14. Saint-Venant bending of beams by terminal load

Recall canonical classifications of S.V. problem. Case IV is characterized by

$$L_1 = L\,,\ L_2 = L_3 = 0\,,\ M_i = 0 \qquad (14.1)$$

(14.1), (13.6) $\Longrightarrow$

$$\sigma_{\alpha\beta} n_\beta = 0 \text{ on } B \text{ (a)} \qquad \sigma_{3\beta} n_\beta = 0 \text{ on } B \text{ (b)} \qquad (14.2)$$

$$\left.\begin{aligned} &\int_{\Pi_2} \sigma_{31}\, dA = L\,,\ \int_{\Pi_2} \sigma_{32}\, dA = 0\,,\ \int_{\Pi_2} \sigma_{33}\, dA = 0 \qquad (a) \\ &\int_{\Pi_2} \sigma_{33} x_\alpha\, dA = 0\,,\ \int_{\Pi_2} (\sigma_{32} x_1 - \sigma_{31} x_2)\, dA = 0 \qquad (b) \end{aligned}\right\} (14$$

~~Assume Π simply connected (removable!) and let~~

Let the x_α-axes be centroidal principal axes of inertia for Π_1, so that (see (13.7), (13.11)),

$$\left.\begin{aligned} &\int_{\Pi} x_\alpha\, dA = 0\,,\ \int_{\Pi} x_1 x_2\, dA = 0 \\ &A = \int_{\Pi} dA\,,\ I = \int_{\Pi} x_1^2\, dA \end{aligned}\right\} (14$$

Using once again S.V.'s semi-inverse approach, we make the following restrictive assumptions on $\underline{\sigma}$, which are motivated by the elementary beam theo

$$\sigma_{\alpha\beta} = 0\,, \quad \sigma_{33} = -\frac{L(\ell - x_3)x_1}{I} \quad \text{on } \overline{R} \qquad (14.$$

Note absence of restrictions on $\sigma_{3\alpha}$. Subst. from (14.5) into (6.3') yields

$$\sigma_{3\alpha,3} = 0\,, \quad \sigma_{3\beta,\beta} = -\frac{Lx_1}{I} \quad \text{on } R \qquad (14.$$

The first of (14.6) $\Rightarrow \sigma_{3\alpha} = \sigma_{3\alpha}(x_1, x_2)$. Subst. from (14.5) into the inverted stress-strain rels. (6.2') to get

$$\left.\begin{aligned} \gamma_{\alpha\alpha} &= \frac{\nu L}{\eta I}(\ell - x_3)x_1 \ (\text{no sum})\,, \quad \gamma_{12} = 0\,, \\ \gamma_{33} &= -\frac{L}{\eta I}(\ell - x_3)x_1\,, \quad \gamma_{3\alpha} = \frac{1}{2\mu}\sigma_{3\alpha}(x_1, x_2) \end{aligned}\right\} \quad (1$$

Recall the strain eqs. of compatibility

$$\gamma_{ij,kl} + \gamma_{kl,ij} - \gamma_{il,jk} - \gamma_{jk,il} = 0 \quad \text{on } R \qquad (2.$$

Subst. from (14.7) into (2.59) gives

$$\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial \gamma_{32}}{\partial x_1}-\frac{\partial \gamma_{31}}{\partial x_2}\right)}_{g}=0\ ,\quad \frac{\partial}{\partial x_2}\underbrace{\left(\frac{\partial \gamma_{32}}{\partial x_1}-\frac{\partial \gamma_{31}}{\partial x_2}\right)}_{g}=-\frac{\nu L}{\eta I}\ \text{on } \Pi \quad (1\text{-}$$

whence,

$$\frac{\partial \gamma_{32}}{\partial x_1}-\frac{\partial \gamma_{31}}{\partial x_2}=\alpha-\frac{\nu L x_2}{\eta I}\ \text{on } \Pi\quad (\alpha=\text{const.}),\qquad (14.$$

which is equivalent to (see later for motivation)

$$\frac{\partial}{\partial x_1}\underbrace{\left(\gamma_{32}-\frac{\alpha x_1}{2}\right)}_{\frac{1}{2}\partial f/\partial x_2}=\frac{\partial}{\partial x_2}\underbrace{\left(\gamma_{31}+\frac{\alpha x_2}{2}-\frac{\nu L x_2^2}{2\eta I}\right)}_{\frac{1}{2}\partial f/\partial x_1}\ \text{on } \Pi\qquad (14.$$

