14. Saint-Venant bending of beams by terminal load

Recall canonical classification of S.V. problem.

Case IV is characterized by

\[ L_1 = L, \quad L_2 = L_3 = 0, \quad M_i = 0 \]  \hspace{1cm} (14.1)

\[(14.1), (13.6) \Rightarrow\]

\[ \sigma_{\alpha\beta} n_\beta = 0 \text{ on } B \ (a) \quad \sigma_{3\beta} n_\beta = 0 \text{ on } B \ (b) \]  \hspace{1cm} (14.5)

\[
\begin{align*}
\int_{\Pi_2} \sigma'_{31} dA &= L, & \int_{\Pi_2} \sigma'_{32} dA &= 0, & \int_{\Pi_2} \sigma'_{33} dA &= 0 \\
\int_{\Pi_2} \sigma_{33} \kappa_\alpha dA &= 0, & \int_{\Pi_2} (\sigma'_{32} \kappa_1 - \sigma'_{31} \kappa_2) dA &= 0
\end{align*}
\]

Assume \( \Pi \) simply connected (unvoidable) and let

let the \( \kappa_\alpha \)-axes be centroidal principal axes of inertia for \( \Pi_1 \), so that (see (13.7), (13.11)),

\[
\begin{align*}
\int_{\Pi} \kappa_\alpha dA &= 0, & \int_{\Pi} \kappa_1 \kappa_2 dA &= 0 \\
A &= \int_{\Pi} dA, & I &= \int_{\Pi} \kappa_1^2 dA
\end{align*}
\]  \hspace{1cm} (14)
Using once again S.V.'s semi-inverse approach, we make the following restrictive assumptions on $\sigma$, which are motivated by the elementary beam theory.

$$
\sigma_{\alpha\beta} = 0, \quad \sigma_{\alpha 3} = -\frac{L (L - k_3) \kappa_1}{I} \text{ on } R
$$

(14)

Note absence of restrictions on $\sigma_{3\infty}$. Subst. from (14.5) into (6.3') yields

$$
\sigma_{3\alpha, 3} = 0, \quad \sigma_{3\beta, \beta} = -\frac{L \kappa_1}{I} \text{ on } R
$$

(14)

The first of (14.6) $\Rightarrow \sigma_{3\alpha} = \sigma_{3\alpha}(k_1, k_2)$. Subst. from (14.6) into the involved stress-strain rels. (6.2') to get

$$
\begin{align*}
\gamma_{\alpha\alpha} &= \frac{\nu L}{\eta I} (L - k_3) \kappa_1 \text{ (no sum)}, \quad \gamma_{12} = 0, \\
\gamma_{33} &= -\frac{L}{\eta I} (L - k_3) \kappa_1, \quad \gamma_{3\alpha} = \frac{1}{2\mu} \sigma_{3\alpha}(k_1, k_2)
\end{align*}
$$

(11)

Recall the strain eqs. of compatibility

$$
\gamma_{ij, kl} + \gamma_{kl, ij} - \gamma_{ie, jk} - \gamma_{jk, ie} = 0 \text{ on } R
$$

(2)

Subst. from (14.7) into (2.59) gives
\[
\frac{\partial}{\partial x_1} \left( \frac{\partial \varphi_{32}}{\partial x_1} - \frac{\partial \varphi_{31}}{\partial x_2} \right) = 0 , \quad \frac{\partial}{\partial x_2} \left( \frac{\partial \varphi_{32}}{\partial x_1} - \frac{\partial \varphi_{31}}{\partial x_2} \right) = -\frac{\nabla f}{\eta I} \text{ on } \Gamma T (14.1)
\]

whence,
\[
\frac{\partial \varphi_{32}}{\partial x_1} - \frac{\partial \varphi_{31}}{\partial x_2} = \alpha - \frac{\nabla L x_2}{\eta I} \text{ on } \Gamma T \quad (\alpha = \text{const.}) \quad (14.2)
\]

which is equivalent to (see later for motivation)
\[
\frac{\partial}{\partial x_1} \left( \frac{\varphi_{32} - \frac{\alpha x_1}{2}}{\frac{\partial \varphi_{32}}{\partial x_1} - \frac{\partial \varphi_{31}}{\partial x_2}} \right) = \frac{\partial}{\partial x_2} \left( \varphi_{31} + \frac{\alpha x_2}{2} - \frac{\nabla L x_2^2}{2 \eta I} \right) \text{ on } \Gamma T \quad (14.3)
\]

