Large deformation and instability in dielectric elastomers

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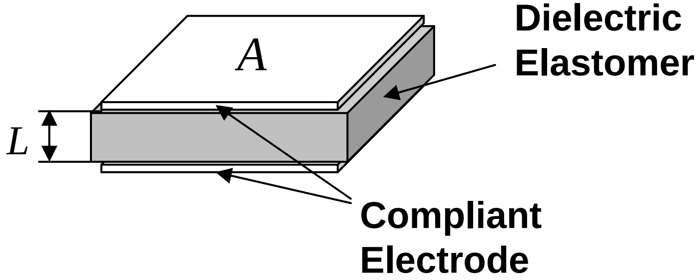
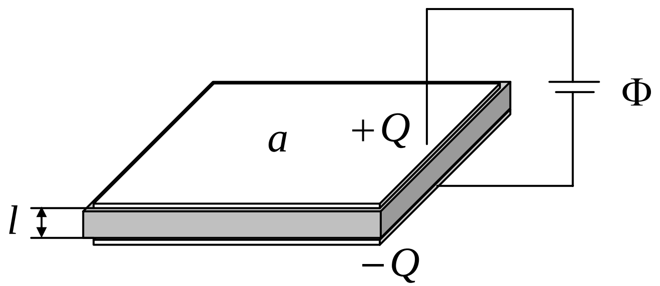
Dielectric elastomer actuators

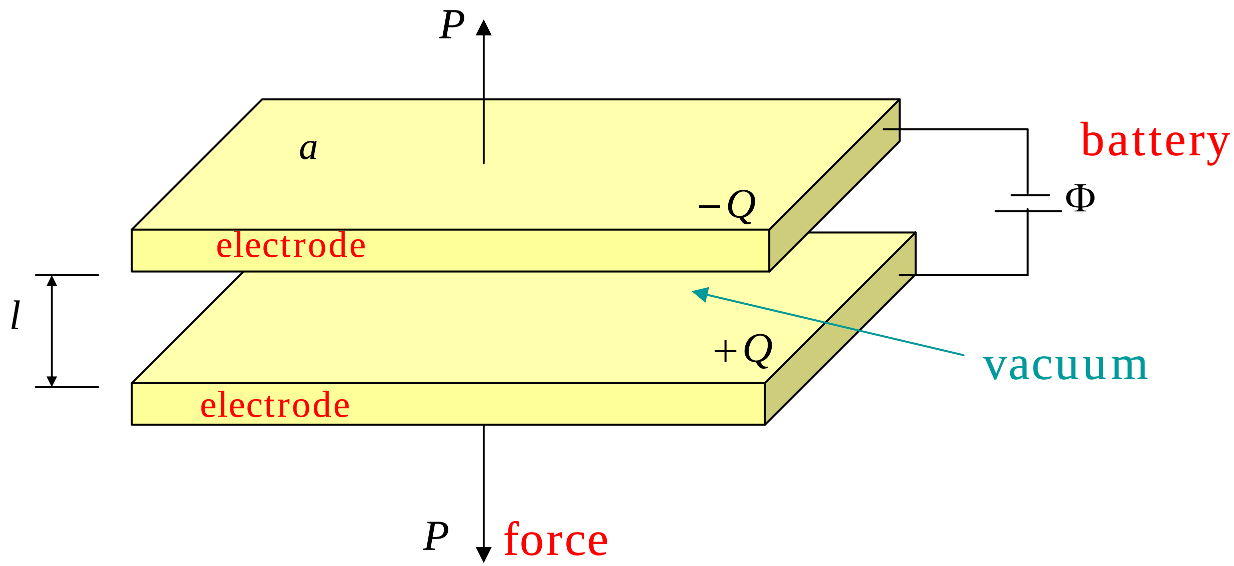


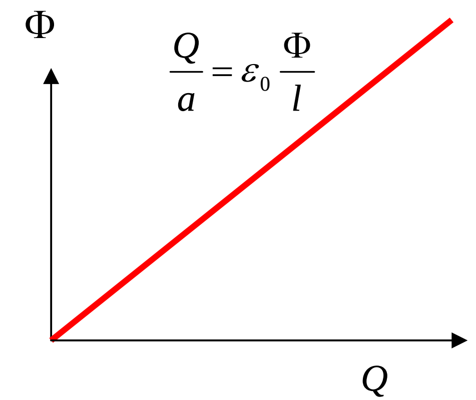


Reference State **Current State**

Pelrine, Kornbluh, Pei, Joseph
High-speed electrically actuated elastomers with strain greater than 100%.
Science 287, 836 (2000).

