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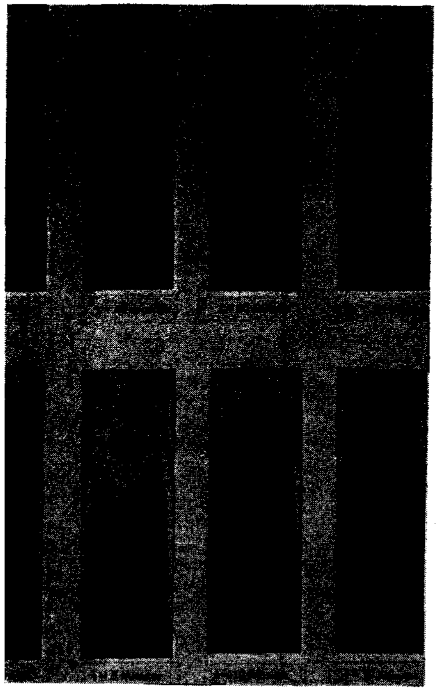

PHOTOTHERMOELASTIC STUDY (DARK FIELD) OF A BEAM WITH $L/d = 0.3$
(Dark specks are due to dry ice.)

tory study was conducted of thermal stresses in short of various $L/d$-ratios exposed to dry ice on one surface.

The results are pictorial in nature as shown in Figs. 7 to 9. Of particular importance are the stresses at the ends of the beam, an effect unimportant for the long beams treated previously in this section. The experimental procedure was the same as used for the long-beam study with the exception that no temperature readings were obtained.

Fig. 7 illustrates the time-dependent nature of the thermal stresses as well as the fairly severe end effects for a beam of $L/d = 3.25$. At the free ends, high shearing stresses can be observed which extend into the beam for a distance approximately equal to the depth of the beam. Such behavior would be anticipated on the basis of the Saint Venant effect.

Figs. 8 and 9 represent elements with beam dimensions of $L/d = 1$ and $L/d = 0.308$, respectively. It is interesting to note that the thermal-shock pattern at the upper edge and free end regions in Figs. 7 to 9 is almost identical for short times. It is only after relatively long times that the fringe pattern becomes characteristically different for the various $L/d$-ratio elements.

## CONCLUSIONS

On the basis of this exploratory study, it appears that the photothermoelastic technique has considerable promise as a quantitative tool for verifying thermal-stress analyses. In addition, the ability to observe the time-dependent behavior of complete thermal-stress fields places photothermoelasticity in a unique position in the experimental thermal-stress-analysis field.

More specifically, the following are the conclusions of this investigation:

(a) The optical and physical properties pertinent to the analysis of thermal stresses have been obtained for Paraplex P-43 over a temperature range from 70 to −40 F.

(b) For the disk and long-beam models which are representative of interference and thermal-gradient type of thermal-stress fields, respectively, good correlation was obtained between the observed and theoretically determined fringe distributions.

(c) For short beams, severe end effects were observed which extend for a distance approximately equal to the beam depth. During the initial stages of sudden temperature application, the thermal-stress field at the upper edge and free-end regions appears to be independent of the beam dimensions.

# Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate

BY G. R. IRWIN,[1] WASHINGTON, D. C.

**A substantial fraction of the mysteries associated with crack extension might be eliminated if the description of fracture experiments could include some reasonable estimate of the stress conditions near the leading edge of a crack particularly at points of onset of rapid fracture and at points of fracture arrest. It is pointed out that for somewhat brittle tensile fractures in situations such that a generalized plane-stress or a plane-strain analysis is appropriate, the influence of the test configuration, loads, and crack length upon the stresses near an end of the crack may be expressed in terms of two parameters. One of these is an adjustable uniform stress parallel to the direction of a crack extension. It is shown that the other parameter, called the stress-intensity factor, is proportional to the square root of the force tending to cause crack extension. Both factors have a clear interpretation and field of usefulness in investigations of brittle-fracture mechanics.**

## INTRODUCTION

DURING and subsequent to the recent World War, investigations of fracturing have shared in the general growth of applied-mechanics research. Among the fracture failures responsible for interest in this field were those of welded ships, gas-transmission lines, large oil-storage tanks, and pressurized cabin planes. The propagation of a brittle crack across one or more plates in which the average tensile stress was thought to be safely below the yield strength is a prominent feature of these examples.