Suppose \( \varphi_{32} \in C^2(\Gamma T) \). Then by (14.9) and the fact that

\[
\varphi_{32} = \frac{\alpha x_1}{2} + \frac{1}{2} \frac{\partial \varphi}{\partial x_2} , \quad \varphi_{31} = -\frac{\alpha x_2}{2} + \frac{\nabla L x_2^2}{2 \eta I} + \frac{1}{2} \frac{\partial \varphi}{\partial x_1} \quad (14.4)
\]

(14.10) and the last of (14.7) \( \Rightarrow \)
\[
\varphi_{31} = -\mu \alpha x_2 + \frac{\nabla L x_2}{\eta I} , \quad \varphi_{32} = \mu \alpha x_1 + \mu \frac{\partial \varphi}{\partial x_1} \quad (14.5)
\]

Subst. from (14.11) into the second of (14.6) to reach
\[
\nabla^2 \varphi = -\frac{L x_1}{\mu I} = \frac{2(1+\alpha)L}{\eta I} x_1 \text{ on } \Gamma T \quad (14.6)
\]

(14.12) is satisfied if one sets
\[ f(\kappa_1, \kappa_2) = H(\kappa_1, \kappa_2) - \frac{L}{\eta I} \left[ \frac{\nu \kappa_1^3}{6} + (1 + \frac{\nu}{2}) \kappa_1 \kappa_2^2 \right] \] on \( \Pi \)

\[ \nabla^2 H = 0 \] on \( \Pi \)

Define a "flexure function" \( F \) by setting

\[ F = \frac{\eta I}{L} (\alpha \varphi - H) \] on \( \Pi \),

where \( \varphi \) is the warping function for \( \Pi \) characterized by (13.24), (13.25), i.e.,

\[ \nabla^2 \varphi = 0 \] on \( \Pi \), \( \frac{\partial \varphi}{\partial n} = \kappa_2 n_1 - \kappa_1 n_2 \) on \( \partial \Pi \)

(14.1)

(14.11), (14.13), (14.4) \( \Rightarrow \)

\[ \sigma_{31} = \mu \alpha \left( \frac{\partial \varphi}{\partial n_2} - \kappa_2 \right) - \frac{L}{2(1+\nu)I} \left[ \frac{\partial F}{\partial \kappa_1} + \frac{\nu \kappa_1^2}{2} + (1 - \frac{\nu}{2}) \kappa_2^2 \right], \]

\[ \sigma_{32} = \mu \alpha \left( \frac{\partial \varphi}{\partial n_1} + \kappa_1 \right) - \frac{L}{2(1+\nu)I} \left[ \frac{\partial F}{\partial \kappa_2} + (2 + \nu) \kappa_1 \kappa_2 \right] \]

\[ \nabla^2 F = 0 \] on \( \Pi \)

(14.4)

Remarks: The stress field \( \sigma \) given by (14.5), (14.16) has a compatible strain field \( \varepsilon \) and conforms to (6.11) provided \( \varphi \) and \( F \) satisfy the first (14.15) and (14.17).
As is clear from (13.23), (13.24), (13.25) the terms involving \( \alpha \) in (14.16) correspond to pure S.V. torics induced by a scalar torque \( M = \alpha K \), where \( K \) is the torsional rigidity of \( \Pi \).

We turn now to the boundary conditions (14.2), (14.2a) o.k. by (14.5). In view of (14.16), (14.15), cond (14.2b) is met provided

\[
\frac{\partial F}{\partial n} = -\left[ \frac{\gamma \kappa_1^2}{2} + (1 - \frac{\gamma}{2}) \kappa_2^2 \right] n_1 - (2 + \gamma) \kappa_1 \kappa_2 n_2 \quad \text{on } \partial \Pi \quad (14.2b)
\]

Thus \( F \) is determined, but for an inessential additive constant, as the sol. the two-dimensional Neumann prob. (14.17), (14.18). Nec. for the existence of sol.

