A Rational Analytic Theory of Fatigue

PAUL C. PARIS
Assistant Professor of Civil Engineering

MARIO P. GOMEZ* and WILLIAM E. ANDERSON
Research Engineers, Boeing Airplane Company

P. C. Paris

Paul C Paris 2/12/2010

M. P. Gomez

W. E. Anderson

A great deal of effort has recently centered around examination of the factors influencing the growth of fatigue cracks. Fatigue has been considered a multi-phase problem: e.g., initiation of a crack and its growth are often considered as separate phenomena. In contrast, the objective of this work is to show that the growth of an initial "crack-like" imperfection to a critical size, which causes static failure of a structure, may be described by a single rational theory.

Two loading parameters, the nature of the stress field near the tip of a crack and the variation of this field, are taken to control the rate of crack extension in a given material. This hypothesis is proven by using it to correlate data from three independent investigators. Since it shows a positive correlation of all available data for crack-extension rates from 10^{-7} to 10^{-2} in. per cycle, the hypothesis may be used to formulate a theory of fatigue that permits computing the structural lives of complicated geometries from simple laboratory tests of material properties.

The Stress Distribution Near the Tip of a Crack

The form of the stress distribution in the vicinity of a crack root was given by Sneddon[1] in 1946 and has recently been expanded by Irwin[2,3] and Williams.[4] The unique character of this form, as Irwin showed,[2] is a controlling factor in attempts to analyze crack extension under static loads. We will show that this same character becomes fundamental in crack extension under cyclic loading upon the addition of new concepts to describe the cyclic nature of the loading.

* Mr. Gomez received his M. S. degree in Metallurgical Engineering in 1958 at the University, after which he worked for Boeing. He is now Senior Scientist at the Missile Systems Division of Lockheed Aircraft Corporation.

Restricting this discussion to cracked bodies in which the geometry and loading of the body are symmetric with respect to the plane of the crack results in very little loss of generality. The nature of cracks

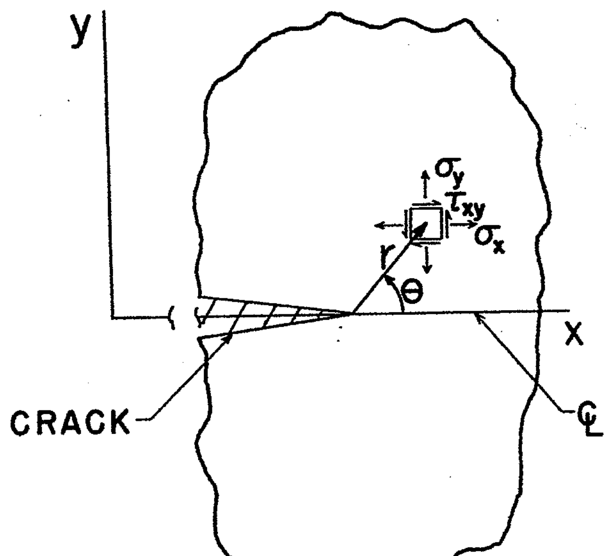

FIG. 1. COORDINATES USED TO DESCRIBE STRESSES NEAR A CRACK TIP (θ; σ_x; τ_y; ν)

is to form most often on such planes, i.e., planes perpendicular to maximum-principle tension stresses. Williams[4] and Irwin[5] have given the required forms of stresses for other cases, but these will not be discussed further in this work.

The coordinates of points in a cracked body with

The Trend in Engineering 13, 9-14 (1961)

SYMBOLS

a = the half crack length

b = plate dimension parallel to a crack

F = a loading force on a body

$f()$ or $F()$ = a function of

K = the stress singularity-intensity factor at the tip of a crack

K_{cr} = the critical value of K for a material associated with a crack extension under static load

L = the length of a plate

N = the number of load cycles since initial loading

P = the loading of a body

r, θ = polar coordinates from the crack tip

x, y = rectangular coordinates centered with respect to a crack

α = the correction factor for K in plates of finite width

β = the ratio of maximum to minimum load on a body during a load cycle

μ = Poisson's ratio

σ_g = gross area stress or nominal stress level

$\sigma_x, \sigma_y, \tau_{xy}$ = components of stress near a crack tip

$\sigma_o(x)$ = the normal stress present at a crack location before the crack appeared

