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where F_n is found by substitution in (1) and hence we obtain

$$W_2 = \frac{4qe^{-cy}}{\pi D_0}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{(n-1)/2}}{n(\gamma_n^4 - \gamma_n^2c^2y)}\cos\gamma_n x \quad (10)$$

Thus the complete solution is

$$W = \frac{4q}{\pi D_0}\sum_{n=1,3,5}^{\infty}\left(Ae^{\lambda_1 y} + Be^{\lambda_2 y} + Ce^{\lambda_3 y} + De^{\lambda_4 y} + \frac{(-1)^{(n-1)/2}e^{-cy}}{n(\gamma_n^4 - \gamma_n^2c^2y)}\right)\cos\gamma_n x \quad (11)$$

where the λ are given by Equations (7) and the constants are obtained from the conditions on $y = 0, b$. The particular case of a uniform thickness plate can, of course, be obtained by writing $c = 0$. Noting that Equations (7) then give two sets of equal roots $\pm\gamma_n$, the solution for uniform thickness is the well-known

$$W = \frac{4q}{\pi D}\sum_{n=1,3,5}^{\infty}\left(Ae^{\gamma_n y} + Be^{-\gamma_n y} + Cye^{\gamma_n y} + Dye^{-\gamma_n y} + \frac{(-1)^{(n-1)/2}}{n\gamma_n^4}\right)\cos\gamma_n x \quad (12)$$

The summations of (11) for specific boundary conditions will involve considerable labor and since they must be repeated for each side ratio a/b and c value, they will not be made here.

Discussion

A Photoelastic Study of Strain Waves Caused by Cavitation[1]

A. J. Durelli.[2] The author emphasizes the advantages of the use in photoelasticity of the material-strain fringe value in respect to the material-stress fringe value. This is particularly true in viscoelastic materials. Substantiation of the usefulness of this approach can be found in papers published by the writer and several of his associates, dealing with various phases of the properties of phenolformaldehydes,[3] of Columbia Resin CR-39,[4] and of epoxy resins.[5]

In some of these papers it is pointed out that it is actually possible to take advantage of the fact that birefringence can be present in a plastic, without a stress. The "crept-in" photoelastic effect can be used a long time after the loads have been removed from the specimen.[6]

More recently it also has been shown that transversal displacements in Marblette plates can be measured a long time after the loads have been removed, to determine the sum of the principal stresses.[7] However, only in a few instances has the writer computed the value of the strain fringe values. Strains have to be measured precisely for this purpose. Poisson's ratio values giver by the author are remarkably accurate.

The results published by the author in Fig. 2 seem very im portant to the writer because they indicate the limit of applicatio of his method of working with CR-39.

The writer has difficulty in following the evaluation of th photoelasticity tests reported by the author. It doesn't seem t be proved that the displacement of the residual fringes in thi pieces of Columbia resin is as sensitive and as accurate a mean of measurement as the author claims. The conclusions obtaine from Formula [21] seem to be premature. The formula depend in a very sensitive way on the value of Poisson's ratio, on th value of the diameter of the strained area, and on the fring order. The author postulates the diameter of the strained ar without giving experimental evidence and uses a value of 0.25 fringe order which again does not seem substantiated by his test

Author's Closure

The author wishes to thank Professor Durelli for his many fi comments. His results concerning the strain-optic coefficient cast epoxy resins are not inconsistent with the present stud Whether or not this is advantageous depends upon the applicatio

With regard to the accuracy of the strain measurements, th strain gages on the specimen were calibrated with Huggenberg extensometers against time.[8] The Huggenberger gages had bee calibrated by means of monochromatic light interference pa terns. Thus the creep strains measurements were accurate to per cent. The precision of Poisson's ratio is indicated in Fig.

For the dynamic experiments (see Fig. 13), no direct stra gage calibration was possible. However, the following indire calibration was used:

Two A-8 strain gages were glued onto the impact specime one on each side. Directly on top of each gage, another A-8 ga was cemented. A comparison of the change in resistance durin impact between the inboard set and the outboard set revealed that the difference was within the manufacturer's stated calibration accuracy. This was evidence that the gages cemented to the bar read the true strain within a few per cent.

Fig. 2 shows the viscoelastic behavior of CR-39 during long-time creep tests. Since the author was mostly concerned with high-speed response, the results presented in Figs. 7, 8, and 9 are the ones which are applicable, while Fig. 2 indicates the limitation of CR-39 for static experiments.

The motion pictures of cavitation were used to obtain the sequence of events leading to the stress pulse, not the magnitude of the fringe order. No accuracy was claimed for the motion picture photoelastic experiment; see page 347 of the paper. The fringe order of 0.25 is reported on the bottom of page 346, and was measured by means of a photomultiplier with 5461Å light; see Fig. 15. The fringe order was determined as follows:

The quarter wave plate and analyser were mounted such that they could be rotated at 30 rev/sec. The positions of the polaroids and quarter wave plates were interchanged, and the analyser was then rotated, with the X sweep of the photomultiplier oscilloscope in operation. The height of the trace on the oscillogram then corresponded to $^1/_2$ fringe order. After reassembly of the apparatus, the cavitation experiment with the photomultiplier was then performed. The amplitude of the spike (Fig. 15) was then measured. Its maximum height corresponds to $^1/_2$ the calibration height ± 10 per cent; thus the fringe order is 0.25 ± 10 per cent.

The diameter of the stressed area is postulated to be the same as that of the small pit, reported at the top of page 347. These individual pits are clearly visible in Fig. 16 (top), surrounding the main pit. The variation of the diameter of the pit was less than 20 per cent.