Suppose $\gamma_{3\alpha}\in\mathcal{C}^1(\Pi)$. Then by (14.9) and the two-dim Stokes thm., $\exists\ f\in\mathcal{C}^2(\Pi)\ \ni$

$$\gamma_{32}=\frac{\alpha x_1}{2}+\frac{1}{2}\frac{\partial f}{\partial x_2},\quad \gamma_{31}=-\frac{\alpha x_2}{2}+\frac{\nu L x_2^2}{2\eta I}+\frac{1}{2}\frac{\partial f}{\partial x_1}\qquad (14.$$

(14.10) and the last of (14.7) $\Rightarrow$

$$\sigma_{31}=-\mu\alpha x_2+\frac{\mu\nu L}{\eta I}x_2^2+\mu\frac{\partial f}{\partial x_1},\ \sigma_{32}=\mu\alpha x_1+\mu\frac{\partial f}{\partial x_2}\qquad (14.$$

Subst. from (14.11) into the second of (14.6) to reach

$$\nabla^2 f=-\frac{L x_1}{\mu I}=-\frac{2(1+\nu)L}{\eta I}x_1\ \ \text{on } \Pi\qquad (14.$$

(14.12) is satisfied if one sets

$$\left.\begin{aligned} f(x_1,x_2) &= H(x_1,x_2) - \frac{L}{\eta I}\left[\frac{\nu x_1^3}{6} + \left(1+\frac{\nu}{2}\right)x_1 x_2^2\right] \text{ on } \Pi \\ \nabla^2 H &= 0 \text{ on } \Pi \end{aligned}\right\} \quad (14.1$$

Define "flexure function" F by setting

$$F = \frac{\eta I}{L}(\alpha\varphi - H) \text{ on } \Pi, \qquad (14.1$$

where φ is the warping function for Π characterized by (13.24), (13.25), i.e.,

$$\nabla^2\varphi = 0 \text{ on } \Pi, \quad \frac{\partial\varphi}{\partial n} = x_2 n_1 - x_1 n_2 \text{ on } \partial\Pi \qquad (14.1$$

(14.11), (14.13), (14.4) $\Longrightarrow$

$$\left.\begin{aligned} \sigma_{31} &= \mu\alpha\left(\frac{\partial\varphi}{\partial x_1} - x_2\right) - \frac{L}{2(1+\nu)I}\left[\frac{\partial F}{\partial x_1} + \frac{\nu x_1^2}{2} + \left(1-\frac{\nu}{2}\right)x_2^2\right], \\ \sigma_{32} &= \mu\alpha\left(\frac{\partial\varphi}{\partial x_2} + x_1\right) - \frac{L}{2(1+\nu)I}\left[\frac{\partial F}{\partial x_2} + (2+\nu)x_1 x_2\right] \end{aligned}\right\} \quad (1$$

$$\nabla^2 F = 0 \text{ on } \Pi \qquad (14.$$

<u>Remarks.</u> The stress field $\underset{\sim}{\sigma}$ given by (14.5), (14.16) has a compatible strain field $\underset{\sim}{\varepsilon}$ and conforms to (6.3 provided φ and F satisfy the first (14.15) and (14.17)

As is clear from (13.23), (13.24), (13.25) the terms involving α in (14.16) correspond to pure S.V. torsion induced by a scalar torque $M = \alpha K$, where K is the torsional rigidity of Π.

We turn now to the <u>boundary conds.</u> (14.2), (1

(14.2a) o.k. by (14.5). In view of (14.16), (14.15), cond (14.2b) is met provided

$$\frac{\partial F}{\partial n} = -\left[\frac{\nu \kappa_1^2}{2} + \left(1-\frac{\nu}{2}\right)\kappa_2^2\right] n_1 - (2+\nu)\kappa_1\kappa_2 n_2 \quad \text{on } \partial\Pi \quad (1$$

Thus F is determined, but for an inessential additi constant, as the sol. the two-dimensional Neuman prob. (14.17), (14.18). Nec. for the existence of sol.

$$\oint_{\partial\Pi} \frac{\partial F}{\partial n} dA = 0 .$$

Indeed, from (14.18), (14.4), and the divergence th

$$\oint_{\partial\Pi} \frac{\partial F}{\partial n} ds = -\int_{\Pi} [\nu\kappa_1 + (2+\nu)\kappa_1] dA = -2(1+\nu)\int_{\Pi} \kappa_1 dA = 0 .$$

