

ES 240

Solid Mechanics

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These notes are largely based on a set of course notes put together by Prof. Suo when he taught ES 240 in 2006, as well as on course notes developed by Prof. Vlassak for ES 246

1. Elements of Linear Elasticity

Reference Text: A classic textbook is *Theory of Elasticity*, by S.P. Timoshenko and J.N. Goodier, McGraw-Hill, New York.

Material particle: A solid is made of atoms, each atom is made of electrons, protons and neutrons, and each proton or neutron is made of... This kind of description of matter is too detailed, and is not used in the theory of elasticity. Instead, we develop a **continuum theory**, in which a material particle contains many atoms, and represents their average behavior.

Displacement field: Any configuration of the material particles can be used as a reference configuration. The displacement is the vector by which a material particle moves relative to its position in the reference configuration. If all the material particles in the body move by the same displacement vector, the body as a whole moves by a rigid-body translation. If the material particles in the body move relative to one another, the body deforms. For example, in a bending beam, material particles on one face of the beam move apart from one another (tension), and material particles on the other face of the beam move toward one another (compression). In a vibrating rod, the displacement of each material particle is a function of time.

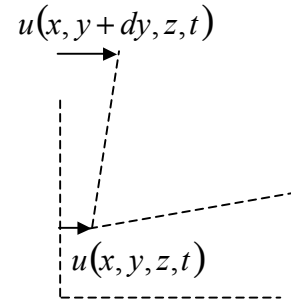
We label each material particle by its coordinates (x, y, z) in the reference configuration. At time t , the material particle (x, y, z) has the displacement $u(x, y, z, t)$ in the x -direction, $v(x, y, z, t)$ in the y -direction, and $w(x, y, z, t)$ in the z -direction. A function of spatial coordinates is known as a field. *The displacement field is a time-dependent vector field.* It is sometimes convenient to write the coordinates of a material particle in the reference configuration as (x_1, x_2, x_3) , and the displacement vector as $u_1(x_1, x_2, x_3, t)$, $u_2(x_1, x_2, x_3, t)$, $u_3(x_1, x_2, x_3, t)$. If we place markers on a body, the motion of the markers visualizes the displacement field and its variation with time.

Strain field: Given a displacement field, we can calculate the strain field. Consider two material particles in the reference configuration: particle A at (x, y, z) and particle B at $(x + dx, y, z)$. In the reference configuration, the two particles are distance dx apart. At a given time t , the two particles move to new locations. The x -component of the displacement of particle A is $u(x, y, z, t)$, and that of particle B is $u(x + dx, y, z, t)$. Consequently, the distance between the two particles elongates by $u(x + dx, y, z, t) - u(x, y, z, t)$. The axial strain in the x -direction is

$$\varepsilon_x = \frac{u(x+dx, y, z, t) - u(x, y, z, t)}{dx} = \frac{\partial u}{\partial x}.$$

This is a strain of material particles in the vicinity of (x, y, z) at time t . The function $\varepsilon_x(x, y, z, t)$ is a component of the strain field of the body.

The **shear strain** is defined as follows. Consider two lines of material particles. In the reference configuration, the two lines are perpendicular to each other. The deformation changes the included angle by some amount. This change in the angle defines the shear strain, γ . We now translate this definition into a strain-displacement relation. Consider three material particles A, B, and C. In the reference configuration, their coordinates are A (x, y, z) , B $(x+dx, y, z)$, and C $(x, y+dy, z)$. In the deformed configuration, in the x -direction, particle A moves by $u(x, y, z, t)$ and particle C by $u(x, y+dy, z, t)$. Consequently, the deformation rotates line AC about axis z by an angle



$$\frac{u(x, y+dy, z, t) - u(x, y, z, t)}{dy} = \frac{\partial u}{\partial y}.$$

Similarly, the deformation rotates line AB about axis z by an angle

$$\frac{v(x+dx, y, z, t) - v(x, y, z, t)}{dx} = \frac{\partial v}{\partial x}.$$

Consequently, the shear strain in the xy plane is the net change in the included angle:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