As a result of these investigations there was a revival of interest in the Griffith theory of fracture strength (1).[2] It was pointed out independently by Orowan (2) and by the author (3) that a modified Griffith theory is helpful in understanding the development of a rapid fracture which is sustained with energy from the surrounding stress field. Expositions of this idea have been given (3, 4, 5) using such terms as fracture work rate and strain-energy release rate.

The basic idea of the modified Griffith theory is that, at onset of unstable fast fracturing, one can equate the fracture work per unit crack extension to the rate of disappearance of strain energy from the surrounding elastically strained material. The term, disappearance of strain energy, refers to the loss of strain energy which would occur if the system were isolated from receiving energy, for example, from movement of the forces applying tension to the material. For convenience this is referred to here as the fixed-grip strain-energy release rate. Since the strain-energy disappearance rate at any moment depends on the load magnitudes rather than on movement of the points of load application, use of the fixed-grip strain-energy release-rate concept is not limited to fixed-grip experiments.

It is the purpose of this paper to describe the relation of these terms to the elastic stresses and strains near the leading edge of a somewhat brittle crack. For purposes of this paper "somewhat brittle" means that a region of large plastic deformations may exist close to the crack but does not extend away from the crack by more than a small fraction of the crack length.

Previous investigations (3–7) have established a viewpoint with respect to the mechanics of fracturing which may be summarized in part as follows:

The fixed-grip strain-energy release rate has the same role as an influence controlling time rate of crack extension as the longitudinal load has in controlling time rate of plastic extension of a tensile bar. In the latter case the force per unit area tending to cause plastic extension is the longitudinal stress. In the former case a motivating force per unit thickness can be defined quite generally in terms of the rate of conversion of strain energy to thermal energy during crack extension. This generalized force is the rate of decrease of the fixed-grip strain energy with crack extension on a unit-thickness basis. Also this energy rate can be regarded as composed of two terms: (a) The strain-energy loss rate associated with nonrecoverable displacements of the points of load application (assumed zero in this discussion); and (b) the strain-energy loss rate associated with extension of the fracture accompanied only by plastic strains local to the crack surfaces. The second of these two terms, herein called $\mathcal{G}$, appears to be the force component most directly related to crack extension and the one with the most practical usefulness.

Determination of characteristic values of $\mathcal{G}$ for onset or arrest of rapid fracturing and the applications of such measurements to "fail-safe" design procedures have been discussed elsewhere (4, 5, 8, 9). It will be shown here that the tensile stresses near the crack tip and normal to the plane of the crack are determined by the force tendency $\mathcal{G}$. The discussion is arranged so as to develop relationships useful in the analysis of fracture experiments whether the purpose of the work is to determine characteristic $\mathcal{G}$-values or simply to determine the stress field near the leading edge of the crack.

The material of this paper is, at one point, related to Sneddon's analysis of stresses near an embedded crack having the shape of a flat circular disk (10). Otherwise, for simplicity and bearing in mind the service fracture failures referred to in the foregoing, discussion is restricted to a straight crack in a plate. It is assumed the plate thickness is small enough compared to the crack length so that generalized plane stress constitutes a useful two-dimensional viewpoint. In addition it is assumed the crack is moving, as brittle cracks generally do move, along a path normal to the direction of greatest tension, so that the component of shear stress resolved on the line of expected extension of the crack is zero.

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[2] Numbers in parentheses refer to the Bibliography at the end of the paper.

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NOTE: Statements and opinions advanced in papers are to be understood as individual expressions of their authors and not those of the Society. Manuscript received by ASME Applied Mechanics Division, February 19, 1956. Paper No. 57—APM-22.