Consider (14.3a). Because of (14.6), (14.4),

$$\int_{\Pi_\ell} \sigma_{31}\, dA = \int_{\Pi} \left(\sigma_{31} + x_1 \sigma_{3\beta,\beta} + \frac{L x_1^2}{I}\right) dA$$

$$= \frac{L}{I}\int_{\Pi} x_1^2 dA + \int_{\Pi} (x_1 \sigma_{3\beta})_{,\beta} dA = L + \underbrace{\oint_{\partial\Pi} x_1 \sigma_{3\beta} n_\beta ds}_{\text{zero by (14.2b)}} = L$$

$$\int_{\Pi_\ell} \sigma_{32} dA = \int_{\Pi} \left(\sigma_{32} + x_2 \sigma_{3\beta,\beta} + \frac{L x_1 x_2}{I}\right) dA$$

$$= \cancel{\frac{L}{I}\int_{\Pi} x_1 x_2 dA} + \int_{\Pi} (x_2 \sigma_{3\beta})_{,\beta} dA = \int_{\partial\Pi} x_2 \sigma_{3\beta} n_\beta ds = 0$$

$\int_{\Pi_\ell} \sigma_{33} dA = 0$ since $\sigma_{33} = 0$ on Π_2 by (14.5).

Consider (14.3b).

$\int_{\Pi_2} \sigma_{33} x_\alpha dA = 0$ since $\sigma_{33} = 0$ on Π_2 by (14.5).

Next, by (14.16), (14.15), (13.27),

$$\int_{\Pi_2} (\sigma_{32} x_1 - \sigma_{31} x_2) dA = \alpha K + \frac{L}{2(1+\nu) I} \int_{\Pi} \Big[x_2 \frac{\partial F}{\partial x_1} - x_1 \frac{\partial F}{\partial x_2} + \left(1 - \frac{\nu}{2}\right) x_2^3$$

$$- \left(2 + \frac{\nu}{2}\right) x_1^2 x_2\Big] dA$$

Thus the last of (14.3b) is met if

$$\left.\begin{aligned}\alpha &= \frac{-L}{2(1+\nu)KI}\int_{\Pi}[x_2\frac{\partial F}{\partial x_1}-x_1\frac{\partial F}{\partial x_2}+(1-\frac{\nu}{2})x_2^3-(2+\frac{\nu}{2})x_1^2x_2]\,dA,\\ K &= \mu\int_{\Pi}[x_1^2+x_2^2+x_1\frac{\partial\varphi}{\partial x_2}-x_2\frac{\partial\varphi}{\partial x_1}]\,dA\end{aligned}\right\}$$

Recapitulate sol. as far as $\underset{\sim}{\sigma}$ is concerned. Note that method used to deduce $\underset{\sim}{\sigma}$ assures its validity only for s.c. Π unless displacements are s.v.. $\underset{\sim}{u}$ follows from (14.16), (14.7) with the aid of (14.15), (14.17) and Thm. 2.17. A lengthy computation yields:

$$\left.\begin{aligned}u_1 &= -\alpha x_2x_3+\frac{L}{\eta I}[\frac{\nu}{2}(l-x_3)(x_1^2-x_2^2)+\frac{1}{2}lx_3^2-\frac{x_3^3}{6}]\\ &\qquad +\mathring{a}_1+\mathring{w}_2x_3-\mathring{w}_3x_2,\\ u_2 &= \alpha x_1x_3+\frac{L\nu}{\eta I}(l-x_3)x_1x_2+\mathring{a}_2+\mathring{w}_3x_1-\mathring{w}_1x_3,\\ u_3 &= \alpha\varphi(x_1,x_2)-\frac{L}{\eta I}[x_1x_3(l-\frac{x_3}{2})+x_1x_2^2+F(x_1,x_2)]\\ &\qquad +\mathring{a}_3+\mathring{w}_1x_2-\mathring{w}_2x_1.\end{aligned}\right\}(14$$

Since $\underset{\sim}{u}$ above is single-valued, sol. is ok even if Π is m. Identify pure-torsion and rigid part of $\underset{\sim}{u}$. Mention direct verification of (14.27).