$\Delta a/\Delta N$ or da/dN = the rate of crack extension

crack-plane symmetric geometry may be described as in Fig. 1. If terms of higher order in r are ignored, the elastic solution for stresses in the vicinity of the crack tip for all such problems is

$$\left.\begin{aligned}\sigma_y &= \frac{K}{\sqrt{2r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right],\\ \sigma_x &= \frac{K}{\sqrt{2r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right],\\ \tau_{xy} &= \frac{K}{\sqrt{2r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2},\\ \tau_{xz} &= \tau_{yz} = 0,\\ &\text{and}\\ \sigma_z &= 0 \text{ (plane stress)}\\ &\text{or}\\ \sigma_z &= \mu(\sigma_x+\sigma_y) \text{ (plane strain)}\end{aligned}\right] \quad (1)$$

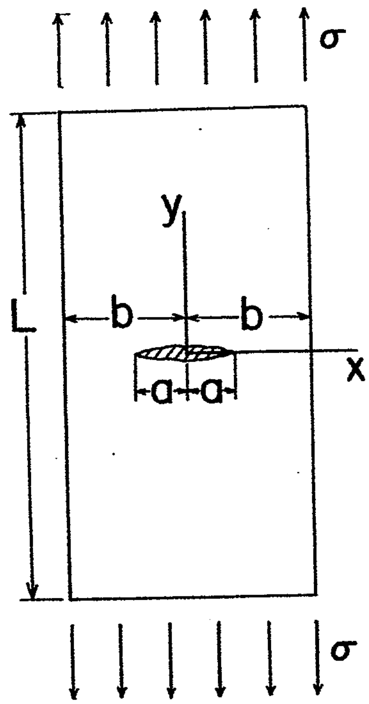

FIG. 2. A TYPICAL FATIGUE-PANEL CONFIGURATION FOR WHICH TEST DATA WERE AVAILABLE

This result implies that the distribution of elastic stress always has the same functional form near the singularity caused by the sharp crack root and that it differs only by a stress singularity-intensity factor, K. This factor is linearly dependent upon the loads on the body and also must contain a geometric factor related to the crack length and other geometric properties of the body. An example is $K = \sigma\sqrt{a}$ for the problem of a crack of length $2a$ in an infinite sheet under a uniform tensile-stress field, σ, perpendicular to the crack.

Now the Griffith-Irwin theory of static strength of bodies containing cracks may be resolved from this discussion in the following fashion: Identical intensity factors of elastic stress will result in identical yield zones near the tips of cracks in the same material if the yield zones are small compared to the region of applicability of the stresses given by Eq. (1). As discussed in previous works,[6] the size of the yield zone may be shown to be small if the nominal stresses in the body are well below the yield point. Therefore, regardless of the appearance of a small yield zone, there will be some critical value of the stress singularity intensity, K_{cr}, near the tip of a crack that will cause static crack extension in a given material. The above hypothesis is the equivalent of the Griffith-Irwin theory, which was originally based on energy considerations.

The Significance of Stress-Intensity Factors in Fatigue

The stress-intensity factor may be considered to be a measure of the effect of the loading and the geometry of a body on the stress intensity near the root of a crack. Therefore, as the loads on a body vary and as the geometry changes by crack extension, the instantaneous values of K reflect the effects of these changes at the crack root.

Let β be the ratio of maximum to minimum load on a cracked body during a cycle of loading. Then, since K is directly proportional to the magnitude of the load, β is also the ratio of K_{max} to K_{min}, regardless of the geometry of the body; that is,

$$\beta = \frac{P_{max}}{P_{min}} = \frac{K_{max}}{K_{min}}. \tag{2}$$

Therefore the stresses near the root of a crack are completely described by K_{max} and β in a given material, since these two parameters give both the intensity and variation of the effects of loading and geometry.