The author hopes that the foregoing description of the calibration, while somewhat lengthy, may be of help to the casual reader who may not be acquainted with some of the ultra high-speed techniques which have been developed at the California Institute of Technology.

The value of 2×10^5 psi cavitation stress should not be surprising to Professor Durelli, in view of other published data which indirectly indicate the order of magnitude of the stress level.[9] In fact, the apparent high level of stress led to the reported investigation in order to obtain a more quantitative estimate of the cavitation stress level.

[1] By G. W. Sutton, published in the September, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 340–348.

[2] Professor, Illinois Institute of Technology, Chicago, Ill. Mem. ASME.

[3] Some Unorthodox Procedures in Photoelasticity," by A. J. Durelli and R. L. Lake, *Proc. SESA*, vol. 9, no. 1, 1951, pp. 97–122.

[4] "Stress Concentrations Produced by Semi-Circular Notches in Infinite Plates Under Uniaxial State of Stress," by A. J. Durelli, R. L. Lake, and E. Phillips, *Proc. SESA*, vol. 10, no. 1, pp. 53–64.

[5] "Experiments for the Determination of Transient Stress and Strain Distributions in Two-Dimensional Problems," by A. J. Durelli and W. F. Riley, JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, 1957, pp. 69–76.

[6] "Stresses in Rotating Parts," by R. L. Lake and A. J. Durelli, *Machine Design*, vol. 25, no. 7, 1953, pp. 128–131.

[7] "Use of Creep to Determine the Sum of the Principal Stresses in Two-Dimensional Problems," by A. J. Durelli and W. F. Riley, *Proc. SESA*, vol. 14, no. 2, 1957, pp. 109–116.

[8] "A Study of the Application of Photoelasticity to the Investiga tion of Stress Waves," by G. W. Sutton, PhD thesis, Californi Institute of Technology, Pasadena, Calif., 1955.

[9] "On the Mechanism of Cavitation Damage," by M. S. Plesset and A. T. Ellis, TRANS. ASME, vol. 77, October, 1955, pp. 1055–1064.

Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate[1]

F. A. McClintock.[2] It is interesting to note how high the applied stresses can be and still satisfy the author's condition that the radius of the plastic zone be small compared to the crack length; i.e., $r_y/a \leqslant r/a < 0.1$. The solid line in Fig. 1 of this discussion shows the distribution of strain in front of a crack in a biaxial tensile field, as given by Equations [3] and [5]. As indicated by the dotted line, plastic yielding will extend at least as far as the radius at which the stress is the yield stress of the material, and actually equilibrium of the y component of force across the x-axis will require that it extend farther. Triaxiality would only inhibit yielding significantly if the thickness of the sheet were greater than about r_y. In the case of a crack subjected to shear, the plastic zone extends twice as far as would be calculated from the theory of elasticity.[3] Taking the radius of the plastic zone to be twice the radius at which $\sigma = Y$ according to elasticity theory, we find

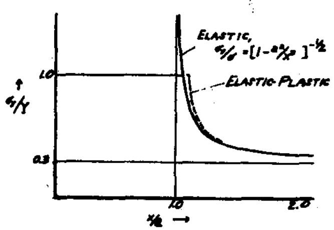

Fig. 1 Stress distributions ahead of crack under biaxial tension, for $\sigma/Y = 0.3$

$$r_p/a = 2[(\sigma/Y)^2/2] < 0.1 \quad \text{or } \sigma/Y < 0.3$$

Hence the nominal applied stress must be well below the yield stress if the radius of the zone of plastic deformation is to be small compared to the crack length. And perhaps one also should require that the zone of plastic deformation should be small compared to an elastic region which is itself small compared to the crack length. This would limit the applied stress even further.

Where the plate thickness is enough greater than r_p so that a plane-strain condition exists at the elastic-plastic boundary, there is a surprising amount of elastic constraint. For, from Equations [2] and [3], except at the very tip of the crack, $\sigma_x = \sigma_y$ and hence for plane strain

$$\epsilon_z = 0 = \sigma_z/E - 2\nu\sigma_x/E \quad \text{or} \quad \sigma_z = 2\nu\sigma_x$$

Thus the maximum difference between the principal stresses, which approximately governs yielding, is

$$Y = \sigma_x - \sigma_z = \sigma_x(1 - 2\nu)$$

For $\nu = 1/3$, the value of σ_x at which yielding occurs would be 3 times that for an unnotched specimen. Thus the author's presentation has made it easy to recognize that important constraints may arise in elastic as well as plastic stress distributions around notches.

It also may be noted that the plastic zone really consists of two regions, one caused by general yielding over macroscopic regions, and the other associated with the ductile fracture required to join cleavage fractures in neighboring grains. It is this latter region, whose radius is of the order of the grain size r_g, which is truly local to the surfaces of the crack. To insure that the general plastic deformation be negligible, it is necessary that

$$r_p = a\left(\frac{\sigma}{Y}\right)^2 < r_g \quad \text{or} \quad \sigma/Y < r_g/a$$

For macroscopic cracks in material with ordinary grain sizes, this condition would be satisfied only for very low values of the

[1] By G. R. Irwin, published in the September, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 361–364.

[2] Associate Professor, Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Mass. Mem. ASME.

[3] "Elastic-Plastic Stress and Strain Distributions Around Sharp Notches Under Repeated Shear," by J. A. H. Hult and F. A. McClintock, Proceedings of the Ninth International Congress for Applied Mechanics, Brussels, Belgium, 1956.

applied stress. Where the general plastic zone is not negligible, it would seem that an understanding of the energy absorbed in plastic deformation around the crack, supplied by $\mathcal{G}$, would require an analysis based on the theory of plasticity. This analysis might involve not only the material, but also the size of the crack, the previous history of loading, and, as noted in reference (7) of the paper and elsewhere,[4] the thickness of the sheet. This limitation on our understanding of the energy absorbed in plastic deformation makes the correlation obtained from the concept of crack extension force seem somewhat fortuitous, but of course does not negate the practical utility of the resulting equations in those cases where they correlate experimental data.