Parallel-plate capacitor





Electric field $E = \frac{\Phi}{l}$

Electric displacement field $D = \frac{Q}{a}$

stress field $\sigma = \frac{P}{a}$

$D = \varepsilon_0 E$

ε_0, permittivity of vacuum

$\sigma = \frac{1}{2}\varepsilon_0 E^2$ Maxwell stress

Field equations in vacuum, Maxwell (1873)

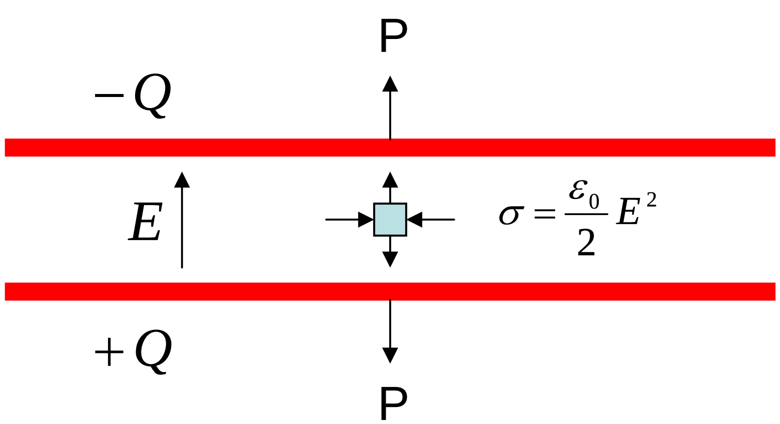
Electrostatic field

$$E_i = -\frac{\partial \Phi}{\partial x_i} \qquad \frac{\partial E_i}{\partial x_i} = \frac{q}{\varepsilon_0}$$

A field of forces maintain equilibrium of a field of charges

$$F_i = qE_i$$

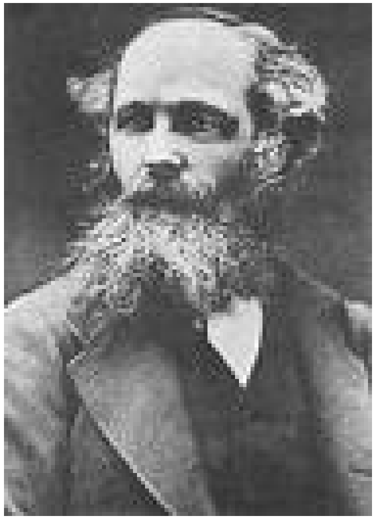
$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon_0 E_j E_i - \frac{\varepsilon_0}{2} E_k E_k \delta_{ij} \right)$$



$$\sigma_{ij} = \varepsilon_0 E_j E_i - \frac{\varepsilon_0}{2} E_k E_k \delta_{ij}$$

Maxwell stress

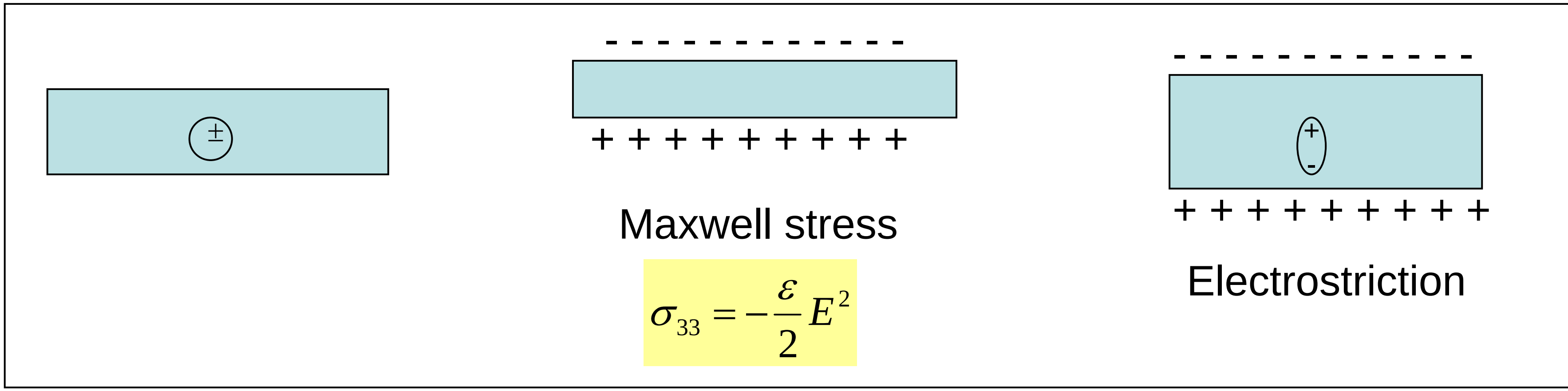
James Clerk Maxwell (1831-1879)