A theory of fatigue crack extension may now be hypothesized as follows: Since, as has been shown, during a cycle of loading the stresses and strains near the tip of a crack are completely specified by K_{max} and β, we can reasonably assume that any phenomena occurring in this region are controlled by these parameters. The amount of crack extension per cycle of loading is just such a phenomenon, or, in functional form,

$$\frac{\Delta a}{\Delta N} = f(K_{max}, \beta). \tag{3}$$

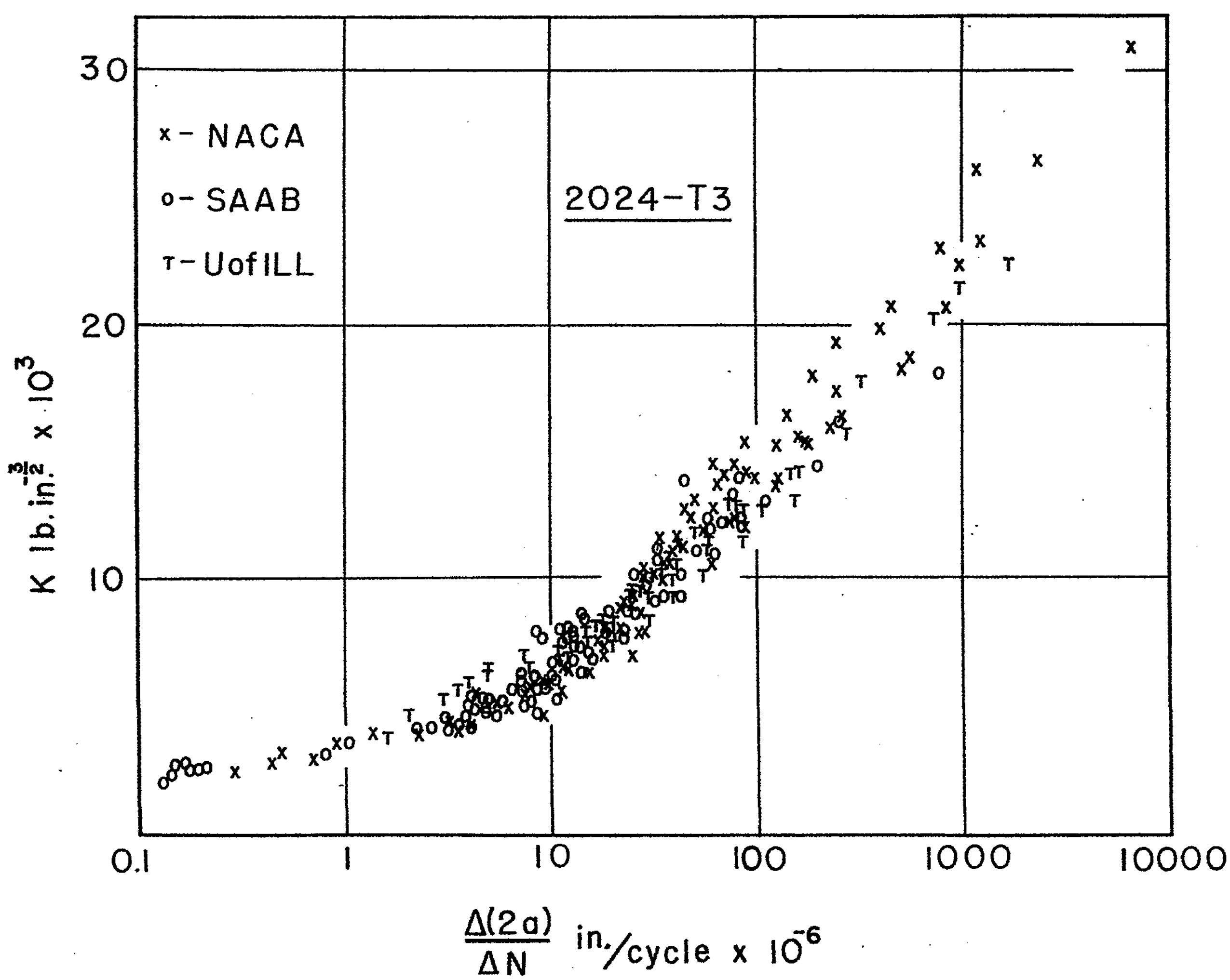

Fig. 3. Crack Extension-Rate Data on 2024-T3 Aluminum Alloy Correlated from Tests by Three Independent Investigators