### Representative Stress Fields Associated With Cracks

A paper by Westergaard (11) gave a convenient semi-inverse method for solving a certain class of plane-strain or plane-stress problems. Let $\bar{\bar{Z}}$, $\bar{Z}$, and $Z'$ represent successive derivatives with respect to $z$ of a function $\bar{\bar{Z}}(z)$, where $z$ is $(x + iy)$. Assume that the Airy stress function may be represented by

$$F = \mathrm{Re}\bar{\bar{Z}} + y\, Im\bar{Z} \qquad [1]$$

then

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = \mathrm{Re}\bar{Z} - y\, Im Z' \qquad [2]$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = \mathrm{Re}\,\bar{Z} + y\, Im Z' \qquad [3]$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -y\, \mathrm{Re}\, Z' \qquad [4]$$

By choices of the function $Z(z)$, Westergaard showed solutions for stress distribution as influenced by bearing pressures or cracks in a variety of situations. The class of problems which can be solved in this way is limited to those such that $\tau_{xy}$ is zero along the $x$-axis.

In particular, if a large plate contains a single crack on the $x$-axis whose length is small compared to the plate dimensions or a colinear series of such cracks, and if the applied loads are such that $\tau_{xy}$ is zero along the $x$-axis, then the stress distribution is readily constructed with the aid of Westergaard's semi-inverse procedures.

Two examples of such problems were given by Westergaard (11) as follows:

1 A central straight crack of length $2a$ along the $x$-axis in an infinite plate with a biaxial field of tension $\sigma$ at large distances from the crack

$$Z(z) = \frac{\sigma}{[1 - (a/z)^2]^{1/2}} \qquad [5]$$

2 A series of equally spaced straight cracks of length $2a$, on the $x$-axis in an infinite plate with biaxial stress $\sigma$, as before, and with the distance between the crack centers, $l$

$$Z(z) = \frac{\sigma}{\left[1 - \left(\dfrac{\sin \pi a/l}{\sin \pi z/l}\right)^2\right]^{1/2}} \qquad [6]$$

Three additional examples obtainable with the semi-inverse procedure suggested by Westergaard are as follows:

3 Single crack along the $x$-axis extending from $-a$ to $a$ with a wedge action applied to produce a pair of "splitting forces" of magnitude $P$ located at $x = b$ (see Fig. 1)

$$Z(z) = \frac{Pa}{\pi(z - b)z}\left[\frac{1 - (b/a)^2}{1 - (a/z)^2}\right]^{1/2} \qquad [7]$$

4 The situation of example 3 with an additional pair of forces of magnitude $P$ at $x = -b$

$$Z(z) = \frac{2Pa}{\pi(z^2 - b^2)}\left[\frac{1 - (b/a)^2}{1 - (a/z)^2}\right]^{1/2} \qquad [8]$$

5 Example 3 repeated along the $x$-axis at intervals $l$, and with the wedge action centered so that $b$ is zero

$$Z(z) = \frac{P \sin \dfrac{\pi a}{l}}{l\left(\sin \dfrac{\pi z}{l}\right)^2}\left[1 - \left(\frac{\sin \dfrac{\pi a}{l}}{\sin \dfrac{\pi z}{l}}\right)^2\right]^{-1/2} \qquad [9]$$

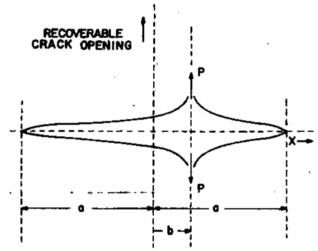


Fig. 1 Opening of a Crack by Wedge Forces

In all of these problems a uniform compression, $-\sigma_{ox}$, may be added to the value of $\sigma_x$ given by Equation [2]. Since linearized elasticity relations are assumed to apply, one may obtain the $Z$-function for combined tension and wedge action by adding the appropriate $Z$-function for tension to the appropriate $Z$-function for a pair of wedge forces.

As an extending crack moves across a plate of finite width the crack may attain sufficient length so that the tensile forces acting to cause crack extension are not sufficiently accurate when obtained using infinite plate relations such as those of Equations [5] and [7]. The major adjustment required is that of the total load across the $x$-axis from the end of the crack to the side of the plate. A convenient way to make this adjustment, if the crack is centered, is to use expressions for $Z$ such as in examples 2 and 5. The side boundaries of the plate would then be taken to occur at $x = -l/2$ and $x = +l/2$. In the stress distributions resulting from examples 2 and 5 the shearing stress $\tau_{xy}$ is zero along the side boundaries. However, the side boundaries are represented as possessing a distribution of $x$-direction loads which should be absent. Depending upon the objectives of the stress analysis this defect may be outweighed in importance by the convenience of having an approximate solution of the problem in compact form.