"I have not been able to make the next step, namely, to account by mechanical considerations for these stresses in the dielectric. I therefore leave the theory at this point…"

A Treatise on Electricity & Magnetism (1873), Article 111

Trouble with Maxwell stress in dielectrics



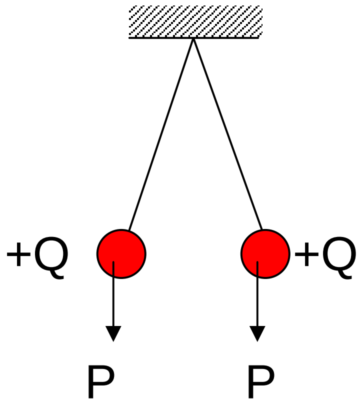
$$\sigma_{33} = -\frac{\varepsilon}{2} E^2$$

Our complaints:

- In general, ε varies with deformation.
- In general, E^2 dependence has no special significance.
- Wrong sign?

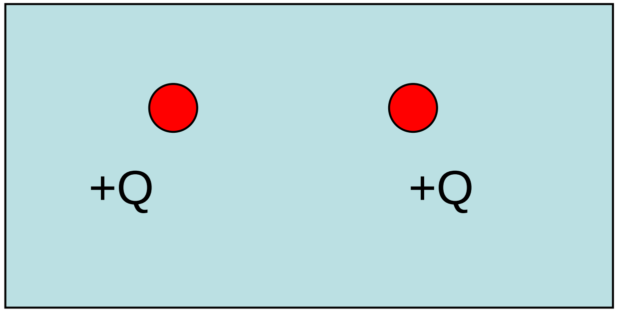
In solids, the Maxwell stress is a bad idea.

Trouble with electric force in dielectrics



In a vacuum,
external force is needed to maintain equilibrium of charges

$$F_i = qE_i$$



In a solid dielectric,
force between charges is NOT an operational concept

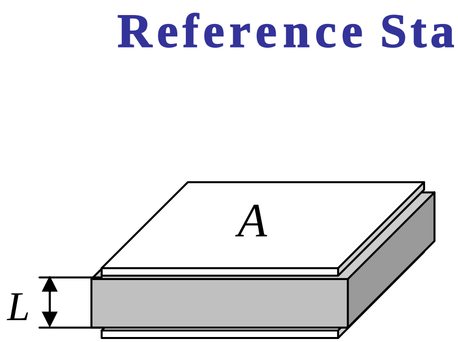
$$\cancel{F_i = qE_i}$$

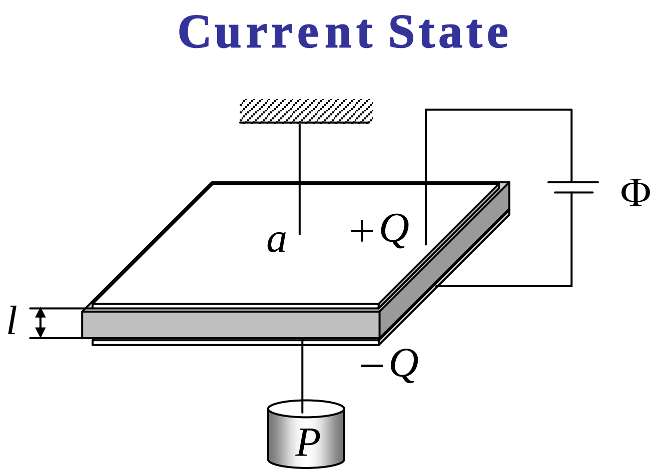
The Feynman Lectures on Physics

Volume II, p.10-8 (1964)