Experimental Evidence

It is pertinent to examine the above hypothesis in the light of available experimental data. Many investigators[7,8,9,10] have measured crack-extension rates due to cyclic loading in aluminum alloys by using the configuration shown in Fig. 2. The majority of these tests have been performed with minimum loads near zero, or $\beta \equiv \infty$; therefore from examination of Eq. (3), these results for a given material should form a single curve on a plot of K_{max} *vs* $\Delta a/\Delta N$. For such a configuration K_{max} may be computed from[3]

$$K = \alpha\sigma_g\sqrt{a},$$

where

$$\alpha = \frac{\sqrt{4 + 2\,(a/b)^4}}{2 - (a/b)^2 - (a/b)^4} \tag{4}$$

if

$$a < b.$$

The results of this attempt at correlation of crack extension-rate data are shown on Figs. 3 and 4. It is worthy of special note that the data on these curves are from three independent investigators, using many specimen sizes, i.e., widths from 1.8 to 12 in., thicknesses from 0.032 to 0.102 in., and lengths from 5 to 35 in. The testing frequencies varied from 50 to 2000 cpm, and the maximum stresses on the gross area varied from 6 to 30 ksi. On each graph, the materials are both clad metals and bare metals. Therefore the correlation shown is surely more than coincidental.

On the presumption that such curves may be obtained for various values of β for a given material from laboratory tests, this discussion proceeds to formulate the necessary elements of an analytic theory of fatigue, these results being applied in the following sections.

An Analytic Theory of Fatigue

Knowledge of the material-property curves in the form of Figs. 3 and 4, with the addition of curves for other β values, implies the functional form of Eq. (3) as given. Further, given the loading and geometry