Suppose Π is such that $\underset{\sim}{0}\in\Pi_0$, i.e. Π contains its centroid. We may dispose over $\underset{\sim}{\mathring{a}}, \underset{\sim}{\mathring{w}}$ in (14.20) to satisfy relaxed fixity conds. at $x_3 = 0$ ("cantilever beam"). Thus

$$\mathring{a}_i = 0\,,\ \ \mathring{w}_i = 0\,,\ \ \varphi(0,0) = 0\,, F(0,0) = 0\,.^{\S}$$

Then (14.20) $\Rightarrow$

$$u_i(\underset{\sim}{0}) = 0\,,\ \ u_{1,3}(\underset{\sim}{0}) = u_{2,3}(\underset{\sim}{0}) = u_{2,1}(\underset{\sim}{0}) = \underset{\sim}{0}$$

and

$$u_1(\underset{\sim}{e}z) = \frac{Lz^2}{2\eta I}\left(\ell - \frac{z}{3}\right) \quad (0 \le z \le \ell)$$

in agreement with elementary beam theory.

The physical significance of α

Recall from (2.57) that

$$w_3 = \tfrac{1}{2}(u_{2,1} - u_{1,2})$$

$w_3(\underset{\sim}{x})$ ... angular displacement of the infinitesimal local rigid rotation about an axis parallel to x_3-axis

Hence,

§ Recall that this normalization of φ and F is possible since both are solutions of Neumann problems.

$$w_{3,3} = \frac{1}{2}(u_{2,13} - u_{1,23}) = \underbrace{\frac{1}{2}(u_{2,3}+u_{3,2})_{,1}}_{\gamma_{32,1}} - \underbrace{\frac{1}{2}(u_{1,3}+u_{3,1})_{,2}}_{\gamma_{31,2}} \quad \text{or}$$

$$\frac{\partial w_3}{\partial x_3} = \frac{\partial \gamma_{32}}{\partial x_1} - \frac{\partial \gamma_{31}}{\partial x_2} \quad \text{(local rate of twist)} \tag{14.21}$$

Now (14.21), (14.8) $\Rightarrow$

$$\frac{\partial w_3}{\partial x_3} = \alpha - \frac{\nu L x_2}{\eta I} \text{ on } \overline{R}, \tag{14.22}$$

so that

$$\alpha = \frac{\partial w_3}{\partial x_3}\Big|_{x_2=0} = \frac{1}{A}\int_{\Pi} \frac{\partial w_3}{\partial x_3}\, dA \tag{14.23}$$

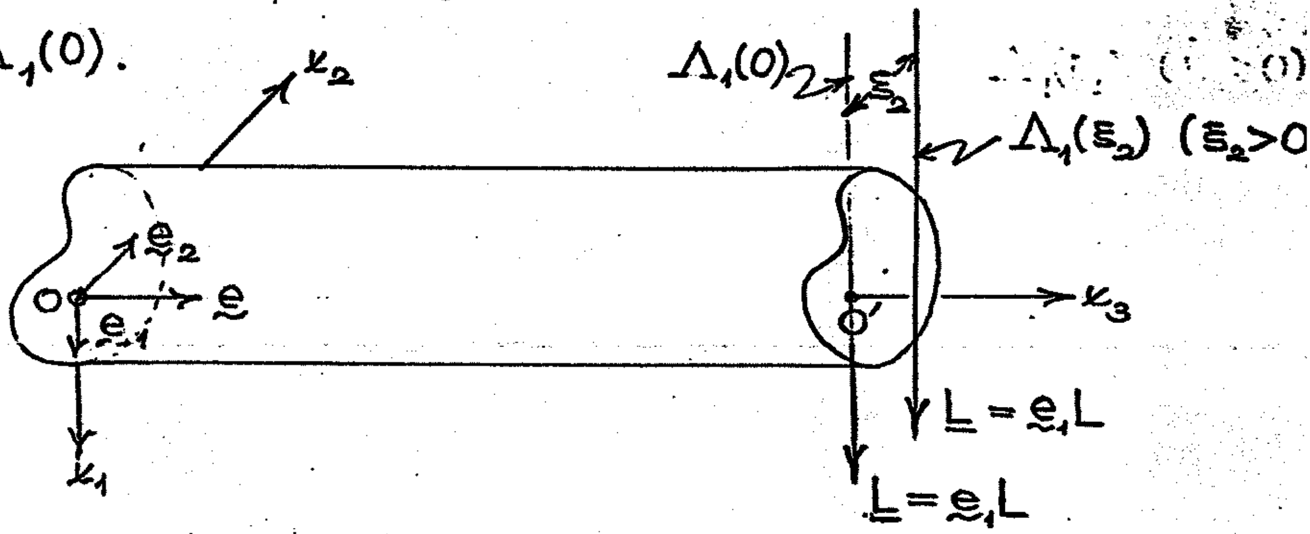
(14.23) yields two alternative interpretations of the parameter α in Case IV. Explain.