\[
\oint_{\partial \Pi} \frac{\partial F}{\partial n} \, dA = 0.
\]

Indeed, from (14.15), (14.4), and the divergence th

\[
\oint_{\partial \Pi} \frac{\partial F}{\partial n} \, ds = -\int_{\Pi} \left[ \gamma \kappa_1 + (2 + \gamma) \kappa_2 \right] \, dA = -2(1 + \gamma) \int_{\Pi} \kappa_1 \, dA = 0.
\]
Consider (14.3a). Because of (14.6), (14.4),

\[
\int_\Omega \sigma_{34} \, dA = \int_\Omega \left( \sigma_{34} + \kappa_1 \sigma_{3\beta\beta} + \frac{L\kappa_2^2}{I} \right) \, dA = \frac{1}{I} \int_\Omega \kappa_1^2 \, dA + \int_\Omega (\kappa_1 \sigma_{3\beta})_\beta \, dA = L + \oint_{\partial\Omega} \kappa_1 \sigma_{3\beta} n_\beta \, ds = L
\]

zero by (14.2b).

\[
\int_\Omega \sigma_{32} \, dA = \int_\Omega \left( \sigma_{32} + \kappa_2 \sigma_{3\beta\beta} + \frac{L\kappa_1\kappa_2}{I} \right) \, dA = \frac{1}{I} \int_\Omega \kappa_2 \kappa_2 \, dA + \int_\Omega (\kappa_2 \sigma_{3\beta})_\beta \, dA = \int_\Omega \kappa_2 \sigma_{3\beta} n_\beta \, ds = 0
\]

\[
\int_\Omega \sigma_{33} \, dA = 0 \quad \text{since } \sigma_{33} = 0 \text{ on } \partial\Omega \text{ by (14.5)}.
\]

Consider (14.3b).

\[
\int_\Omega \sigma_{33} \kappa_\alpha \, dA = 0 \quad \text{since } \sigma_{33} = 0 \text{ on } \partial\Omega \text{ by (14.5)}.
\]

Next, by (14.16), (14.15), (13.27),

\[
\int_\Omega \left( \sigma_{32} \kappa_1 - \sigma_{31} \kappa_2 \right) \, dA = \alpha K + \frac{L}{2(1+\nu)I} \int_\Omega \left[ \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Phi}{\partial x_2} + (1-\frac{\nu}{2}) \kappa_1^2 \right] \, dA
\]

\[-(2 + \frac{\nu}{2}) \kappa_1^2 \kappa_2 J \, dA\]
Thus the last of (14.36) is met if

\[ \alpha = \frac{-L}{2(1+\nu)K} \int \left[ \frac{\partial F}{\partial \kappa_1} - \kappa_1 \frac{\partial F}{\partial \kappa_2} + \left(1 - \frac{\nu}{2}\right) \kappa_2^3 - \left(2 + \frac{\nu}{2}\right) \kappa_1 \kappa_2^2 \right] dA, \]

\[ K = \mu \int \left[ \kappa_1^2 + \kappa_2^2 + \kappa_1 \frac{\partial \phi}{\partial \kappa_2} - \kappa_2 \frac{\partial \phi}{\partial \kappa_1} \right] dA \]

Recapitulate sol. as far as \( \delta \) is concerned. Note that method used to deduce \( \delta \) assures its validity only for \( \Pi \) unless displacements are s.v. \( \delta \) follows from (14.16) (14.7) with the aid of (14.15), (14.17) and Thm. 2.17. A lengthy computation yields:

\[ u_1 = -\alpha \kappa_2 \kappa_3 + \frac{1}{\eta} \left[ \frac{\nu}{2} (l - \kappa_3) (\kappa_1^2 - \kappa_2^2) + \frac{1}{2} \kappa_3^2 - \frac{\kappa_2^3}{3} \right] \]

\[ + \hat{\alpha}_4 + \hat{\omega}_3 \kappa_3 - \hat{\omega}_3 \kappa_2, \]

\[ u_2 = \alpha \kappa_4 \kappa_3 + \frac{1}{\eta} \left[ (l - \kappa_3) \kappa_1 \kappa_2 + \hat{\alpha}_2 + \hat{\omega}_3 \kappa_3 - \hat{\omega}_1 \kappa_2 \right], \]

\[ u_3 = \alpha \phi(\kappa_1, \kappa_2) - \frac{1}{\eta} \left[ \kappa_1 \kappa_3 \left( l - \frac{\kappa_3}{2} \right) + \kappa_1 \kappa_2^2 + F(\kappa_1, \kappa_2) \right] \]

\[ + \hat{\alpha}_3 + \hat{\omega}_2 \kappa_2 - \hat{\omega}_2 \kappa_4. \]