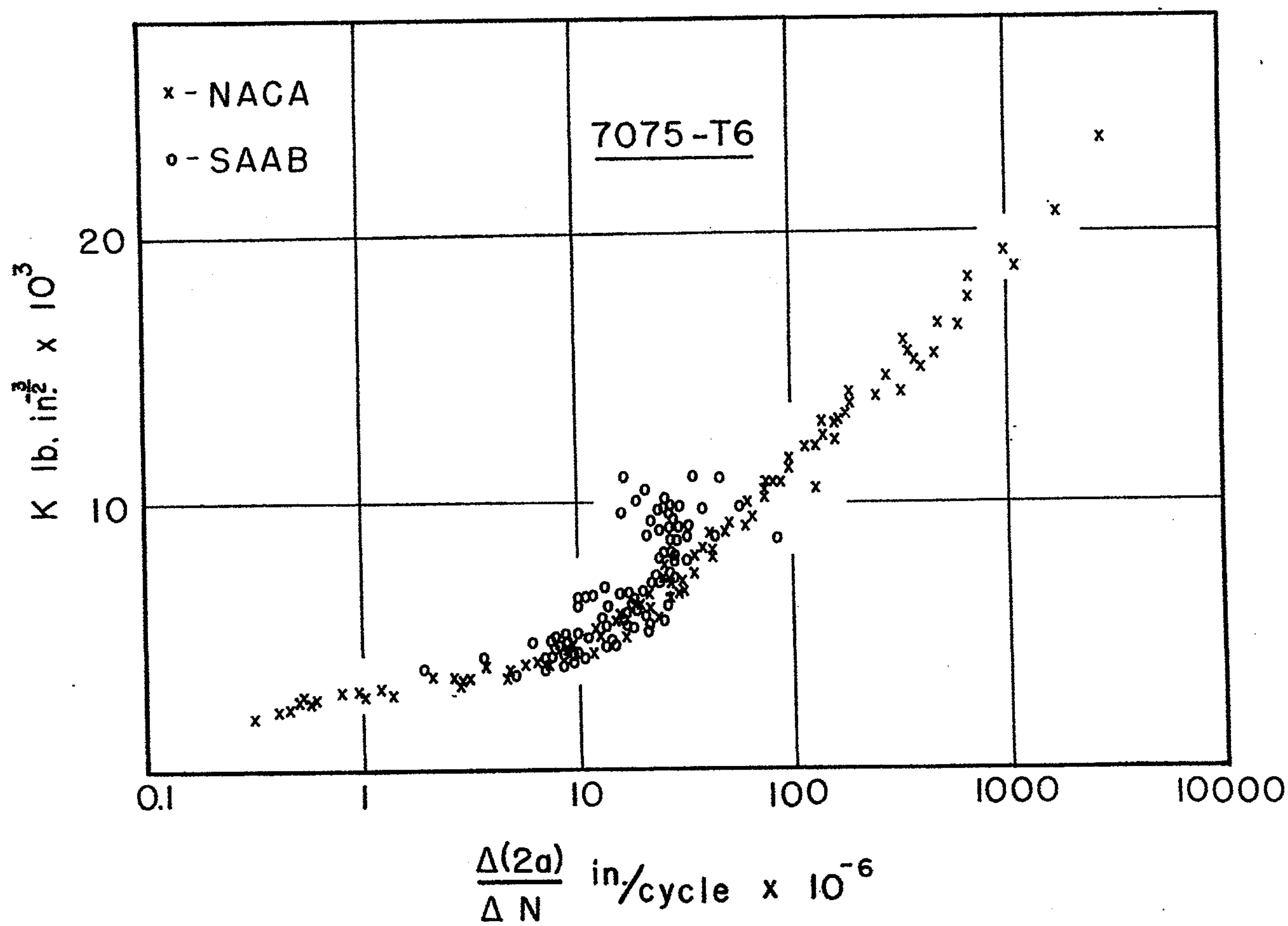

Fig. 4. Crack Extension-Rate Data Correlation for 7075-T6 Aluminum Alloy

of a structure, β is known from the load ratio during any cycle, N, or

$$\beta = \beta(N). \quad (5)$$

Moreover, K may be computed for any crack length, a, and the maximum load as given during the Nth cycle, or

$$K_{max} = K_{max}(N,a). \quad (6)$$

Therefore, for a problem with the above specification, Eq. (5) and Eq. (6) may be substituted into Eq. (3) to give

$$da/dN = F(N,a), \quad (7)$$

where the functional form of F is known, point by point, from the data given, and effects of loading history are neglected.

The solution to Eq. (7) may be found provided an initial crack size may be specified, or some equivalent condition may be stated. In practice, maximum imperfection sizes may be stipulated on the basis of material quality, production methods, and inspection technique, for in practice we know that fatigue cracks grow from just such imperfections. Then Eq. (7) may be integrated, at least by numerical procedures, to generate a complete crack history for the structure. The most difficult phase of this analysis is the computation of K for a given load and crack length, as will be commented upon later.

Relationships to Classical Fatigue Theory

Suppose the material-property curves of the form of Figs. 3 and 4 are known for a specimen that has been subjected to an ordinary fatigue test and has developed an easily measurable crack in a given number of cycles. By using these curves and the measured crack length, the crack-extension rates may be integrated backward to determine an effective initial imperfection size.

Using this computed initial imperfection size makes it possible to calculate the number of load cycles required for failure of the specimen at any stress level. Therefore the complete S-N curve for a material may be computed by this process; moreover, the whole S-N curve can be obtained from a single specimen, since curves of the material properties for several β values may be generated during the same test.

An accumulative damage theory is automatically present in the preceding analysis, which replaces Miner's empirical hypothesis of damage. The Soderberg or Goodman diagrams being likewise empirical, this paper presents a rational analytic theory. As is evident, the form of the analysis lends itself well to statistical analyses.