<u>The center of flexure</u>

Return to (14.2), (14.3). Let $\Lambda_1(\xi_2)$, $\Lambda_2(\xi_1)$ be defined by

$$\left.\begin{aligned} \Lambda_1(\xi) &= \{\underline{x} \mid -\infty < x_1 < \infty,\ x_2 = \xi,\ x_3 = l\},\ \xi \in (-\infty,\infty) \\ \Lambda_2(\xi) &= \{\underline{x} \mid x_1 = \xi,\ -\infty < x_2 < \infty,\ x_3 = l\},\ \xi_1 \in (-\infty,\infty) \end{aligned}\right\} \text{(1}$$

Thus Λ_α is the family of all straight lines parallel to the x_α-axis and lying in the plane $x_3 = l$. Conds. (14.3) require that the tractions on Π_2 be statically equivalent to a single force whose line of action is $\Lambda_1(0)$.



Consider the generalized problem in which the tractions on Π_2 are statically equivalent to a force $e_1 L$ with the line of action $\Lambda_1(\xi_2)$, $\xi_2 \in (-\infty, \infty)$. The only change in boundary conds. thus arising is in the last of (14.3b), which must now be replaced by

$$\int_{\Pi_2} (\sigma_{32} x_1 - \sigma_{31} x_2)\, dA = -L\,\xi_2 . \tag{14.25}$$

Clearly, the previous solution remains valid, provided

(14.??) is replaced by

$$\alpha = \frac{-L}{2(1+\nu)IK} \int_{\Pi} [x_2 \frac{\partial F}{\partial x_1} - x_1 \frac{\partial F}{\partial x_2} + (1-\frac{\nu}{2}) x_2^3 - (2+\frac{\nu}{2}) x_1^2 x_2] dA - \frac{L \xi_2}{K} \quad (14$$

where K is again the torsional rigidity of Π.

Evidently, $\alpha = 0$ if $\xi_2 = \xi_2^*$, where

$$\xi_2^* = \frac{-1}{2(1+\nu)I} \int_{\Pi} [x_2 \frac{\partial F}{\partial x_1} - x_1 \frac{\partial F}{\partial x_2} + (1-\frac{\nu}{2}) x_2^3 - (2+\frac{\nu}{2}) x_1^2 x_2] dA \quad (14.$$

$\Lambda_1(\xi_2^*)$ ... "line of flexure" parallel to x_1-axis. In view of (14.23), tractions on Π_2 that are statically equivalent to a single force with the line of action $\Lambda_1(\xi_2^*)$ produ[ce] zero average rate of twist according to S.V.'s solutio[n] of the corresponding S.V. problem. L

The solution of the generalized prob. for tractions on Π_2 that are statically equivalent to a force $\underline{e}_2 L$ with the line of action $\Lambda_2(\xi_1)$ is immediate from the sol. established above and symmetry.

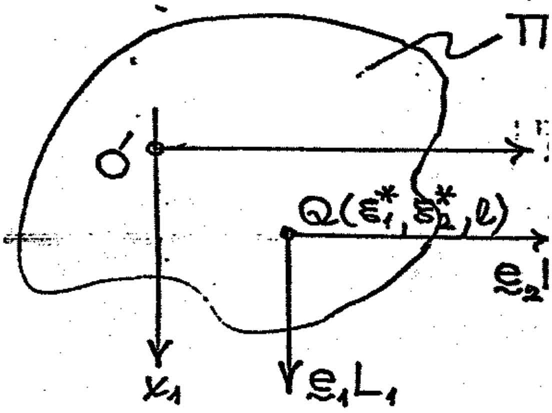
$\Lambda_2(\xi_1^*)$ ... line of flexure parallel to x_2-axis. Explain.

$$Q(\xi_1^*, \xi_2^*, \ell) = \Lambda_1(\xi_2^*) \cap \Lambda_2(\xi_1^*) \quad \text{(14.28)}$$

Q ... "Center of flexure" ("Center of shear").

Explain significance of Q. Note that location of Q depends on shape of Π and on ν but is independent of load.

If Π is symmetric about $x_2=0$, (14.17), (14.18) $\Rightarrow$ $F(x_1,-x_2) = F(x_1,x_2) \ \forall\ (x_1,x_2) \in \overline{\Pi}$. In this instance the integrand in (14.27) is an odd function of x_2 and thus $\xi_2^* = 0$. Hence: if Λ is an axis of symm. of Π_2 then $Q \in \Lambda$.