Since \( u_3 \) above is single-valued, sol. is ok even if \( \Pi \) is m

Identify pure-lower and rigid part of \( \Sigma \). Mention due verification of (14.27).

Suppose \( \Pi \) is such that \( Q \in \Pi_0 \), i.e. \( \Pi \) contains its

antithroid. We may dispose over \( \hat{\alpha}_3, \hat{\Sigma} \) in (14.20) to satisfy
relaxed fixity cond. at \( \kappa_3 = 0 \) ("cantilever beam"). Thus \( \dot{\theta}_v = 0 \), \( \dot{\theta}_v = 0 \), \( \varphi(0,0) = 0 \), \( F(0,0) = 0 \). §

Then (14.20) \( \Rightarrow \)

\[
\begin{align*}
\tilde{w}_v(0) = 0, \quad w_{1,1}(0) = w_{2,2}(0) = w_{2,4}(0) = 0.
\end{align*}
\]

and

\[

t_l(\xi z) = \frac{Lz^2}{2\eta I} \left( 1 - \frac{Z}{z} \right) \quad (0 \leq z \leq l)
\]

in agreement with elementary beam theory.

---

The physical significance of \( \alpha \)

Recall from (2.57) that

\[
\tilde{w}_g = \frac{1}{2} (w_{2,11} - w_{1,4})
\]

\( w_3(\xi) \) ... angular displacement of the infinitesimal local rigid rotation about an axis parallel to \( x_3 \)-axis.

Hence,

\[
\tilde{w}_g = \frac{1}{2} \left( \frac{Lz^2}{2\eta I} \left( 1 - \frac{Z}{z} \right) \right) \quad (0 \leq z \leq l)
\]

Recall that this normalization of \( \varphi \) and \( F \) is possible since both are solutions of Neumann problems.
\[ w_{2,3} = \frac{1}{2}(u_{2,13} - u_{1,23}) = \frac{1}{2}(u_{2,3} + u_{3,2}) \frac{1}{j_{31,2}} \frac{1}{j_{32,1}} \]

\[ \frac{\partial w_3}{\partial x_3} = \frac{\partial z_{e_2}}{\partial x_2} - \frac{\partial z_{e_1}}{\partial x_2} \quad \text{(local rate of hoist)} \quad (14.21) \]

Now (14.21), (14.8) \Rightarrow

\[ \frac{\partial w_3}{\partial x_3} = \alpha - \frac{\pi L x_3}{\eta I} \quad \text{on } \Omega , \quad (14.22) \]

so that

\[ \alpha = \left. \frac{\partial w_3}{\partial x_3} \right|_{x_3=0} = \frac{1}{A} \int_{\Pi} \frac{\partial w_3}{\partial x_3} \, dA \quad (14.2) \]

(14.23) yields two alternative interpretations of the parameter \( \alpha \) in Case IV. Explain.

The center of flexure

Return to (14.2), (14.3). Let \( \Lambda_1(\xi_2), \Lambda_2(\xi_2) \) be defined by

\[ \Lambda_1(\xi) = \{ x_3 | -\infty < x_1 < \infty, x_2 = \xi, x_3 = \xi \} \quad (\xi \in (-\infty, \infty)) \]

\[ \Lambda_2(\xi) = \{ x_3 | x_1 = \xi, -\infty < x_2 < \infty, x_3 = \xi \} \quad (\xi \in (-\infty, \infty)) \]
Thus $\Delta_\infty$ is the family of all straight lines parallel to the $x_3$-axis and lying in the plane $x_3=2$. Conditions (14.3) require that the tractions on $\Pi_2$ be statically equivalent to a single force whose line of action is $\Lambda_4(0)$. 

