u_z^{(n)}(\rho, z_n) = -\alpha_n + o(1)$ as $\rho^2 + z_n^2 \to \infty, \; \alpha = \alpha_1 + \alpha_2$

(d) $\int_\Lambda p \, dA = \ell$ where $p(\underset{\sim}{x}) = -\sigma_{zz}^{(n)}(\rho,0) \;\; \forall \; \underset{\sim}{x} \in \Lambda$

$$\Lambda = \{(x_1, x_2) \mid 0 \le x_1^2 + x_2^2 \le a^2\}$$

Note that

$\alpha = \alpha_1 + \alpha_2 = -[u_z^{(1)}(\rho,\infty) + u_z^{(2)}(\rho,\infty)] \ldots$ "distance of app

Method of solution: Assume

$$s^{(n)} = [\underline{u}^{(n)}, \underline{\tau}^{(n)}, \underline{\sigma}^{(n)}] = \bar{s}^{(n)} + [\mathring{\underline{u}}^{(n)}, \underline{0}, \underline{0}] \text{ on } \bar{R} \qquad (12.$$

where $\mathring{u}_\rho^{(n)} = \mathring{u}_\phi^{(n)} = 0$, $\mathring{u}_z^{(n)} = -\alpha_n$ (rigid translatio

while $\bar{s}^{(n)}$ is the appropriate Boussinesq sol. for the half-space R_n under an axisymm. normal load distribution p over Λ. Then try to determine $a, \alpha,$ from (*), (d). Subsequently check on (**). All remaining requirements will be automatically met.

(12.26), (12.13), (12.12) $\Rightarrow$

$$u_z^{(n)}(\rho,0) = \frac{1-\nu_n}{2\pi\mu_n} \int_\Lambda \frac{p(\underline{\xi})}{|\underline{x}-\underline{\xi}|} dA_{\underline{\xi}} - \alpha_n \quad \forall\, \underline{x} \in \Lambda \qquad (n=1,$$

so that (*) becomes

$$\left.\begin{aligned} &(k_1+k_2)\int_\Lambda \frac{p(\underline{\xi})}{|\underline{x}-\underline{\xi}|} dA_{\underline{\xi}} = \alpha - \beta\rho^2 \quad \forall\, \underline{x} \in \Lambda \\ &|\underline{x}-\underline{\xi}| = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}\,, \quad k_n = \frac{1-\nu_n}{2\pi\mu_n} = \frac{1-\nu_n^2}{\pi E_n} \end{aligned}\right\} (1$$

If a, α are known, (12.27) is a singular integral eq. for p. One finds that (12.27) has a solution given by

$$p(\underset{\sim}{x}) = q(\rho) = p_0\sqrt{1-\left(\frac{\rho}{a}\right)^2} \quad (0 \le \rho \le a), \quad p_0 = p(\underset{\sim}{0}), \qquad (12.$$

provided

$$a \overset{\checkmark}{=} \frac{\pi^2 p_0 (k_1+k_2)}{4\beta}, \quad \underset{\alpha = 2\beta a^2}{\alpha \overset{\checkmark}{=} \frac{1}{2}\pi^2 a\, p_0 (k_1+k_2)} \qquad (12.$$

Further, (12.28), (d) $\Rightarrow$ $\ell = 2\pi p_0 \int_0^a \sqrt{1-(\rho/a)^2}\,\rho\, d\rho = 2\pi p_0 a^2/3 \Rightarrow$

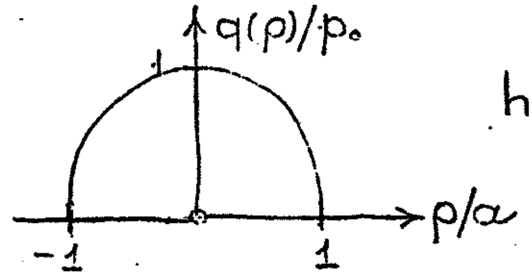
$$p_0 = \frac{3\ell}{2\pi a^2} \qquad (12$$

(12.29), (12.30), (12.22) $\Rightarrow$

$$a = \sqrt[3]{\frac{3\pi \ell (k_1+k_2) r_1 r_2}{4(r_1+r_2)}}, \quad \alpha = \sqrt[3]{\frac{9\pi^2 \ell^2 (k_1+k_2)^2 (r_1+r_2)}{16 r_1 r_2}} \qquad (12$$

Exercise 39. Verify that p given by (12.28) satisfies (12.27) if (12.29) hold.

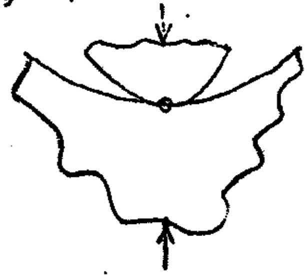
Discussion. Sketch contact pressure distribution



hemi-spheroidal

Note nonlinear in ℓ (12.31).

Mention check on (**), completion of sol. by Huber in closed elementary form. Experimental verification!

Generalizations:  asymmetric geometry (see Love, Lurie)

For more general loadings, friction, slippage see

Cattaneo, Rendiconti, Accad. dei Lincei, Ser. 6, 27 (19

Mindlin, J. Appl. Mech. 16, p.259 (1949)