F. R. Steinbacher.[5] It is of interest to note that the author's solution, although applying Westergaard's (author's reference 11) method differently, agrees with certain phases of those obtained by Williams[6] recently. For example, both show that for a symmetrically loaded crack the energy density is not a maximum along the crack direction but at an angle of approximately ±70 deg. However, it would be of interest to obtain experimental verification of this from strain gages located near the crack in a pattern similar to the one suggested in Fig. 2 of the paper. It is of interest to note that Liu and Carpenter,[7] in plotting the energy distribution on the surface of a steel plate containing a symmetrically loaded central crack, did obtain patterns of equal energy lines which agree reasonably well with the author's energy distribution as a function of r and θ.

The use of the symbol $\mathcal{G}$ as a force tending to cause crack extension instead of dW/dA, the strain-energy rate, appears to change Equation [12] from the one quoted in the author's earlier work only symbolically; namely

$$S_y = \left[\frac{E\, dW/dA}{\pi 2x}\right]^{1/2} \qquad [1]$$

Attempts to evaluate either $\mathcal{G}$ or dW/dA experimentally has shown that either value is very susceptible to material and to the configurations of the specimen. As a matter of fact, Kies[8] and more recently Frisch[9] have shown that for some materials values for $\mathcal{G}$ may vary as much as 3 to 1. Our experience has shown that such variations can be reduced somewhat by using $(E\mathcal{G}/\pi)^{1/2}$. Nevertheless, since neither term is too satisfactory for predicting crack propagation characteristics in aircraft fail-safe design, reliable modifications and corrections for either Equation [12] or [17] must be found.

It is quite possible, of course, that much of this variation could be due to the differences which exist between the characteristics of a stationary crack and those of a rapidly moving crack. There is sure to be as much difference as there is in the characteristics of starting friction and sliding friction. Even a better illustration may be that of the removal of metal from a specimen, as with a lathe. In each case the force needed to start the action is always greater than that required to keep it going. Lockheed is presently investigating this phase of crack propagation.

[4] "Some Design Considerations for Tear-Resistant Airplane Structures," by A. Sorensen, Institute of the Aeronautical Sciences Preprint No. 618, Jan., 1956.

[5] Group Engineer, Structural Development Group, Lockheed Aircraft Corporation, Burbank, Calif.

[6] "On the Stress Distribution at the Base of a Stationary Crack," by M. L. Williams, Journal of Applied Mechanics, Trans. ASME, vol. 79, 1957, pp. 109–114.

[7] A Study of Plastic Deformation and Fracturing by Strain Energy Distribution," by S. I. Liu and S. T. Carpenter, Ship Structure Committee, Bureau of Ships, Report SSC-38, December 20, 1950.

[8] "The Resistance of Materials to Fracture Propagation and Gun Fire Damage," by J. A. Kies, Naval Research Laboratory Memo Report 594, May, 1956.

[9] "Comparison of Semi-Empirical Solutions for Crack Propagation with Experiments," by J. Frisch, Trans. ASME, vol. 80, 1958, pp. 921–928.

M. L. Williams.[10] The author has presented another of his illuminating papers on the subject of stress fields around a crack. To extend his results, an additional interpretation of his quantity $\mathcal{G}$, "the strain-energy-loss rate associated with extension of the fracture accompanied only by plastic strains local to the crack surfaces" may be suggested. Also the subsequent relation draws attention to another characteristic dimensional quantity, namely, the radius of curvature at the base of the crack during deformation which at fracture may perhaps be associated with a critical "notch radii" or "base particle width."[11]

The first term in the displacement variation normal to the crack for the particular case discussed by the author, or the more general series expansion,[6] shows that locally the free edges of the crack deform according to the positive square root of the distance from the base of the crack, i.e.

$$v(x, 0) = Kx^{1/2} + \ldots$$

from which the local radius of curvature R at the base of the crack is found to be finite and equal to $R = (1/2)K^2$ and thus

$$v(x, 0) = (2R)^{1/2}x^{1/2}$$

comparing the author's expression

$$v(x, 0) = \frac{2}{E}\left(\frac{E\mathcal{G}}{\pi}\right)^{1/2}(2x)^{1/2}$$

there is the association

$$\mathcal{G} = \frac{\pi E}{4} R$$

which may be of additional assistance in physical interpretation, and perhaps related to Neuber's work on equivalent particle size from which effective stress-concentration factors are deduced.

In the latter part of the paper, the apparent dissimilarity with other results[6] as to the circumferential location of the maximum normal stress, σ_y —73.4 or 60 deg—is quickly resolved by noting that the maximum stresses were evaluated holding y and r constant, respectively.

Author's closure

The author wishes to thank those who submitted discussions for their suggestions and for their interest in his paper. In this reply the discussion by Williams is considered first, since those by McClintock and by Steinbacher refer to experimental matters, whereas the paper dealt primarily with theoretical viewpoints.

By Williams' method[6] for developing the stress distribution at the base of a stationary crack, one may obtain the stresses associated with a shear-type displacement of the crack surfaces along with the stresses associated with separation-type crack-surface displacements. Only stresses of the latter type were discussed by the writer. The stress relations for a crack opening subjected to shear are, however, easily represented in terms of the stress function method of the writer's paper. For example, if one assumes the Airy stress function is

$$F = -y\,\mathrm{Re}\bar{Z}$$

[10] Associate Professor, California Institute of Technology, Pasadena, Calif.