"It is a difficult matter, generally speaking, to make a unique distinction between the electrical forces and mechanical forces due to solid material itself. Fortunately, no one ever really needs to know the answer to the question proposed. He may sometimes want to know how much strain there is going to be in a solid, and that can be worked out. But it is much more complicated than the simple result we got for liquids."

All troubles are gone if we relate measurable quantities





$$\lambda = l / L$$

$$s = P / A$$

$$\tilde{E} = \Phi / L$$

$$\tilde{D} = Q / A$$

Equilibrium condition

$$\delta F = P \delta l + \Phi \delta Q$$

Free-energy density

$$\frac{\delta F}{AL} = \frac{P \delta l}{AL} + \frac{\Phi \delta Q}{LA} \qquad \delta W = s \delta \lambda + \tilde{E} \delta \tilde{D}$$

Equations of state

$$s = \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \lambda} \qquad \tilde{E} = \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \tilde{D}}$$

Extend the theory to 3D inhomogeneous field

Historical work

- Toupin (1956)
- Eringen (1963)
- Tiersten (1971)

……

Recent work

- Dorfmann, Ogden (2005)
- McMeeking, Landis (2005)

……

Suo, Zhao, Greene, J. Mech. Phys. Solids 56, 467 (2008)

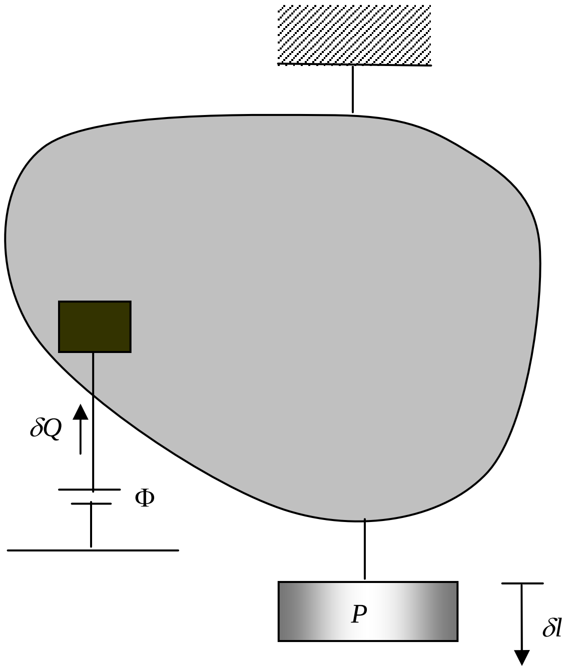
Two ways of doing work

A field of weights do work through displacements:

$$\int B_i \delta x_i dV + \int T_i \delta x_i dA$$

A field of batteries do work through changing:

$$\int \Phi \delta q dV + \int \Phi \delta \omega dA$$



Equilibrium condition

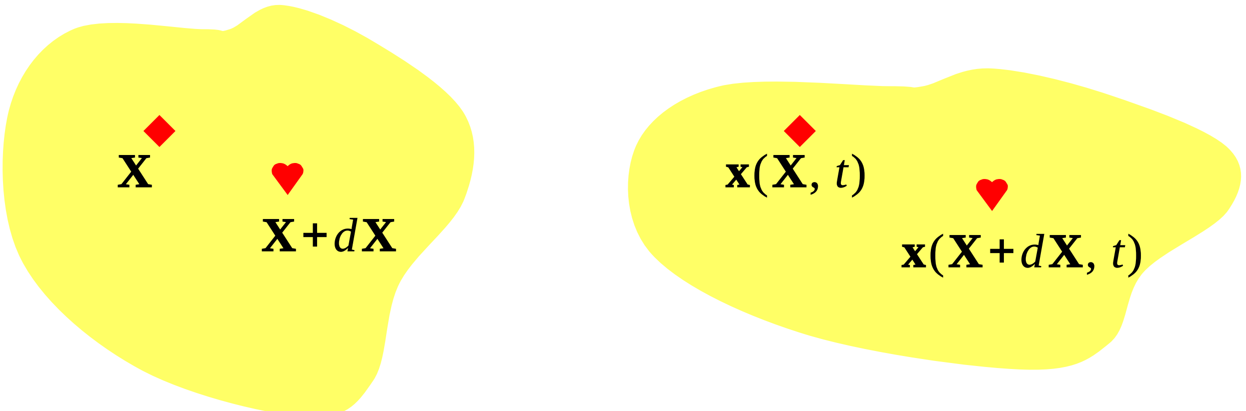
$$\int \delta W dV = \int B_i \delta x_i dV + \int T_i \delta x_i dA + \int \Phi \delta q dV + \int \Phi \delta \omega dA$$