Suppose, next, that the situation to be studied is a crack extending across a finite-width plate from one of the plate side boundaries. Let the intersection of the crack with the side boundary be the origin of co-ordinates and let the line of crack extension be the positive portion of the $x$-axis, the end of the crack being at $x = a$. It will be assumed that weights or blocks have been set against the side boundaries so as to prevent or greatly reduce the tendency of these boundaries to move in the negative $x$-direction as the crack extends. In this event $Z$-functions similar to those of examples 2 and 5 may again be employed as a convenient means for obtaining a compact approximation to the stress distribution. In this situation the side boundaries of the plate would be assumed to be at $x = 0$ and at $x = l/2$.

In any of the foregoing examples the only stress acting at the edges of the crack is the optional added stress in the $x$-direction $-\sigma_{ox}$. An uncertainty as to proper choice of $\sigma_{ox}$ exists for the example discussed previously of a crack extending from one side of a finite-width plate. In addition, if the crack moves rapidly, determination of the stress distribution away from the crack will require a dynamic-stress analysis.

### Stress Environment of the End of the Crack

However, the stress distribution near the end of the crack can be expressed ($a$) independently of uncertainties of both magnitude of applied loads and of the dynamic unloading influences, and ($b$) in such a way that records from several strain gages placed near the end of the crack serve to determine the "crack-tip stress distribution."



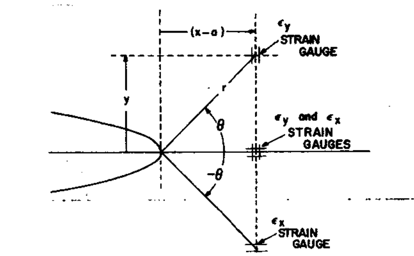


Fig. 2 Relation of $r$ and $\theta$ to $y$ and $(x - a)$ and Examples of Locations for Strain Gages

Consider for all of the five examples the substitution of variables

$$z = a + re^{i\theta}$$

where

$$r^2 = (x - a)^2 + y^2 \text{ and } \tan \theta = y/(x - a)$$

as shown in Fig. 2.

If one assumes quantities such as $r/a$ and $r/(a - b)$ may be neglected in comparison to unity, one finds in each case

$$\sigma_y = \left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} \frac{\cos \theta/2}{\sqrt{(2r)}}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad [10]$$

and

$$\sigma_x = \left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} \frac{\cos\dfrac{\theta}{2}}{\sqrt{(2r)}}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \sigma_{ox} \qquad [11]$$

where $E$ is Young's modulus. $\mathcal{G}$ is independent of $r$ and of $\theta$ and will be discussed in following sections of this paper.

For a crack traversing a plate, the thickness of which is considerably smaller than the crack length, a generalized plane-stress viewpoint is appropriate and $\sigma_z$ is zero. However, for comparison with results obtained by Sneddon (10) one may consider for the moment the set of three extensional stresses which would pertain to a plane-strain analysis. Sneddon studied the stress distribution predicted by linear elastic theory in the vicinity of a "penny-shaped" crack embedded in a much larger solid material and subjected to tension perpendicular to the plane of the crack. For the extensional stresses in the close neighborhood of the crack outer boundary, Sneddon gave expressions identical to Equations [10] and [11] with regard to the functional relationship of $\sigma_x$ and $\sigma_y$ to $r$ and $\theta$. A third extensional stress directed parallel to the outer boundary of the penny-shaped crack was given by Sneddon with the remark that no counterpart to this third extensional stress existed in a two-dimensional analysis of stresses near a crack. However, the remark applies only to the two-dimensional analysis assuming generalized plane stress. For the two-dimensional analysis assuming plane strain the third extensional stress, which is Poisson's ratio times the sum of $\sigma_x$ and $\sigma_y$, as in Equations [10] and [11], is the counterpart to Sneddon's third extensional stress component. Thus for any small region around the outer boundary of Sneddon's penny-shaped crack, the stresses, strains, and displacements correspond to a situation which is locally one of plane strain. The preceding comment becomes intuitively obvious when one considers that, in Sneddon's example, all particle displacements lie in planes which contain the axis of symmetry. These planes would approximate to a set of parallel planes within any region whose dimensions are very small compared to distance from the region to the axis of symmetry.