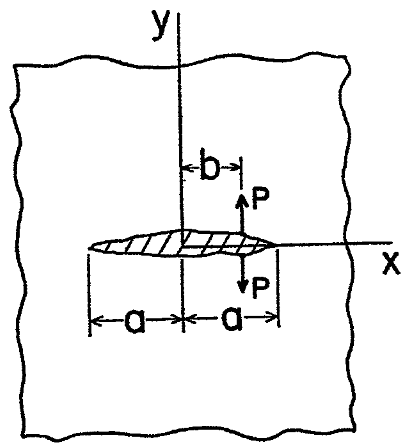

FIG. 5. A CRACK IN AN INFINITE SHEET WITH CONCENTRATED FORCES APPLIED TO CRACK SURFACE

Computation of K, the Stress-Intensity Factor

By use of data on materials, in the form of Fig. 3, the inherently nonlinear problem of structural life has been resolved to require only the computation of instantaneous values of K as the crack propagates through the structure. Since K is the elastic-stress intensity factor for stresses near a crack tip, the problem has been essentially linearized from the point of view of mechanics, a method that permits the use of well-known techniques in attacking the problem.

Irwin[3] has given the solution to the problem shown in Fig. 5, which may be applied by superposition technique (as a Green's Function) to solve all problems of sheet-skins containing internal cracks. This method, which is described in detail in Reference 6, will be outlined here.

Consider the general and arbitrary x-axis symmetric plane-stress problem containing a crack, illustrated by Fig. 6(a). This may be considered to be a free body removed from a gross structure, and may be chosen large enough in size that the crack itself has very little influence on the magnitude and distribution of the boundary forces and stresses shown. Thus we may proceed to solve the plane-stress problem for this sheet, but we find considerable difficulty in problems of doubly connected regions. As an alternative, this problem may be considered to be the sum of two problems, i.e., those in Fig. 6 (b) and (c). The first, (b), is the solution to the same problem with no crack present; the second, (c), is the solution with the same geometry as the original problem, (a), but with loads on the crack surfaces only, of equal and opposite intensity to the stresses that occur at the crack location in (b). This approach ensures that the sum of the boundary forces in (b) and (c) are identical to those in (a), and thus their sum is the solution to (a).

Now we desire only to determine the stress inten-

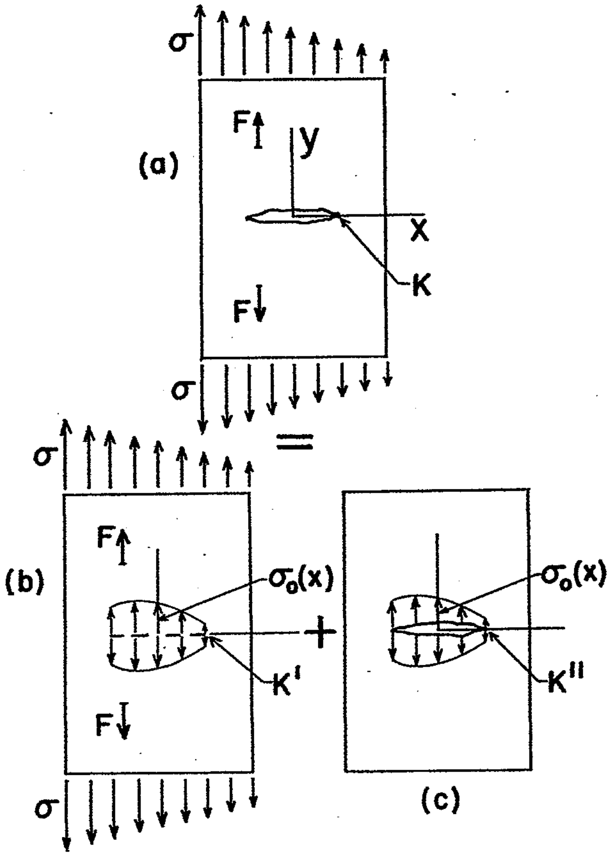

Fig. 6. The Reduction of a Problem, (a), into Two Simpler Problems, (b) and (c), for Computations of Stress Singularity - Intensity Factors