[11] "Theory of Notch Stresses," by H. Neuber, Edward Bros., Inc., Ann Arbor, Mich., 1946.

then the five stress functions Z of the examples in the paper provide the solution to five specific problems of cracks subjected to shearing forces. The crack-extension force associated with each of these stress fields tends to produce extension of the crack as a shear dislocation.

Williams' discussion suggests that the crack-extension force $\mathcal{G}$ may be related to Neuber's plastic particle hypothesis. The relationship of this hypothesis to the Griffith equation was discussed by Orowan.[12]

The general relationship of $\mathcal{G}$ to notch stress theory can be stated concisely as follows. Assume the stress intensity factor represented by the symbol $\mathcal{K}$ where

for plane strain

$$\mathcal{K}^2 = \frac{E\mathcal{G}}{(1-\nu^2)\pi}$$

and for plane stress

$$\mathcal{K}^2 = \frac{E\mathcal{G}}{\pi}$$

with E and ν being Young's modulus and Poisson's ratio. Then for any notch, internal or external, and having nearly zero flank angle

$$\mathcal{K} = \lim_{R \to 0} \frac{1}{2}\sigma_m\sqrt{R}$$

where σ_m is the maximum tensile stress at the root of the notch and R is the notch root radius of curvature.

In terms of Neuber's plastic particle hypothesis

$$\mathcal{K} = \sigma_p\sqrt{\frac{\epsilon}{2}}$$

where σ_p is the average tensile stress across a small segment of length ϵ beyond the end of the crack. Since fracture strength experiments do not determine critical values of σ_p and $\sqrt{\epsilon}$ separately but only a critical value of their product, those using the Neuber hypothesis must assume a value of σ_p, usually the ultimate tensile strength, in order to obtain an experimental value for a critical plastic particle size ϵ.

By substituting "empirical" for "fortuitous" in his final sentence the author can agree with all of McClintock's remarks. However, readers should not confuse a theoretical estimate of the limits of applicability of brittle fracture theory with what appears to be the actual situation practically. Fracture strengths, $\mathcal{G}_c$, measured with flat sheets in uniform tension have shown no significant error due to local plastic yielding when the net section stress was less than the yield stress. For the nonuniform stress field of a spinning disk with the crack in the central region of greatest stress, Winne and Wundt[13] found that plastic yielding influenced the measured values of $\mathcal{G}_c$ only when the average net section stress exceeded 65 per cent of the yield stress. Thus the suggested crack-extension force analysis is applicable when, as in common engineering practice, the net section stress has been made substantially less than the yield stress.

Difficulties with variations in $\mathcal{G}_c$ or dW/dA for the same material have been experienced at the author's laboratory as well as by Steinbacher and his associates. These difficulties are of two kinds: (*a*) The stress distribution actually present in the test specimen may differ from that assumed in the calculation of $\mathcal{G}_c$. For example, a thin sheet in uniaxial tension with a central starting crack may buckle across the span of the crack unless supported by face plates whereas the equation used for calculation of $\mathcal{G}_c$ assumes the sheet is flat; (*b*) under conditions of plane strain (thick sections) $\mathcal{G}_c$ is much less than for conditions of plane stress (thin sections). In general, whether due to triaxiality, temperature, or strain rate, changes which cause an increase of yield stress cause a decrease of $\mathcal{G}_c$.

Different values of $\mathcal{G}_c$ or dW/dA for the same material due to (*b*) are to be expected from the physical nature of the property and the variability itself is of definite interest. Inconsistent results due to (*a*) can be eliminated with growth in testing experience.

[12] "Energy Criteria of Fracture," by E. Orowan, *The Welding Journal Research Supplement*, March, 1955.

[13] "Application of the Griffith-Irwin[3] Theory of Crack Propagation to the Bursting Behavior of Disks," by D. H. Winne and B. M. Wundt, ASME Paper No. 57—A-249.

A Direct Method for Determining Airy Polynomial Stress Functions[1]

J. N. Goodier.[2] Problems of the kind illustrated in this paper by the two examples worked out are not completely determinate boundary-value problems in the theory of elasticity because the loadings, or reactions, or both, are specified only as resultants. The distribution of traction forming the resultant is not specified. It follows that there is no unique solution. The method proposed appears to lead to a unique solution, and this means that one solution has somehow been singled out. Examination of Equations [11] and [16] shows that they do not demand $C_{mn} = 0$ as the text following Equation [17] would have it. They merely allow such a solution out of an infinity of solutions. Thus an arbitrary choice has occurred here. Being the choice of the simplest solution, it has led to the well-known solution usually obtained by seeking the polynomial stress function of lowest degree which meets the required resultant conditions. A similar comment applies to the second example, and in general.

It also may be of interest to point out that polynomial stress functions can be written down at once, avoiding the necessity of obtaining relations between coefficients as in Equation [8]. The function of the nth degree (only) is obtained from the form

$$\phi = \mathrm{Re}[\bar{z}\psi(z) + \chi(z)], \qquad z = x + iy$$

by taking

$$\psi(z) = (a_n + ib_n)z^{n-1}, \qquad \chi(z) = (c_n + id_n)z^n$$

where a_n, b_n, c_n, d_n are arbitrary real constants.

G. Horvay.[3] The author furnishes a new method of solution to the old problem of finding the biharmonic polynomial which properly represents the effects of prescribed tractions applied to the boundary of a rectangle. It may be mentioned that Donnell on one hand,[4] Boley and Tolins[5] on the other, also tackled the

[1] By Ching-Yuan Neou, published in the September, 1957, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 79, pp. 387–390.