### Force Tending to Cause Crack Extension

As the crack extends, an energy transfer from mechanical or strain energy into other forms occurs in the vicinity of the crack. The process is such that transfer of strain energy to heat dominates.

$\mathcal{G}$ is the magnitude of this energy exchange associated with unit extension of the crack and may be regarded as the force tending to cause crack extension. This may be seen as follows:

The linear elasticity relations resulting from Equations [5] through [11] correspond to a parabolic shape for the crack opening near the crack tip. In Fig. 3 the origin of $x$, $y$-co-ordinates has been shifted so that the crack opening, shown by the dashed line,

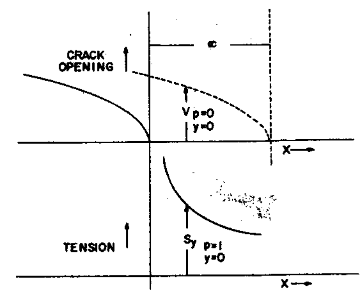


Fig. 3 Linear-Elastic-Theory Crack Openings and Stresses Near End of a Crack

extends to $x = \alpha$. It is assumed $\alpha$ is very small compared to the length of the crack. If $y$-direction tensions given by

$$S_y(p) = p\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2} \frac{1}{\sqrt{(2x)}} \qquad [12]$$

are exerted on the edges of the crack from $x = 0$ to $x = \alpha$, and $p$ is increased from zero to 1, the crack is closed up so that the crack opening appears to end at the origin as shown by the full line. The factor $p$ may be regarded as a proportional loading parameter. To the same approximation as Equation [10], the crack opening from $x = 0$ to $x = \alpha$ at any time during the closure operation is given by

$$v(p) = (1 - p)\frac{2}{E}\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2}\sqrt{[2(\alpha - x)]} \qquad [13]$$

Since the degree of closure is a linear function of $S_y$ the work done by the closing forces as $p$ is varied from zero to 1 is given by

$$\int_0^\alpha S_y(1)v(0)dx = \frac{2\mathcal{G}}{\pi}\int_0^\alpha\left(\frac{\alpha - x}{x}\right)^{1/2} dx = \alpha\mathcal{G} \qquad [14]$$

Thus $\alpha\mathcal{G}$ is the "fixed grip" loss of energy from the strain-energy field as the crack extends by the amount $\alpha$ and the generalized force interpretation of $\mathcal{G}$ is apparent.

For mathematical simplicity the foregoing calculation was based upon the linear elasticity stresses and crack-opening displacements in the immediate vicinity of the crack tip. One should not assume, however, that local stress relaxation and crack-opening distortion by plastic flow necessarily change the rate of loss of strain energy with crack extension from that indicated in the foregoing by an appreciable amount. The procedure leading to Equation [14] is equivalent to finding the derivative with respect to crack length of the total strain energy under fixed-grip conditions. The contribution to this calculation from a small circular region enclosing the crack tip is relatively small. In situations such as those of Equations [5] through [9], the fraction of $\mathcal{G}$ contributed from this region is, in fact, only about a third of the ratio of the outer radius of the region to the half length $a$ of the crack. Thus if plastic strains near a crack affect the stress field only within distances from the crack, small in relation to the crack length, then the influence of these plastic strains on the calculation of $\mathcal{G}$ is correspondingly small.

