



Stress field and momentum balance. Imagine the three-dimensional body again. At time t , the material particle (x, y, z) is under a state of stress $\sigma_{ij}(x, y, z, t)$. Denote the distributed external force per unit volume by $\mathbf{b}(x, y, z, t)$. An example is the gravitational force, $b_z = -\rho g$. The stress and the displacement are time-dependent fields. Each material particle has the acceleration vector $\partial^2 u_i / \partial t^2$. Cut a small differential element, of edges dx , dy and dz . Let ρ be the density. The mass of the differential element is $\rho dx dy dz$. Apply Newton's second law in the x -direction, and we obtain that

$$\begin{aligned} & dydz[\sigma_{xx}(x+dx, y, z, t) - \sigma_{xx}(x, y, z, t)] \\ & + dx dz[\sigma_{yx}(x, y+dy, z, t) - \sigma_{yx}(x, y, z, t)] \\ & + dx dy[\sigma_{zx}(x, y, z+dz, t) - \sigma_{zx}(x, y, z, t)] + b_x dx dy dz = \rho dx dy dz \frac{\partial^2 u_x}{\partial t^2} \end{aligned}$$

Divide both sides of the above equation by $dx dy dz$, and we obtain that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = \rho \frac{\partial^2 u_x}{\partial t^2}.$$

This is the momentum balance equation in the x -direction. Similarly, the momentum balance equations in the y - and z -direction are

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

When the body is in equilibrium, we drop the acceleration terms from the above equations.

Using the summation convention, we write the three equations of momentum balance as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

Hooke's law. For an isotropic, homogeneous solid, only two independent constants are needed to describe its elastic property: Young's modulus E and Poisson's ratio ν . In addition, a thermal expansion coefficient α characterizes strains due to temperature change. When temperature changes by ΔT , thermal expansion causes a strain $\alpha\Delta T$ in all three directions. The combination of multi-axial stresses and a temperature change causes strains

$$\varepsilon_{xx} = \frac{1}{E} \left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) \right] + \alpha\Delta T$$

$$\varepsilon_{yy} = \frac{1}{E} \left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx}) \right] + \alpha\Delta T$$

$$\varepsilon_{zz} = \frac{1}{E} \left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) \right] + \alpha\Delta T$$

The relations for shear are

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \sigma_{xy}, \gamma_{yz} = \frac{2(1+\nu)}{E} \sigma_{yz}, \gamma_{zx} = \frac{2(1+\nu)}{E} \sigma_{zx}.$$

Recall the notation $\varepsilon_{xy} = \gamma_{xy}/2$, and we have

$$\varepsilon_{xy} = \frac{1+\nu}{E} \sigma_{xy}, \varepsilon_{yz} = \frac{1+\nu}{E} \sigma_{yz}, \varepsilon_{zx} = \frac{1+\nu}{E} \sigma_{zx}$$

Index notation and summation convention. The six stress-strain relation may be written as

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}.$$

The symbol δ_{ij} stands for 0 when $i \neq j$ and for 1 when $i = j$. We adopt the convention that a repeated index implies a summation over 1, 2 and 3. Thus, $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.

Homogeneity. When talking about homogeneity, you should think about at least two length scales: a large (macro) length scale, and a small (micro) length scale. A material is said to be *homogeneous* if the macro-scale of interest is much larger than the scale of microstructures. A fiber-reinforced material is regarded as homogeneous when used as a component of an airplane, but should be thought of as *heterogeneous* when its fracture mechanism is of interest. Steel is usually thought of as a homogeneous material, but really contains numerous voids, particles and grains.

Isotropy. A material is isotropic when response in one direction is the same as in any other direction. Metals and ceramics in polycrystalline form are isotropic at macro-scale, even though their constituents—grains of single crystals—are anisotropic. Wood, single crystals, uniaxial fiber reinforced composites are *anisotropic* materials.

Example: a rubber layer pressed between two steel plates. A very thin elastic layer, of Young's modulus E and Poisson's ratio ν , is well bonded between two perfectly rigid plates. A thin rubber layer between two thick steel plates is a good approximation of the situation. The thin layer is compressed between the plates by a known normal stress σ_z . Calculate all the stress and strain components in the thin layer.

Solution. The stress state at the edges of the elastic layer is complicated. We will neglect this edge effect, and focus on the field away from the edges, where the field is uniform. This emphasis makes sense if we are interested in, for example, the displacement of one plate relative to the other. Of course, this emphasis is misplaced if we are concerned of debonding of the layer from the plates, as debonding may initiate from the edges, where stresses are high.