sity factor, K, for the problem, which is the sum of the stress intensity factors, K' and K'', for problems (b) and (c). Since K is the intensity of the singularity of stresses at the crack tip—note $1/\sqrt{r}$ in Eq. (1)—and this singularity is not present, in (b), then

$$K' = 0. \qquad (8)$$

The task remaining is to determine K'' for problem (c), since, observing Eq. (2),

$$K = K''. \qquad (9)$$

Thus the solution to all problems of the type of (a) is reduced to finding the stress, σ_o (x), on the crack axis with no crack present, as in (b), and using this stress as the loading in (c).

To find the value of the stress singularity, K'', for (c), Irwin's solution[3] for the stress singularity in Fig. 5 gives

$$K_P = \frac{P(a+b)^{\frac{1}{2}}}{\pi(a)^{\frac{1}{2}}(a-b)^{\frac{1}{2}}}. \qquad (10)$$

Taking (10) as a Green's Function, the stress singularity, K'', is given by

$$K = K'' = \frac{1}{\pi\sqrt{a}} \int_{-a}^{a} \frac{\sigma_o(x)\,(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}\,dx, \qquad (11)$$

which may be further simplified by *y-axis symmetry,*

$$K = \frac{2\sqrt{a}}{\pi} \int_{o}^{a} \frac{\sigma_o(x)}{(a^2-x^2)^{\frac{1}{2}}}\,dx. \qquad (12)$$

Hence all the general problems of the form of Fig. 6(a) may be attacked by solving the problem for the stresses, σ_o (x), along the crack axis with the crack absent, and integrating the result by Eq. (11) or Eq. 12.

Equally powerful techniques may be presented for other general classes of problems. Therefore we conclude that, in general, K may be computed easily in most of the problems of interest for engineering purposes.

Conclusion

On the basis of the experimental data given, it is evident that rates of crack growth—for example, those in 2024-T3 and 7075-T6 skins of aircraft structure—may be computed by the theory presented over a wide range of nominal stress levels and crack sizes. The ramifications of such broad correlation imply an analytic theory of fatigue based on a concept of growth from initial imperfections through which structural life may be predicted.

REFERENCES

1. I. N. Sneddon, "The Stress Distribution in the Neighborhood of a Crack in an Elastic Solid," *Proc. Royal Soc. of London,* Vol. A-187, 1946, pp. 229-260.
2. G. R. Irwin, "Relation of Stresses Near a Crack to the Crack Extension Force," N.R.L., 1956. (Also *Proc. IXth International Congress of Applied Mech.*, Paper No. 101).
3. G. R. Irwin, "Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate," *Jour. of Appl. Mech.,* June, 1957.
4. M. L. Williams, "On the Stress Distribution at the Base of a Stationary Crack," *Jour. of Appl. Mech.,* March, 1957, Vol. 24, No. 1, p. 109.
5. G. R. Irwin, "Fracture Mechanics," *Proc. Symposium on Naval Structural Mechanics,* Stanford University, Aug. 11, 1958. (See also "Fracture," *Encyclopedia of Physics,* Vol. VI, Springer, 1958.)
6. P. C. Paris, "The Mechanics of Fracture Propagation and Solutions to Fracture Arrestor Problems," Report D2-2195, The Boeing Airplane Company, Sept., 1957.
7. D. E. Martin and G. M. Sinclair, "Crack Propagation Under Repeated Loading," *Proc. Third U. S. Cong. Appl. Mech.,* June, 1958, pp. 595-604.
8. A. J. McEvily, Jr., and Walter Illg, "The Rate of Fatigue-Crack Propagation in Two Aluminum Alloys," NACA TN 4394, Sept., 1958.
9. Waloddi Weibull, "The Propagation of Fatigue Cracks in Light Alloy Plates," SAAB TN 25, 1954.
10. Waloddi Weibull, "Effect of Crack Length and Stress Amplitude on Growth of Fatigue Cracks," Mem. 65, Aero. Res. Institute of Sweden, 1956.