Suppose: $\underset{\sim}{L} = \underset{\sim}{e}_\alpha L_\alpha$

$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{\sigma}]$ ... S.V. flexure sol. for $\underset{\sim}{L}$ at O

$\mathcal{S}' = [\underset{\sim}{u}', \underset{\sim}{E}', \underset{\sim}{\sigma}']$ ... S.V. flexure sol. for $\underset{\sim}{L}$ at Q

Then, evidently, $\mathfrak{s} = \mathfrak{s}' + \mathfrak{s}''$ on $\bar{R}$, where $\mathfrak{s}'' = [\underset{\sim}{u}'', \underset{\sim}{\tau}'', \underset{\sim}{\sigma}'']$ is S.V. torsion solution corresponding to the torque $\underset{\sim}{M} = \underset{\sim}{e}_3(\xi_2 L_1 - \xi_1 L_2)$.

Mention Trefftz' energetic definition of the shear-cent
See Z. Ang. Math. Mech., 15, p. 220, 1935.

Resultant shear stress, lines of shear stress

The stresses (14.5) coincide with those in the elementary beam theory. Not so for $\sigma_{3\alpha}$ given by (14.16). Set

$$\underset{\sim}{\sigma}_3(\underset{\sim}{x}) \equiv \underset{\sim}{S}(\underset{\sim}{x}, \underset{\sim}{e}_3) = \underset{\sim}{\sigma}(\underset{\sim}{x}) + \underset{\sim}{e}_3 \sigma_{33}(\underset{\sim}{x}), \quad \underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{e}_\alpha \sigma_{3\alpha}(\underset{\sim}{x}) \ \forall \underset{\sim}{x}$$

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{\sigma}(x_1, x_2) \ \ldots \text{ resultant shear stress on } \Pi$$

$$\sigma = |\underset{\sim}{\sigma}| = \sqrt{\sigma_{31}^2 + \sigma_{32}^2}$$

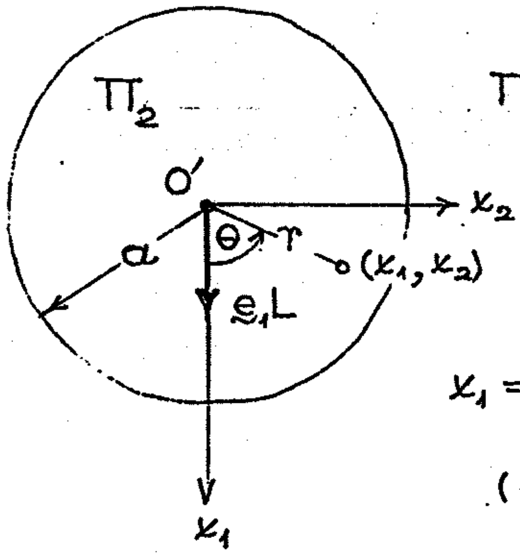
Note ν-dependence of $\underset{\sim}{\sigma}$ (lost in elementary theory). The trajectories of $\underset{\sim}{\sigma}(x_1, x_2)$ ("lines of shear stress") obey the differential eq.

$$\frac{dx_2}{dx_1} = \frac{\sigma_{32}}{\sigma_{31}} \qquad (*)$$

From (14.16), if $\alpha = 0$, (*) becomes

$$\frac{dx_2}{dx_1} = \frac{2\partial F/\partial x_2 + 2(2+\nu)x_1 x_2}{2\partial F/\partial x_1 + \nu x_1^2 + (2-\nu)x_2^2} \qquad (14.2$$

Application: S.V. bending of a beam of circular cross-section.



$$\left.\begin{aligned} \Pi &= \{(x_1, x_2) \mid 0 \le x_1^2 + x_2^2 < a^2\} \\ I &= \int_\Pi x_1^2 \, dA = \frac{\pi a^4}{4} \end{aligned}\right\} \quad (14$$

$$\left.\begin{aligned} &x_1 = r\cos\theta, \; x_2 = r\sin\theta \\ &(0 \le r \le a, \; 0 \le \theta < 2\pi) \end{aligned}\right\} \quad (14$$