A field of markers: stretch

$$\lambda = \frac{l}{L}$$

Reference state

Current state

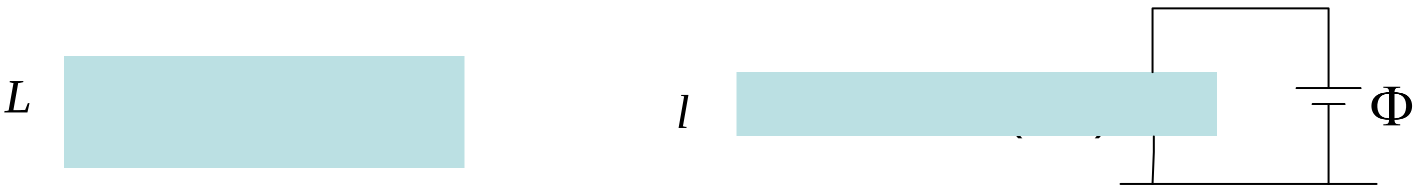


$$\frac{x_i(\mathbf{X}+d\mathbf{X},t)-x_i(\mathbf{X},t)}{dX_K}$$

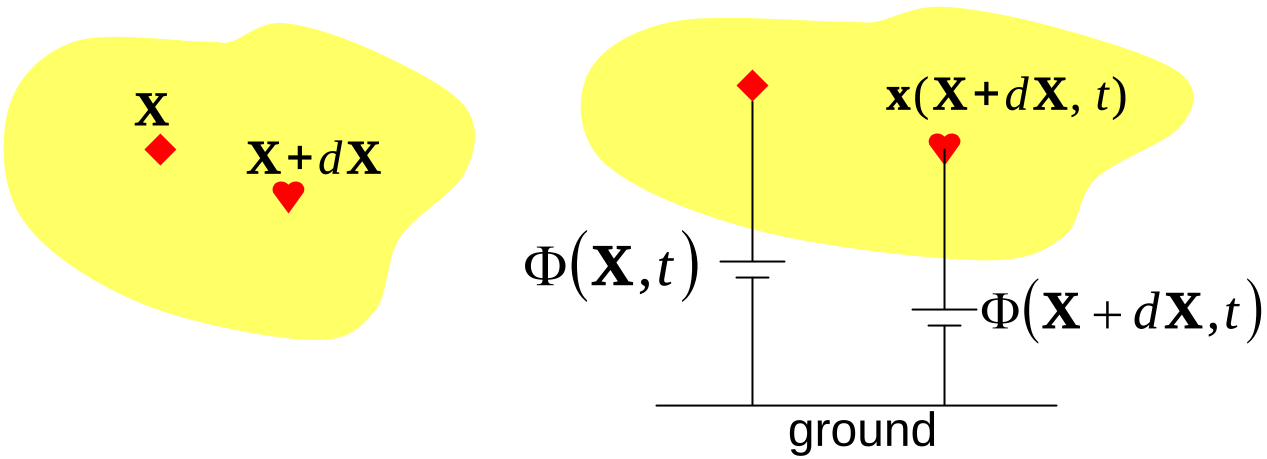
$$F_{iK} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K}$$