For a body in the three-dimensional space, the strain state of a material particle is described by a total of six components. The strains relate to the displacements as

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

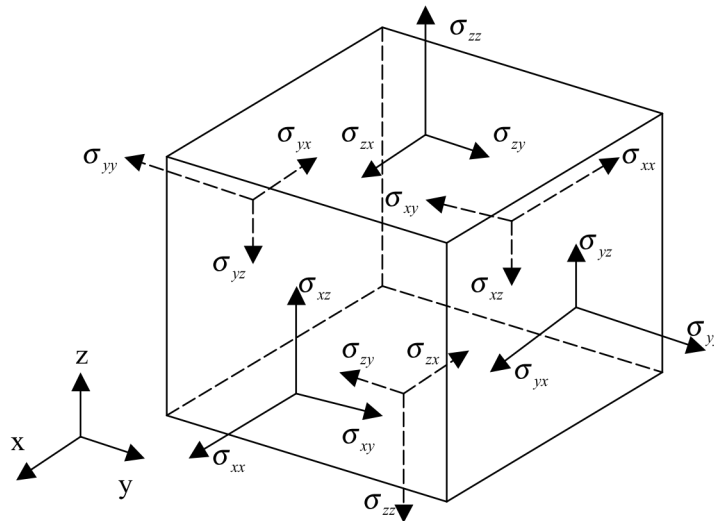
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Another definition of the shear strain relates the definition here by $\varepsilon_{xy} = \gamma_{xy}/2$. With this new definition, we can write the six strain-displacement relations neatly as

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The state of stress at a material particle. Imagine a three-dimensional body. The body may not be in equilibrium (e.g., the body may be vibrating). The material property is unspecified (e.g., the material can be solid or fluid). Imagine a material particle inside the body. What *state of stress* does the material particle suffer from?

To talk about internal forces, we must expose them by drawing a **free-body diagram**. Represent the material particle by a small cube, with its edges parallel to the coordinate axes. Cut the cube out from the body to expose all the forces on its 6 faces. Define stress as force per unit area. On each face of the cube, there are three **stress components**, one normal to the face (normal stress),



and the other two tangential to the face (shear stresses). Now the cube has six faces, so there are a total of 18 stress components. A few points below get us organized.

Notation: σ_{ij} . The first subscript signifies the direction of the vector normal to the face. The second subscript signifies the direction of the stress component. We label the coordinates as (x, y, z) . It is sometimes convenient to write the coordinates as (x_1, x_2, x_3) .

Sign convention. On a face whose normal is in the positive direction of a coordinate axis, the stress component is positive when it points to the positive direction of the axis. On a face whose normal is in the negative direction of a coordinate axis, the stress component is positive when it points to the negative direction of the axis.

Equilibrium of the cube. As the size of the cube shrinks, the forces that scale with the volume (gravity, inertia effect) are negligible. Consequently, the forces acting on the cube faces must be in static equilibrium. Normal stress components form pairs. Shear stress components form quadruples. Consequently, only 6 independent stress components are needed to describe the state of stress of a material particle.

Write the six stress components in a 3 by 3 symmetric matrix:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}.$$

Traction vector: Imagine a plane inside a material. The plane has the unit normal vector \mathbf{n} , with three components n_1, n_2, n_3 . The force per area on the plane is called the **traction**. The traction is a vector, with three components:

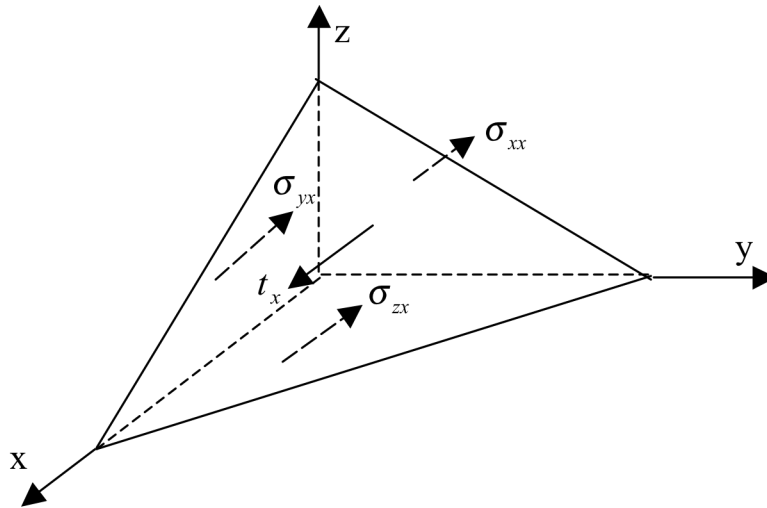
$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}.$$

Question: There are infinite many planes through a point. How do we determine the traction vectors on all these planes?

Answer: You can calculate the traction vector from

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

We now see the merit of writing the components of a state of stress as a matrix. Also, the six components are indeed sufficient to characterize a state of stress, because the six components allow us to calculate the traction vector on any plane.



Proof of the stress-traction relation. This traction-stress relation is the consequence of the equilibrium of a **tetrahedron** formed by the particular plane and the three coordinate planes. Denote the areas of the four triangles by A , A_x , A_y , A_z . Recall a relation from geometry:

$$A_x = An_x, A_y = An_y, A_z = An_z$$

Balance of the forces in the x -direction requires that

$$t_x A = \sigma_{xx} A_x + \sigma_{xy} A_y + \sigma_{xz} A_z.$$

This gives the desired relation

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z.$$

This relation can be rewritten using the index notation:

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3.$$

It can be further rewritten using the summation convention:

$$t_1 = \sigma_{1j} n_j.$$

Similar relations can be obtained for the other components of the traction vector:

$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$

$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

The three equations for the three components of the traction vector can be written collectively in the matrix form, as give in the beginning of this section. Alternatively, they can be written as

$$t_i = \sigma_{ij} n_j.$$

Here the summation is implied for the index j . The above expression represents three equations.

Example: stress and traction. A material particle is in a state of stress with the following components:

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 5 & 6 & 4 \end{bmatrix}.$$

- Compute the traction vector on a plane intersecting the axes x , y and z at 1, 2 and 3, respectively.
- Compute the magnitude of the normal stress on the plane.
- Compute the *magnitude* of the shear stress on the plane.
- Compute the *direction* of the shear stress on the plane.

Solution. We need to find the unit vector normal to the plane. This is a problem in analytical geometry. The equation of a plane intersecting the axes x , y and z at 1, 2 and 3 is

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

Alternatively, a plane can be defined by a given point on the plane, \mathbf{x}_0 , and a unit vector normal to the plane, \mathbf{n} . For any point \mathbf{x} on the plane, $\mathbf{x} - \mathbf{x}_0$ is a vector lying in the plane, so that $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, or

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

A comparison of the two equations of the plane shows that the normal vector is in the direction $\left[\frac{1}{1}, \frac{1}{2}, \frac{1}{3} \right]$. Normalizing this vector, we obtain the unit vector normal to the plane:

$$\mathbf{n} = \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix}.$$

(a) The traction vector on the plane is

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 22/7 \\ 33/7 \\ 56/7 \end{bmatrix}$$

(b) Normal stress on the plane is the traction vector projected on to the normal direction of the plane

$$\sigma_n = \mathbf{t} \cdot \mathbf{n} = \frac{22}{7} \times \frac{6}{7} + \frac{33}{7} \times \frac{3}{7} + \frac{56}{7} \times \frac{2}{7} = 7$$

(c) and (d) The shear stress in the plane is a vector:

$$\boldsymbol{\tau} = \mathbf{t} - \sigma_n \mathbf{n} = \frac{1}{7} \begin{bmatrix} -20 \\ 12 \\ 42 \end{bmatrix}.$$

The direction of this vector is the direction of the shear stress on the plane. The magnitude of the shear stress is 6.86.