In view of polar symm. of Π, $Q = O'$, $\alpha = 0$. Also here $\varphi = 0$ on $\overline{\Pi}$ (no warping in S.V. torsion). Set

$$F(x_1, x_2) = F(r\cos\theta, r\sin\theta) = G(r, \theta) \qquad (14$$

Then, from (14.17), (14.18), (14.31), G obeys

$$\nabla^2 G = 0 \quad on\ \Pi ,$$

$$\left.\frac{\partial G}{\partial r}\right|_{(r=a)} = -a^2\cos\theta\left[\frac{\nu}{2}\cos^2\theta + (1-\frac{\nu}{2})\sin^2\theta\right] - (2+\nu)a^2\sin^2\theta\cos$$

or, on using elementary trigonometric identities,

$$\nabla^2 G = 0 \ on\ \Pi , \left.\frac{\partial G}{\partial r}\right|_{(r=a)} = -(\tfrac{3}{4}+\tfrac{\nu}{2})a^2\cos\theta + \frac{3a^2}{4}\cos 3\theta \quad (14$$

To solve the Neumann prob. (14.33) consider the analytic functions

$$z = x_1 + ix_2 = r(\cos\theta + i\sin\theta), \quad z^3 = r^3(\cos 3\theta + i\sin 3\theta) \quad (14.$$

and recall that $\nabla^2 \Re\{f(z)\} = 0$ on Π if f anal. on Π
Hence (14.33) has the solution

$$G(r,\theta) = -(\tfrac{3}{4}+\tfrac{\nu}{2})a^2 r\cos\theta + \frac{r^3}{4}\cos 3\theta .$$

$$= -(\tfrac{3}{4}+\tfrac{\nu}{2})a^2\Re\{z\} + \tfrac{1}{4}\Re\{z^3\} \quad (14$$

(14.35), (14.34), (14.32) yield the required flexure function

$$F(x_1,x_2) = -(\tfrac{3}{4}+\tfrac{\nu}{2})a^2 x_1 + \tfrac{1}{4}(x_1^3 - 3x_1x_2^2) \ \forall \ (x_1,x_2)\in\overline{\Pi} \quad (14.$$

(14.36), (14.5), (14.16), (14.30) yield the stress field

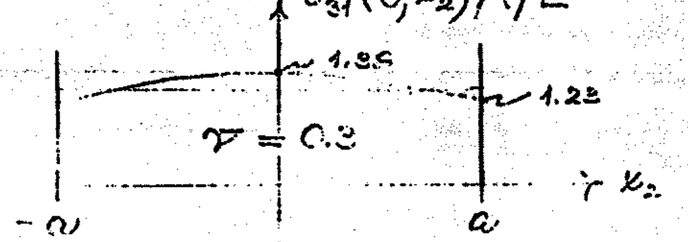
$$\left.\begin{aligned} &\tau_{11} = \tau_{22} = \tau_{12} = 0 \ , \quad \tau_{33} = -\frac{4L(l-x_3)x_1}{\pi a^4} \\ &\tau_{31} = \frac{(3+2\nu)L}{2\pi a^4(1+\nu)}\left(a^2 - x_1^2 - \frac{1-2\nu}{3+2\nu}x_2^2\right), \ \tau_{32} = -\frac{(1+2\nu)L x_1 x_2}{\pi a^4(1+\nu)} \end{aligned}\right\} (14.$$

Note ν-dependence of $\tau_{3\alpha}$, cf. torsion. In particular,

$$\left.\begin{aligned} &\tau_{31}(0,x_2) = \frac{(3+2\nu)L}{2\pi a^4(1+\nu)}\left(a^2 - \frac{1-2\nu}{3+2\nu}x_2^2\right) \ , \quad \tau_{32}(0,x_2) = 0 \\ &\tau_{31}(0,0) = \max_{[-a,a]}\tau_{31}(0,\cdot) = \frac{(3+2\nu)L}{2(1+\nu)A} \ , \quad A = \pi a^2, \ L > 0 \\ &\tau_{31}(0,\pm a) = \frac{1+2\nu}{1+\nu}\frac{L}{A} \end{aligned}\right\} (1$$