Consider the generalized problem in which the tractions on $\Pi_2$ are statically equivalent to a force $\varepsilon_1 L$ with the line of action $\Lambda_4(\varepsilon_2)$, $\varepsilon_2 \in (-\infty, \infty)$. The only change in boundary conditions, thus arising, is in the last of (14.3b), which must now be replaced by

$$\int_{\Pi_2} (\sigma_{32} x_1 - \sigma_{31} x_2) \, dA = -L \varepsilon_2.$$

(14.25)

Clearly, the previous solution remains valid, provided...
\[ \alpha = \frac{-L}{2(1+\nu)IK} \int \left[ \kappa_2 \frac{\partial F}{\partial x_1} - \kappa_4 \frac{\partial F}{\partial x_2} + \left(1 - \frac{\nu}{2}\right) \kappa_2^3 - \left(2 + \frac{\nu}{2}\right) \kappa_2^2 \kappa_4 \right] dA - \frac{L \varepsilon_2^*}{K} \]  

where \( K \) is again the torsional rigidity of \( \Pi \).

Evidently, \( \alpha = 0 \) if \( \varepsilon_2 = \varepsilon_2^* \), where

\[ \varepsilon_2^* = \frac{-1}{2(1+\nu)IK} \int \left[ \kappa_2 \frac{\partial F}{\partial x_1} - \kappa_4 \frac{\partial F}{\partial x_2} + \left(1 - \frac{\nu}{2}\right) \kappa_2^3 - \left(2 + \frac{\nu}{2}\right) \kappa_2^2 \kappa_4 \right] dA \]  

\( \Lambda_4(\varepsilon_2^*) \) "line of flexure" parallel to \( \kappa_4 \)-axis. In view of \( (14.23) \), tractions on \( \Pi_2 \) that are statically equivalent to a single force with the line of action \( \Lambda_4(\varepsilon_2^*) \) produce average rate of twist according to S.V.'s solution of the corresponding S.V. problem.

The solution of the generalized prob. for tractions on \( \Pi_2 \) that are statically equivalent to a force \( \varepsilon_2 \) with the line of action \( \Lambda_2(\varepsilon_1) \) is immediate from the sol. established above and symmetry.
\( \Lambda_2(\varepsilon^*_1) \) ... line of flexure parallel to \( x_2 \)-axis. Explain.

\[ Q(\varepsilon^*_1, \varepsilon^*_2, \ell) = \Lambda_2(\varepsilon^*_2) \cap \Lambda_2(\varepsilon^*_1) \]  

(14.24)

\( Q \) ... "Center of flexure" ("Center of shear").

Explain significance of \( Q \). Note that location of \( Q \) depends on shape of \( \Pi \) and on \( \varepsilon \) but is independent of load.

If \( \Pi \) is symmetric about \( x_2 = 0 \), (14.17), (14.18) ⇒

\( F(\varepsilon_1, -\varepsilon_2) = F(\varepsilon_1, \varepsilon_2) \) \( \forall (\varepsilon_1, \varepsilon_2) \in \Pi \). In this instance the integrand in (14.27) is an odd function of \( \varepsilon_2 \) and thus \( \varepsilon^*_2 = 0 \). Hence: if \( \Lambda \) is an axis of symm. of \( \Pi_2 \) then \( Q \in \Lambda \).

Suppose: \( \mathbf{b} = \varepsilon \mathbf{a} \) \( \mathbf{b} \)

\( f = [\mathbf{u}, \mathbf{v}, \mathbf{r}] \) ... S.V. flexure sol.

for \( \mathbf{b} \) at \( O \)

\( f' = [\mathbf{u}', \mathbf{v}', \mathbf{r}'] \) ... S.V. flexure sol.

for \( \mathbf{b} \) at \( Q \)
Then, evidently, \( s = s' + s'' \) on \( \mathbb{F} \), where \( s'' = [\xi^2, \xi^3, \sigma] \) is S.V. torsion solution corresponding to the torque
\[ M = \varepsilon_3 (\varepsilon_2 L_1 - \varepsilon_1 L_2). \]

Mintz Treffz' energetic definition of the shear-ant.