### Remarks on Measurement Methods

Consider the situation suggested earlier of a crack moving in the $x$-direction across a large plate. As the crack moves under and beyond a strain gage placed close to its path for $\epsilon_y$ measurement, Fig. 2, the gage output is expected to rise and then fall to a small value. An uncertainty in interpretation of the gage record in terms of stress will exist if $\sigma_{ox}$ is uncertain. If it can be assumed that $\sigma_{ox}$ is zero then, using Equations [10] and [11]

$$E\epsilon_y = \sigma_y - \nu\sigma_x =$$
$$\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2}\frac{\cos\theta/2}{\sqrt{(2r)}}\left[(1-\nu) + (1+\nu)\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \ldots\ldots [15]$$

where $\nu$ is Poisson's ratio. By putting $r = y \csc\theta$ and differentiating with respect to $\theta$ with $y$ constant, one finds $\epsilon_y$ should be greatest when the gage position relative to the end of the crack is at $\theta = 70$ deg. This result is quite insensitive to the assumed value of $\nu$ (unpublished calculations by L. McFadden and J. H. Hancock based upon Equation [9]).

A better situation for analysis purposes exists if both $\epsilon_y$ and $\epsilon_x$ are measured. In this event one has

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \ldots\ldots\ldots\ldots\ldots [16]$$

Differentiating Expression [10] for $\sigma_y$ with respect to $\theta$ holding $y$ constant, one finds the maximum will occur when the measurement position is at 73.4 deg. As a crack moves under and beyond the position of measurement of $\epsilon_y$ and $\epsilon_x$ the quantity $(\epsilon_y + \nu\epsilon_x)$ plotted against time should have a maximum at that angle. Thus with $\theta$, $r$, and $(\epsilon_y + \nu\epsilon_x)$ known for a particular location of the crack, the stress-intensity factor $(E\mathcal{G}/\pi)^{1/2}$, and the crack extension force $\mathcal{G}$ existing at the moment of that crack location can be calculated.

The stresses near the crack tip predicted by linear elasticity theory are calculable from Equations [10] and [11] except for knowledge of the intensity factor $(E\mathcal{G}/\pi)^{1/2}$, which appears in the expressions both for $\sigma_x$ and for $\sigma_y$ and the additive uniform stress factor $-\sigma_{ox}$, which appears in the expression for $\sigma_x$. Any arrangement of strain gages which permits determination of these two factors serves to determine the crack-tip stress distribution. The region in which the stresses are thus represented is an annular region which excludes any large distortions close to the crack but which extends outward only a small fraction of the crack length.

### Conclusions

The stress field near the end of a somewhat brittle tensile fracture, in situations of generalized plane stress or of plane strain, can be approximated by a two-parameter set of equations. The most significant of these parameters, the intensity factor, is $(E\mathcal{G}/\pi)^{1/2}$ for plane stress where $\mathcal{G}$ is the force tending to cause crack extension.[3] When the experimental situation permits use of strain gages at distances from the crack tip, small compared to the crack length, values of $\mathcal{G}$ and $\sigma_{ox}$ may be evaluated conveniently by measuring local strain at selected positions.

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[3] For plane strain, one substitutes $E/(1-\nu^2)$ for $E$ in the expression for the stress-intensity factor. No change in the magnitude of the stress-intensity factor occurs because $\mathcal{G}$, for plane strain, is $(1-\nu^2)$ times $\mathcal{G}$ for plane stress.

# Stresses in a Perforated Strip

By CHIH-BING LING,[1] TAIWAN, CHINA

**This paper presents an analytic solution of the classical problem dealing with the stresses in an infinite strip having an unsymmetrically located perforating hole. The solution is applicable to any stress system acting in the strip, which is symmetrical with respect to the line of symmetry of the strip. The required stress function is constructed by using four series of biharmonic functions and a biharmonic integral. The four series of biharmonic functions are formed from a class of periodic harmonic functions specially constructed for the purpose. The solution can be regarded as a complete solution of the problem in the sense that, unlike the previous solutions by Howland, Stevenson, and Knight for a symmetrically perforated strip, it is valid in the entire strip. Numerical examples are given for the fundamental cases of longitudinal tension and transverse bending.**

### Introduction

The stresses in an infinite strip having a symmetrically located perforating hole when the strip is under a longitudinal tension were investigated by Howland (1)[2] in 1930. The method used in his solution is essentially one of successive approximations such that the required solution is the sum of a series of biharmonic functions any one of which cancels the normal and tangential stresses on one of the boundaries due to the previous solution. Later, Howland in 1934 with the collaboration of Stevenson (2) extended the method to unsymmetrical stress systems acting in the same symmetrically perforated strip, and worked out in detail the cases of transverse bending with or without shear. The longitudinal tension case as just mentioned also was solved by Knight (3) in 1934 in a way which is more direct than that used by Howland. Nevertheless, all the solutions are valid in the neighborhood of the hole only.