[2] Professor of Engineering Mechanics, Stanford University, Stanford, Calif. Mem. ASME.

[3] General Electric Research Laboratory, Schenectady, N. Y. Mem. ASME.

[4] "Bending of Rectangular Beams," by L. H. Donnell, Journal of Applied Mechanics, Trans. ASME, vol. 74, 1952, p. 123.

[5] "On the Stresses and Deflections of Rectangular Beams," by B. A. Boley and I. S. Tolins, Journal of Applied Mechanics, Trans. ASME, vol. 78, 1956, pp. 339–342.

problem, but in a different manner, using the method of progressive approximations by biharmonic polynomials.[6] This algorithm automatically breaks off when the applied tractions are polynomials. The DBT approach appears preferable to the Neou method, at least in principle, because the calculations may be terminated whenever the improvement produced by the successive polynomials is judged to be negligible. (In the author's method one has to complete the calculations before the solution is usable.) In practice the author's method may very well lead more quickly to the goal. This has not been checked.

The present comments aim to re-emphasise the oft-forgotten fact that the biharmonic-polynomials approach is inapplicable to rectangles which are nearly squares. In Fig. 1(a) we show a square, subject to the traction

$$\sigma_y(x, \pm 1) = P_2(x) = \frac{3x^2 - 1}{2} \qquad [a]$$

on top and bottom; P_k denotes the kth Legendre polynomial. Writing the harmonic polynomials[7]

$$V_n = \mathscr{I}(z^n), \qquad U_n = \mathscr{R}(z^n)$$

i.e.

$$V_1 = y \qquad U_2 = x^2 - y^2$$

$$V_3 = 3x^2y - y^3 \qquad U_4 = x^4 - 6x^2y^2 + y^4$$

$$V_5 = 5x^4y - 10x^2y^3 + y^5 \quad U_6 = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$$

. . . .

one verifies that

$$\Phi_a = \frac{1}{8} U_4 - \frac{1}{4} U_2 + \frac{1}{4} yV_3 \qquad [a]$$

is the biharmonic polynomial which produces the prescribed $\sigma_y(x, \pm 1)$ traction, but creates, at the same time, an equally large

$$\sigma_x(\pm 1, y) = -\frac{3y^2 - 1}{2} \quad [\tau(\pm 1, y) = 0] \qquad [a]$$

stress on the left and right edges. In this case therefore the best one can do is to write

$$\Phi_{\text{approx}} = \frac{1}{2} \Phi_a$$

That leaves residuals

$$\sigma_y(x, \pm 1) = \frac{3x^2 - 1}{4}, \qquad \sigma_x(\pm 1, y) = -\frac{3y^2 - 1}{4}$$

on the four edges; these residuals no longer can be reduced by the biharmonic-polynomial approach, but must be removed by an eigenfunction method.[8,9]

The cases shown in Figs. 1(b, c) of this discussion illustrate the same fact. When self-equilibrating tractions, such as

$$\tau(x, \pm 1) = \pm \bar{P}_2(x) \equiv \pm \int_{-1}^{x} P_2(\xi) d\xi = \pm \frac{x^3 - x}{2} \qquad [b]$$

or

$$\sigma_y(x, \pm 1) = P_4(x) = \frac{35x^4 - 30x^2 + 3}{8} \qquad [c]$$

act on the horizontal edges of the square, the polynomial biharmonic functions

$$\Phi_b = \frac{1}{16} U_4 + y \left[-\frac{1}{80} V_5 + \frac{1}{12} V_3 + \frac{19}{240} V_1 \right] \qquad [b]$$

$$\Phi_c = \frac{7}{48} U_6 - \frac{15}{16} U_4 - 2U_2 + y \left[\frac{7}{16} V_5 + \frac{5}{6} V_3 - \frac{7}{12} V_1 \right] \qquad [c]$$

create stresses along the vertical edges, to wit

$$\sigma_x(\pm 1, y) = -\frac{3}{35} P_4(y) + \frac{13}{21} P_2(y); \qquad \tau(\pm 1, y) = \mp 2\bar{P}_3(y) \qquad [b]$$

$$\sigma_x(\pm 1, y) = 2P_4(y) - \frac{65}{3} P_2(y); \qquad \tau(\pm 1, y) = \pm 35\bar{P}_5(y) \qquad [c]$$

[6] There are still further types of approaches floating around in the literature.

[7] $\mathscr{R}$, $\mathscr{I}$ denote real and imaginary parts.

[8] "The Use of Self-Equilibrating Functions in Solution of Beam Problems," by G. Horvay and J. S. Born, Proc. Second U. S. Congress of Applied Mechanics, ASME, 1954, pp. 267–276.

[9] "Saint Venant's Principle: A Biharmonic Eigenvalue Problem," Journal of Applied Mechanics, Trans. ASME, vol. 79, 1957, pp. 381–386.

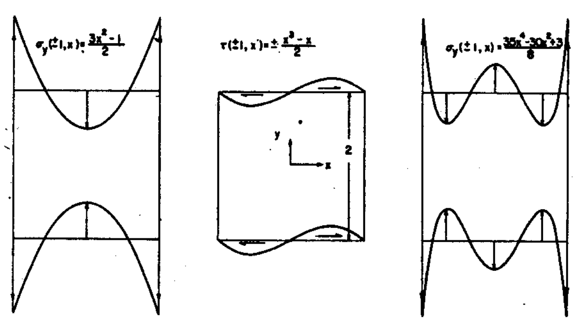

Fig. 1(a) Fig. 1(b) Fig. 1(c)

which again cannot be eliminated by biharmonic polynomials without restoring the tractions that produced them.