**Resultant shear stress, lines of shear stress**

The stresses (14.5) coincide with those in the elementary beam theory. Not so for \( \sigma_{3x} \) given by (14.16). Set
\[ \sigma_3(x) = \varepsilon(x, \xi_3) = \gamma(x) + \varepsilon_3 \sigma_{33}(x), \quad \sigma(x) = \varepsilon_\alpha \sigma_{3\alpha}(x) + x \]
\[ \gamma(x) = \gamma(x_1, x_2) \ldots \text{resultant shear stress on } TT \]
\[ \sigma = ||\gamma|| = \sqrt{\sigma_{31}^2 + \sigma_{32}^2} \]

Note \( x \)-dependence of \( \gamma \) (lost in elementary theory).
The trajectories of \( \gamma(x_1, x_2) \) ("lines of shear stress") obey the differential eq.
\[ \frac{dx_2}{dx_1} = \frac{\sigma_{32}}{\sigma_{31}} \quad (*) \]
From (14.16), if $\alpha = 0$, (*) becomes

$$\frac{d\kappa_2}{d\kappa_1} = \frac{2\partial F/\partial \kappa_2 + 2(2+\gamma)\kappa_1\kappa_2}{2\partial F/\partial \kappa_1 + \gamma \kappa_1^2 + (2-\gamma)\kappa_2^2}$$

(14.2)

Application: S.V. bending of a beam of circular cross-section.

$$\Pi = \{(\kappa_1, \kappa_2) | 0 \leq \kappa_1^2 + \kappa_2^2 < a^2\}$$

$$I = \frac{\int_{\Pi} \kappa_1^2 dA}{\pi a^4} = \frac{\pi a^4}{4}$$

$$\kappa_1 = r \cos \theta, \kappa_2 = r \sin \theta$$

(0 ≤ r ≤ a, 0 ≤ θ < 2π)

In view of polar symm. of $\Pi$, $O = O'$, $\alpha = 0$. Also here $\varphi = 0$ on $\Pi$ (no warping in S.V. torsion).

Set

$$F(\kappa_1, \kappa_2) = F(r \cos \theta, r \sin \theta) = G(r, \theta)$$

(14)

Then, from (14.17), (14.18), (14.31), $G$ obeys
\[ \nabla^2 G = 0 \text{ on } \mathcal{T}, \]

\[ \left. \frac{\partial G}{\partial r} \right|_{r=\alpha} = -\alpha^2 \cos \theta \left[ \frac{2}{2} \cos^2 \theta + (1 - \frac{2}{2}) \sin^2 \theta \right] - (2 + \nu) \alpha^2 \sin^2 \theta \frac{\partial \psi}{\partial \theta} \]

or, on using elementary trigonometric identities,

\[ \nabla^2 G = 0 \text{ on } \mathcal{T}, \left. \frac{\partial G}{\partial r} \right|_{r=\alpha} = -\left( \frac{3}{4} + \frac{\nu}{2} \right) \alpha^2 \cos \theta + \frac{3 \alpha^2}{4} \cos^3 \theta \]

(14.33)

To solve the Neumann prob. (14.33) consider the analytic functions

\[ z_1 = z_1 + i z_2 = r (\cos \theta + i \sin \theta), \quad z_2 = r^3 (\cos 3\theta + i \sin 3\theta) \]

(14.34)

and recall that \( \nabla^2 \text{Re}(z_3)^3 = 0 \text{ on } \mathcal{T} \) if \( f \text{ anal. on } \mathcal{T} \)

Hence (14.33) has the solution

\[ G(r, \theta) = -\left( \frac{3}{4} + \frac{\nu}{2} \right) \alpha^2 r \cos \theta + \frac{r^3}{4} \cos^3 \theta. \]

(14.35)

(14.36), (14.34), (14.32) yield the required flexure function

\[ F(\kappa_1, \kappa_2) = -\left( \frac{3}{4} + \frac{\nu}{2} \right) \alpha^2 \kappa_1 + \frac{1}{4} (\kappa_1^3 - 3 \kappa_1 \kappa_2^2) \quad (\kappa_1, \kappa_2) \in \mathcal{T} \]

(14.36)