In this paper an analytic solution for the stresses in an infinite strip having an unsymmetrically located perforating hole will be presented. The solution is applicable to any stress system acting in the strip, which is symmetrical with respect to the line of symmetry of the strip. The required stress function is constructed by using four series of biharmonic functions and a biharmonic integral. The four series of biharmonic functions are formed from a class of periodic harmonic functions as defined in Appendix 1. The present solution can be regarded as a complete solution of the problem in the sense that, unlike the previous solutions, it is valid in the entire strip. In what follows, the method of solution is first described and then numerical examples are given for the fundamental cases of longitudinal tension and transverse bending.

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Discussion of this paper should be addressed to the Secretary, ASME, 29 West 39th Street, New York, N. Y., and will be accepted until October 10, 1957, for publication at a later date. Discussion received after the closing date will be returned.

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### Method of Solution

Consider an infinite strip of uniform width $2b$, perforated unsymmetrically by a circular hole of radius $\lambda b$. Let the origin of the co-ordinates be at the center of the hole and the $x$-axis parallel to the edges of the strip. For convenience, the rectangular co-ordinates $(x, y)$ henceforth will be regarded as dimensionless quantities referring to a typical length $b$ or one half of the width of the strip. The edges of the strip will be represented by the lines $y = 1 - c$ and $(-1 - c)$, respectively, as shown in Fig. 1; $c$ being a quantity numerically less than unity.

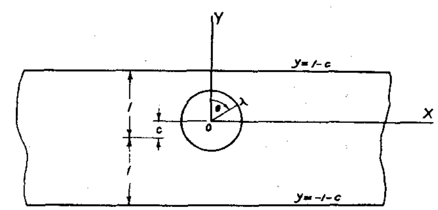


Fig. 1 The Strip, Lengths Being Measured by $b$

Suppose that in the absence of the hole the stresses in the strip are derived from a basic stress function $\chi_0$. In order to allow for the effect of the hole, an auxiliary stress function $\chi_1$ is constructed, using series of suitable biharmonic functions and a biharmonic integral, such that the corresponding stresses vanish at $x$ infinity or the ends of the strip as well as along the edges. This is added to $\chi_0$ so that the required stress function is given by

$$\chi = \chi_0 + \chi_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots [1]$$

The boundary conditions at the rim of the hole are then satisfied by adjusting the coefficients of superposition attached to the series of biharmonic functions. The biharmonic functions have singularities at the origin. Such singularities eventually will be excluded from the material of the strip by the perforating hole.

Suppose further that the basic stress function is even in $x$; that is, it is symmetrical with respect to the $y$-axis or the line of symmetry of the strip. Consequently, the auxiliary stress function is also even in $x$. It will be constructed as follows

$$\chi_1 = b^2\sum_{s=0}^{\infty} A_{2s}S_{2s}(x, y) + b^2\sum_{s=1}^{\infty} D_{2s}S_{2s-1}(x, y)$$
$$+ b^2\sum_{s=1}^{\infty}(y+c)B_{2s}S_{2s-1}(x, y) + b^2\sum_{s=1}^{\infty} C_{2s}\{(y+c)S_{2s}(x, y)$$
$$+ S_{2s-1}(x, y)/(2s-1)\} + b^2\int_0^{\infty}\{\psi_1(m)\cosh m(y+c)$$
$$+ \psi_2(m)(y+c)\sinh m(y+c) + \psi_3(m)\sinh m(y+c)$$
$$+ \psi_4(m)(y+c)\cosh m(y+c)\}\cos mx\, dm \ldots\ldots [2]$$

where $S_s$ are periodic harmonic functions defined in Appendix 1; $A_{2s}, B_{2s}, \ldots$ being coefficients of superposition and $\psi_n$ arbitrary functions. The auxiliary stress function thus constructed satis-