By symmetry, the field has only the normal components and has no shear components. Also by symmetry, we note that

$$\sigma_x = \sigma_y$$

Because the elastic layer is bonded to the rigid plate, the two strain components vanish:

$$\varepsilon_x = \varepsilon_y = 0.$$

Using Hooke's law, we obtain that

$$0 = \varepsilon_x = \frac{1}{E} (\sigma_x - \nu\sigma_y - \nu\sigma_z),$$

or

$$\sigma_x = \frac{\nu}{1-\nu} \sigma_z.$$

Using Hooke's law again, we obtain that

$$\varepsilon_z = \frac{1}{E} (\sigma_z - \nu\sigma_y - \nu\sigma_x) = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} \sigma_z.$$

Consequently, the elastic layer is in a state of uniaxial strain, but all three stress components are nonzero. When the elastic layer is incompressible, $\nu = 0.5$, it cannot be strained in just one direction, and the stress state will be hydrostatic.

Summary of Elasticity Concepts

Players: Fields

Stress tensor: stress state must be represented by 6 components (directional property).

Strain tensor: strain state must be represented by 6 components (directional property).

Stress field: stress state varies from particle to particle (positional property).

Stress field is represented by 6 functions, $\sigma_{xx}(x, y, z; t)$, $\sigma_{xy}(x, y, z; t)$...

Displacement field is represented by 3 functions, $u(x, y, z; t)$, $v(x, y, z; t)$, $w(x, y, z; t)$.

Strain field is represented by 6 functions, $\varepsilon_{xx}(x, y, z; t)$, $\varepsilon_{xy}(x, y, z; t)$,...

Rules: 3 elements of solid mechanics

Momentum balance

Deformation geometry

Material law

Complete equations of elasticity:

Partial differential equations

Boundary conditions

- Prescribe displacement.

- Prescribe traction.

Initial conditions: For dynamic problems (e.g., vibration and wave propagation), one also need prescribe initial displacement and velocity fields.

Solving boundary value problems: ODE and PDE

Idealization, analytical solutions: e.g., S.P. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York

Handbook solutions: R.E. Peterson, *Stress Concentration Factors*, John Wiley, New York, 1974. 2nd edition by W.D. Pilkey, 1997

Brute force, numerical methods: finite element methods, boundary element methods

3D Elasticity: Collected Equations

Momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = \rho \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho \frac{\partial^2 w}{\partial t^2}$$

Strain-displacement relation

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_{zx} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

$$\varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

Hooke's Law

$$\begin{aligned} \varepsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha \Delta T, & \varepsilon_{zx} &= \frac{1+\nu}{E} \sigma_{zx} \\ \varepsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_z + \sigma_x)] + \alpha \Delta T, & \varepsilon_{xy} &= \frac{1+\nu}{E} \sigma_{xy} \\ \varepsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha \Delta T, & \varepsilon_{yz} &= \frac{1+\nu}{E} \sigma_{yz} \end{aligned}$$

Stress-traction relation

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

3D Elasticity: Equations in other coordinates

1. Cylindrical Coordinates (r, θ, z)

Momentum balance

u, v, w are the displacement components in the radial, circumferential and axial directions, respectively. Inertia and body force terms are neglected.

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} &= 0\end{aligned}$$

Strain-displacement relation

$$\begin{aligned}\varepsilon_r &= \frac{\partial u}{\partial r}, & \varepsilon_{\theta z} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \\ \varepsilon_\theta &= \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right), & \varepsilon_{zr} &= \frac{1}{2} \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right) \\ \varepsilon_z &= \frac{\partial w}{\partial z}, & \varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right)\end{aligned}$$

2. Spherical Coordinates (r, θ, ϕ)

θ is measured from the positive z -axis to a radius; ϕ is measured round the z -axis in a right-handed sense. u, v, w are the displacements components in the r, θ, ϕ directions, respectively. Inertia terms are neglected.

Momentum balance

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r} (2\sigma_r - \sigma_\phi - \sigma_\theta - \sigma_{r\theta} \cot \theta) = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} [(\sigma_\theta - \sigma_\phi) \cot \theta + 3\sigma_{r\theta}] = 0$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_\phi}{\partial \phi} + \frac{1}{r} (3\sigma_{r\phi} + 2\sigma_{\theta\phi} \cot \theta) = 0$$

Strain-displacement relation

$$\varepsilon_r = \frac{\partial u}{\partial r}$$

$$\varepsilon_{\theta\phi} = \frac{1}{2r} \left(\frac{\partial w}{\partial \theta} - w \cot \theta + \frac{1}{\sin \theta} \frac{\partial v}{\partial \phi} \right)$$

$$\varepsilon_\theta = \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right)$$

$$\varepsilon_{\phi r} = \frac{1}{2} \left(\frac{\partial w}{\partial \theta} - \frac{w}{r} + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \right)$$

$$\varepsilon_\phi = \frac{1}{r \sin \theta} \left(\frac{\partial w}{\partial \phi} + u \sin \theta + v \cos \theta \right)$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right)$$