(14.36), (14.5), (14.16), (14.30) yield the stress field
\[ \tau_{11} = \tau_{22} = \tau_{42} = 0, \quad \tau_{33} = -\frac{4L(l-v_2)}{\pi a^4} \]

\[ \tau_{31} = \frac{(3+2\nu)L}{2\pi a^4(1+\nu)} \left( a^2 - \nu^2 - \frac{1-2\nu}{3+2\nu} \nu^2 \right), \quad \tau_{32} = -\frac{(1+2\nu)L}{\pi a^4(1+\nu)} \nu \]

Note: \( \nu \)-dependence of \( \tau_{30} \), cf. torsion. In particular,

\[ \tau_{31}(0, \nu_2) = \frac{(3+2\nu)L}{2\pi a^4(1+\nu)} \left( a^2 - \frac{1-2\nu}{3+2\nu} \nu^2 \right), \quad \tau_{32}(0, \nu_2) = 0 \]

\[ \tau_{31}(0,0) = \max_{[0, \alpha]} \tau_{31}(0, \nu) = \frac{(3+2\nu)L}{2(1+\nu)A}, \quad A = \pi a^2, \quad L > 0 \]

\[ \tau_{31}(0, \pm \alpha) = \frac{1+2\nu}{1+\nu} \frac{L}{A} \]

In contrast, the elementary beam theory predicts

\[ \tau_{31}(0, \nu_2) = \frac{4L}{3A} \quad (-\alpha \leq \nu_2 \leq \alpha) \]

\[ \frac{\tau_{31}(0, \nu_2)A}{L} \]

- Graph with \( v = 0.3 \) and \( v = 1.22 \) maximum error approx. 4%.
Lines of shearing stress: By (14.29) here

\[ \frac{d\kappa_2}{d\kappa_1} = \frac{\tau_{ab}}{\tau_{14}} = \frac{2(1+2\nu)\kappa_1 \kappa_2}{(3+2\nu)[-\alpha^2 + \kappa_1^2 + (1-2\nu)\kappa_2^2/(3+2\nu)]} \]  \hspace{1cm} (14.39)

whose complete sol. is

\[ \kappa_1^2 + \kappa_2^2 = \alpha^2 + k \kappa_2^{(3+2\nu)/(1-2\nu)} \]  \hspace{1cm} (k...const.) (14.40)

Note that \( k = 0 \) yields the eq. of C.

\[ \text{Deformations. Assume freely, conds. on p.58. The deflection curve} \]
\[ \text{i.e. } u, (0,0,\kappa_3), \text{ according to (14.20), is independent of} \]
\[ \text{shape of } TT \text{ and of } \nu; \text{ it was discussed earlier.} \]

Consider now the axial displacement \( u_3 \). By (14.20), (14.36), and since here \( \phi = 0 \) on TT, one has
\[ u_3(\kappa_1, \kappa_2, \kappa_3) = \frac{L}{\pi \rho \alpha^3 E} \left[ (3+2\nu) \kappa_2^2 - 2\kappa_3 (2\kappa_2 - \kappa_3) \right] \kappa_3 \]

\[ - \frac{L}{\pi \rho \alpha^3 E} \frac{( \kappa_1^2 + \kappa_2^2 ) \kappa_1}{\kappa_1 \kappa_2} \]

\[ \delta(\kappa_1, \kappa_2) \]

The term linear in \( \kappa_1 \) evidently represents a rigid tilting of cross-sections about its centroidal axis par. \( \kappa_2 \)-axis. \( \delta(\kappa_1, \kappa_2) \) represents warping.

Contour lines of \( \delta(\kappa_1, \kappa_2) \):

Mention work cite for IT., ellipse, equilateral triangle, semi-circle, semicircular segment of circle, circular sector, rectangle. Conformal mapping, numerical methods. Time-honored function, see book analogy.
Remarks. See Bogy, J.A.M., 89, 1, p. 176 (1967) for an
alternative sol. of the preceding prob. within S.V.'s form-
dulation (quantitative evidence in support of S.V.'s pri-
mary

Mention exact sols. for the following cross-sections:
ellipse, equilateral Δ, segment of circle, circular sector,
rectangle. Timoshenko's stress functions and membran-
analogy for Case IV (see Timoshenko-Goodier).

Refer to minimum energy characterization of S.V.'s
solutions to the relaxed S.V. prob. See Knowles & Co.,