In contrast, the elementary beam theory predicts

$$\tau_{31}(0,x_2) = \frac{4L}{3A} \qquad (-a \le x_2 \le a)$$



maximum error approx. 4%

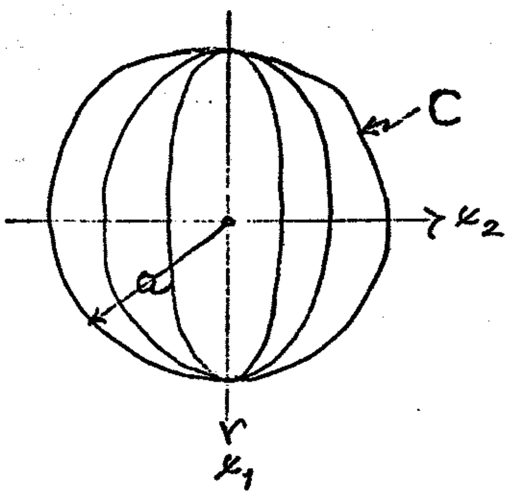
<u>Lines of shearing stress</u>: By (14.29) here

$$\frac{dx_2}{dx_1} = \frac{\tau_{32}}{\tau_{31}} = \frac{2(1+2\nu)x_1x_2}{(3+2\nu)\left[-a^2+x_1^2+(1-2\nu)x_2^2/(3+2\nu)\right]}, \qquad (14.39)$$

whose complete sol. is

$$x_1^2 + x_2^2 = a^2 + k\,x_2^{(3+2\nu)/(1-2\nu)} \qquad (k\ldots\text{const.}) \qquad (14.40)$$

Note that $k=0$ yields the eq. of C.



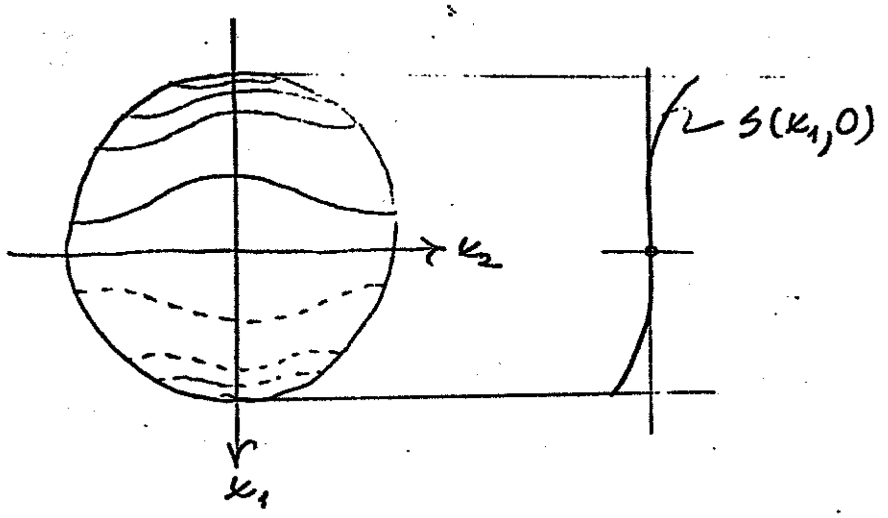
<u>Deformations</u>. Assume fixity conds. on p.58. The deflection curve i.e. $u_1(0,0,x_3)$, according to (14.20), is independent of shape of Π and of ν; it was discussed earlier.

Consider now the axial displacement u_3. By (14.20), (14.36), and since here $\varphi = 0$ on Π, one has

$$
\left.\begin{aligned}
u_3(x_1,x_2,x_3) &= \frac{L}{\pi a^4 E}\left[(3+2\nu)a^2 - 2x_3(2l-x_3)\right]x_1 \\
&\quad \underbrace{-\frac{L}{\pi a^4 E}\left(x_1^2+x_2^2\right)x_1}_{\psi(x_1,x_2)}
\end{aligned}\right\} \quad (14.4)
$$

The terms linear in x_1 evidently represents a rigid tilting of cross-section about its centroidal axis par. x_2-axis. $\psi(x_1,x_2)$ represents warping;

Contour lines of $\psi(x_1,x_2)$:



~~Mention exact solns. for $\Pi_{..}$, ellipse, equilateral Δ, semi-circ segment of circle, circular sector, rectangle. Conformal mapping, numerical methods. Timoshenko stress functions, membrane analogy;~~

<u>Remarks</u>. See Bogy, J.A.M., <u>89</u>, 1, p.175 (1967) for an alternative sol. of the preceding prob. within S.V.'s formulation (quantitative evidence in support of S.V.'s princip

Mention exact sols. for the following cross-sections: ellipse, equilateral Δ, segment of circle, circular sector rectangle. Timoshenko's stress function and membrane analogy for Case IV (See Timoshenko-Goodier).

Refer to minimum energy characterization of S.V.'s solutions to the relaxed S.V. prob. See Knowles & Co., ARMA, <u>21</u>, 2, p.89 (1966).