When the rectangle is narrow like a beam, then the story is the same, but the ending is different. Figs. 4(a) and 5(a) of Horvay and Born[8] illustrate the case of a rectangle with x, y-dimensions 2/5 and 2, respectively. A traction

$$\sigma_x\left(\frac{1}{5}, y\right) = 4(3y^2 - 1)$$

is applied; the three other edges are free. The maximum residual left by the biharmonic-polynomial solution on the short edges is found to be

$$\sigma_y\left(-\frac{1}{5}, \pm 1\right) = -1.73$$

as contrasted with the maximum stress in the rectangle

$$\sigma_y(1, 0) = -38.5$$

As in the previous cases, the self-equilibrating residuals cannot be eliminated by biharmonic polynomials (only an eigenfunction approach can eliminate them), but they may now be neglected in a first approximation. (In the reference, however, they are not neglected but are eliminated by means of a variationally determined eigenfunction expansion. The results so obtained are then compared with the results of a calculation where the variational method replaces the biharmonic-polynomial method from the outset and automatically suppresses the residuals along the edges; the second phase of the problem is thereby eliminated. The agreement of the results by the two methods is found to be very good.)

A. L. Ross,[10] H. A. Eagle.[11] In reading the author's solution for finding the coefficients $C_{mn} = 0$, it is indicated that it results from a unique solution of Equations [11], [16], [12], and [17]. However, an examination of, say, Equations [11] and [16] for a specific value of m, shows that there are n unknown coefficients C_{mn} and only two equations for their solution, which of course means that a unique solution cannot be determined from these equations alone. Of course, an examination of the results of the first example shows that for this problem the trivial solution, for $m > 2$, is indeed the unique and correct one. It therefore appears that a more detailed discussion of the method used in proving that these coefficients actually are zero is in order. The Recursion Relationship [8] and the two pairs of Equations [11], [16], [12], and [17] will be used to determine the coefficients of the three sections of the matrix shown as Fig. 2 of this discussion. The coefficients in the three regions will be determined by three distinct patterns of solution. For a matrix of $M \times N$ size the coefficients of Region I can be shown to be zero by the use of the Recursion Relationships [8] since all coefficients outside of the region, i.e., for $n > N$ or $m > M$ are assumed to be identically zero. In determining the coefficients of Region II, one makes use of Equations [11] and [16] to find the second and fourth columns and uses Equations [12] and [17] to find the first and third columns of this region. The coefficients of Region III are shown to be zero by a pattern along diagonals (Region IIIa) and four column rows (Region IIIb) and then progressing upward. Coefficients of Regions IIIa can be shown to be identically zero by the use of the Recursion Equation [8] as was done for Region I. The coefficients of Region IIIb can then be shown to be identically zero in the same manner as was used for Region II. Proceeding in this manner, the coefficients in Regions I, II, III of the matrix can be shown to be identically zero. The remaining coefficients are then determined as outlined in a clear manner by the author.

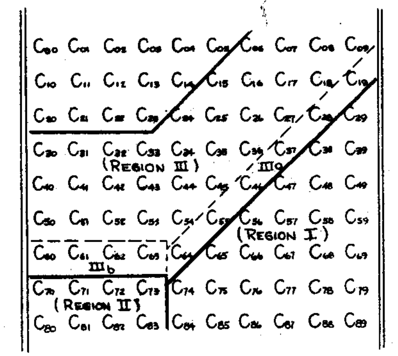

Fig. 2

C. J. Thorne.[12] It is well known that the real and imaginary parts of $f(z)$, $f(\bar{z})$, $\bar{z}f(z)$, and $zf(\bar{z})$ with $z = x + iy$ are biharmonic functions. With $f(z) = z^n/n!$ we have a unique set of biharmonic polynomials convenient under differentiation. It may be of interest to know that these polynomials have been tabulated for many values over the unit square.[13]

Author's Closure

The author wishes to thank Drs. Goodier, Horvay, Ross, Eagle, and Thorne for their helpful discussions. Besides, he wishes to point out that the sentence following Equation [17] on page 388 should read: "Equations [11] for $m \geqslant 3$ and [16]. . ." instead of "Equation [17] for $m \geqslant 3$ and [16]. . .".

The question of taking $C_{mn} = 0$ for $m \geqslant 3$ as unique solutions to the Equations [11], [16], [12], and [17] might well be answered by the Ross-Eagle proof, if we could only establish the fact that the matrix is finite and all the C_{mn} outside of the region of $n > N$ or $m > M$ are assumed to be identically zero. In fact, the matrix is finite and the question is where to draw the lines in our infinite matrix. Many investigators use a polynomial stress function up to $m + n = 5$, without giving any reasons for leaving out the terms of higher power which might be important.

Equations [11] for $m \geqslant 3$ and [16] for a particular value of m can be satisfied either by $C_{mn} = 0$ or by an infinite number of sets of solutions which correspond to an infinite number of combinations of boundary conditions. Although an alternative boundary condition of $\sigma_x = 0$ at the free end in Example 1 may be used to replace the self-equilibrating conditions, the number of combinations of boundary conditions is certainly not infinite. Moreover, although there are more than two unknowns in the two simultaneous Equations [11] and [16], the indeterminacy, however, can be removed if we take into account all the Equations

[10] Principal Engineer, Aircraft Nuclear Propulsion Department, General Electric Company, Cincinnati, Ohio.

[11] Technical Engineer, Aircraft Nuclear Propulsion Department, General Electric Company.

[12] U. S. Naval Ordnance Test Station, China Lake, Calif.

[13] "A Table of Harmonic and Biharmonic Polynomials and Their Derivatives," by C. J. Thorne, Supplement to Bulletin No. 39 of the Utah Engineering Experiment Station, University of Utah, November, 1949.

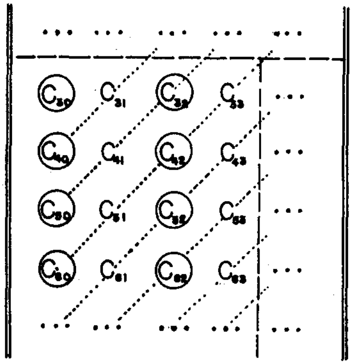

Fig. 3

[11] and [16] for $m = 3, 4, 5 \ldots, M$ (where M is a very large number or infinite) and the interdependence of the various C_{mn}. By successive substitutions from the recursion Equation [8], it is possible to express C_{mn} for $n \geqslant 4$ in terms of the two C_{mn} at the lower ends of the diagonals in the Matrix [4]. For example

$$C_{34} = -15C_{30} - 2C_{32},$$

and

$$C_{36} = 2C_{30} + \frac{1}{5} C_{32}.$$

Now, for $M - 2$ rows (below the dotted line in Fig. 3 of the discussion), we have $M - 2$ Equations [11] and [16], respectively, for $M - 2$ unknowns C_{m0} and C_{m2} respectively (shown by O in Fig. 3), i.e., the number of independent unknowns is now equal to the number of independent equations. The solution is thus

$$C_{m0} = C_{m2} = 0 \quad \text{for } m = 3, 4, 5 \ldots$$

unless the determinant of C_{m0} and C_{m2} of the powers of $x^m y^n$ in the $2(M - 2)$ equations is zero which is most unlikely. The same reasoning can be used to obtain $C_{mn} = 0$ in Equations [12] for $m \geqslant 3$ and [17].

Further support for $C_{mn} = 0$ for $m \geqslant 3$ and n = even number can be rallied from the obvious fact that because of the asymmetric loadings at $y = \pm a$, the polynomial terms even in y must be excluded and, by virtue of Equation [16] or [8], C_{m0} for $m \geqslant 3$ must all be zero.

The author fully realizes the existence of many excellent methods other than the biharmonic polynomial approach, such as the method of complex potentials, the Donnell-Boley-Tolins method of successive approximation, variational principle, etc., as mentioned by various discussers. The complex stress function,

$$\phi = \text{Re} \sum_{n=2}^{N} [\bar{Z}(a_n + ib_n)Z^{n-1} + (c_n + id_n)Z^n],$$

($N = \infty$, theoretically) is an elegant, but more complex, way for representing a polynomial stress function of Equation [3]. Although it enjoys the advantage of being no longer subjected to the compatibility Equation [8], the expansion of, for instance,

$\text{Re}[\bar{Z}(a_5 + ib_5)Z^4 + (c_5 + id_5)Z^5]$ into

$$(a_5 + c_5)x^5 - (3b_5 + 5d_5)x^4y - (2a_5 + 10c_5)x^3y^2 - (2b_5 - 10d_5)x^2y^3 - (3a_5 - 5c_5)xy^4 + (b_5 - d_5)y$$

is no simpler than the use of Equation [3] subjected to the restriction of Equation [8], which, for most of the time, is used to set one C_{mn} to zero by inspection when the other two coefficients in the Equation [8] are zero. Besides, to obtain σ_x, σ_y, and τ_{xy} from:

$$\sigma_x + \sigma_y = 4\text{Re}\psi'(Z)$$

and

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{Z}\psi''(Z) + \chi''(Z)]$$

is certainly more tedious than to obtain Equation [5], [6], and [7] from Equation [3].

Since it is clearly indicated in the paper that the present method applies only to a long narrow rectangular beam under continuous loadings, we must rule out the case of a square plate or a rectangular strip having a small length-to-depth ratio; otherwise the influences of the extraneous end tractions could hardly get a chance to fade out, as are clearly illustrated in Dr. Horvay's discussion, and consequently the results would find little or no place to be held, by virtue of Saint Venant's principle as good approximations to the exact stress conditions in the beam.

Interaction Curves for Shear and Bending of Plastic Beams[1]

E. T. Onat.[2] The static analysis of perfectly plastic beams shows that bending usually predominates so that the concept of simple plastic hinges is sufficient for most cases. As the author's results indicate once more, the shearing action is of importance only for extremely short beams. Therefore it might seem at first that the establishment of approximate interaction curves for bending and shear of plastic beams is hardly justified since the structures in question cannot any longer be labeled as beams. However, the analysis of the plastic behavior of the beams under impulsive loading[3] indicates that shear action may be important even for technical beams at the initial stage of loading. Therefore any attempt to include the shear action into the analysis in an approximate manner (such as the author's) may be of great help for a better understanding of the dynamic behavior of beams. However, the author's approach to establish interaction curves for shear and bending is open to some objections.

The author states at the outset that the derivation of the interaction curve is first approached from the view of statically admissible stress fields. It is difficult to see that statically admissible fields are indeed used since no effort is made to extend the field beyond the critical section considered. Moreover, it is seen from Equations [4] and [1] that the stress-boundary conditions are violated at $y = \pm h$ since τ_{xy} is not zero there.

On the other hand, the assumption that the deformation in the critical section of the beam is the composite of a pure bending and uniform shear does not necessarily result in a kinematically admissible velocity field that leads to the rate of energy dissipation given by Equation [16]. It is easily seen that, if no effort is made to extend the strain-rate field of Equation [15] beyond the critical cross section considered, no claim should be made that this strain-rate field is kinematically admissible. As a matter of fact, the success of the interaction curves for bending and stretching of plastic beams[4] is related to the fact that physically significant kinematically admissible velocity fields may be found which can be associated with the desired rates of curvature and extension at the critical cross section under consideration. Even the simple plastic hinge does not correspond to a discontinuity which acts within the section but it spreads out over a distance equal to the depth of the beam.[5,6] It can be stated in conclusion that the introduction of generalized stresses and strains and associated strain rates cannot alone constitute a basis for approximate analysis.

Two-dimensional stress and velocity fields must be considered in order to obtain a realistic inclusion of shear effects.

[1] By P. G. Hodge, Jr., published in the September, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 453–456.

[2] Associate Professor of Engineering, Brown University, Providence, R. I.

[3] "Impact of Finite Beams of Ductile Metal," by P. S. Symonds and C. F. A. Leth, *Journal of the Mechanics and Physics of Solids*, vol. 2, 1954, p. 92.

[4] "The Influence of Axial Forces on the Collapse Load of Frames," by E. T. Onat and W. Prager, Proceedings First Midwest Conference on Solid Mechanics, University of Illinois, Urbana, Ill., 1953, pp. 40–42.

[5] "The Effect of Shear on the Plastic Bending of Beams," by D. C. Drucker, JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 78, 1956, pp. 509–514.

[6] "Plane Plasticity," by B. B. Hundy, Paper MW/B/54 of the Solid Mechanics Group of the Mechanical Working Division, British Iron & Steel Research Association.

Author's Closure

The author wishes to thank Professor Onat for his illuminating comments on the possible effect of shear in dynamic plasticity problems. Certainly, it was not the author's intention to imply that the approximate treatment in the paper was a completely accurate answer to the problem. Naturally, two-dimensional stress and velocity fields would be preferable, if they could be found. In fact, to be strictly accurate, three-dimensional fields must be constructed.

However, the author cannot agree with the rather discouraging picture painted by Professor Onat. Just as a simple bending theory is *sufficiently* accurate for *many* beam problems, so may the theory presented in the paper be useful in cases for which shear is of some importance. In view of the fact that very few two-dimensional solutions are available, it would seem worth while to at least consider the proposed approximate theory. Certainly, it would be desirable to compare the results of the approximate theory with a more exact theory and with experiments in both static and dynamic problems. Until this has been done, it is difficult to make a final assessment of the value of the approximate theory.

Forced Vibration of Systems With Nonlinear, Nonsymmetrical Characteristics[1]

F. R. Arnold.[2] Contrary to the author's statement, his method does not yield an exact solution for the free-vibration case ($P = 0$) of the nonlinear problem. Even for this case for which the exact range and period of oscillation may be determined by ordinary methods, the exact solution is not a pure harmonic as is assumed, and the amplitude of the harmonic is only approximate in that it is one half the range of oscillation. At best, this free-vibration "solution" is characterized by a range of oscillation equal in magnitude to the exact one and a period equal to the exact period. The details of the exact motion are not there. It is entirely possible that another method would come closer to the details of the motion and be slightly off in amplitude for any selected frequency of vibration.

The author may be interested in papers by K. Klotter and others on the Ritz method as listed in the bibliography of the paper by the writer in the JOURNAL OF APPLIED MECHANICS, December, 1955.

The author's method requires that the free-vibration solution be known first. This is often quite a trick to obtain. The Ritz method needs nothing but the differential equation as the starting point.

The method as presented is, however, another interesting approach to approximate solutions of nonlinear problems which well deserve all the methods that can be devised.

[1] By S. Mahalingam, published in the September, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 435–439.

[2] Associate Professor of Mechanical Engineering, Stanford University, Stanford, Calif.

Author's Closure

The author wishes to thank Professor Arnold for his comments. In the expression $F(a) = \omega_a^2 a$ given in the paper, a is half the range of free vibration and ω_a the corresponding frequency. In the case of a bilinear characteristic or a characteristic of the form $f(x) = \alpha x + \beta x^3$, the relationship between a and ω_a can be determined exactly by direct methods. It is then assumed that a is the amplitude of the one-term approximation for free vibrations. The graphical construction then yields an approximate value for one half the range of forced vibration. In the special case $P = 0$, however, the method gives an exact value for the magnitude of one half the range of oscillation. It was in this sense that the term "exact solution" was used in the paper. As Professor Arnold states, the method does not give any indication regarding the details of the motion. The method has been put forward as an improvement of the Martienssen method and, as such, it has the usual limitations of one-term approximate solutions.

A Photoelastic Study of Maximum Tensile Stresses in Simply Supported Short Beams Under Central Transverse Impact[1]

A. J. Durelli.[2] This paper is a new contribution in the field of wave propagation, a field which is being explored very intensively at present in many laboratories in many countries. The particular experience of the authors will certainly be found useful by other investigators.

A few comments, basic in nature, seem appropriate:

1 The authors give 154 psi as a fringe value for the Castolite they used. This is considerably less than the average value 215 psi given by Flynn (reference of the paper) for Castolite. Although the authors do not give any details about the way the fringe value was determined, they mentioned that SR-4 strain gages were used for the purpose of calibration. The basic question to be raised is the interpretation of the readings of the SR-4 gage cemented to a plastic model. The authors are referred to the

[1] By A. A. Betser and M. M. Frocht, published in the December, 1957, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, vol. 79, pp. 509–514.

[2] Supervisor, Armour Research Foundation, and Professor, Illinois Institute of Technology, Chicago, Ill. Mem. ASME.