A field of batteries: electric field

Reference state　　　　Current state



$$\tilde{E} = \frac{\Phi}{L}$$

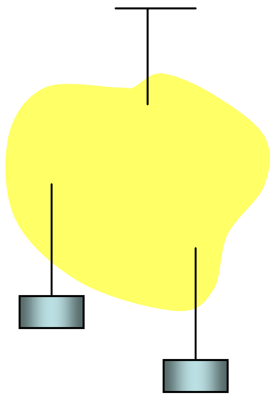


$$\frac{\Phi(\mathbf{X}+d\mathbf{X},t)-\Phi(\mathbf{X},t)}{dX_K}$$

$$\tilde{E}_K = -\frac{\partial \Phi(\mathbf{X},t)}{\partial X_K}$$

A field of weights: stress

Define the stress s_{iK}, such that



$$\int s_{iK} \frac{\partial \xi_i}{\partial X_K} dV = \int B_i \xi_i dV + \int T_i \xi_i dA$$

holds for any test function $\xi_i(\mathbf{X})$

Apply divergence theorem, one obtains that

$$\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) = 0$$

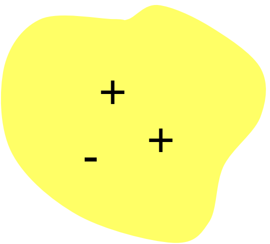
in volume

$$\left(s_{iK}^{-}(\mathbf{X},t) - s_{iK}^{+}(\mathbf{X},t)\right) N_K(\mathbf{X},t) = T_i(\mathbf{X},t)$$

on interface

A field of charges: electric displacement

Define the electric displacement $\tilde{D}_K$, such that



$$-\int \frac{\partial \zeta}{\partial X_K} \tilde{D}_K dV = \int \zeta q dV + \int \zeta \omega dA$$

holds for any test function $\zeta(\mathbf{X})$.

Apply divergence theorem, one obtains that

$$\frac{\partial \tilde{D}_K(\mathbf{X},t)}{\partial X_K} = q(\mathbf{X},t) \qquad \left(\tilde{D}_K^+(\mathbf{X},t) - \tilde{D}_K^-(\mathbf{X},t)\right) N_K(\mathbf{X},t) = \omega(\mathbf{X},t)$$

in volume

on interface

Material model

$$W = W\left(\mathbf{F}, \tilde{\mathbf{D}}\right)$$

$$\delta W = s_{iK}\,\delta F_{iK} + \tilde{E}_K\,\delta \tilde{D}_K$$

$$s_{iK}\left(\mathbf{F}, \tilde{\mathbf{D}}\right) = \frac{\partial W\left(\mathbf{F}, \tilde{\mathbf{D}}\right)}{\partial F_{iK}}, \qquad \tilde{E}_K\left(\mathbf{F}, \tilde{\mathbf{D}}\right) = \frac{\partial W\left(\mathbf{F}, \tilde{\mathbf{D}}\right)}{\partial \tilde{D}_K}$$

A field theory

A field of markers

$$F_{iK}\left(\mathbf{X},t\right)=\frac{\partial x_i\left(\mathbf{X},t\right)}{\partial X_K}$$

A field of weights

$$\frac{\partial s_{iK}\left(\mathbf{X},t\right)}{\partial X_K}+B_i\left(\mathbf{X},t\right)=0$$

A field of batteries

$$\tilde{E}_K\left(\mathbf{X},t\right)=-\frac{\partial \Phi\left(\mathbf{X},t\right)}{\partial X_K}$$

A field of charges

$$\frac{\partial \tilde{D}_K\left(\mathbf{X},t\right)}{\partial X_K}=q\left(\mathbf{X},t\right)$$

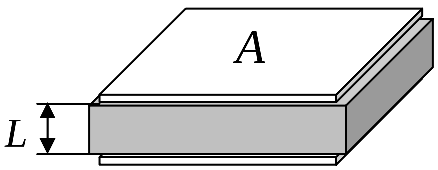
Equations of state

$$s_{iK}\left(\mathbf{F},\tilde{\mathbf{D}}\right)=\frac{\partial W\left(\mathbf{F},\tilde{\mathbf{D}}\right)}{\partial F_{iK}} \qquad \tilde{E}_K\left(\mathbf{F},\tilde{\mathbf{D}}\right)=\frac{\partial W\left(\mathbf{F},\tilde{\mathbf{D}}\right)}{\partial \tilde{D}_K}$$

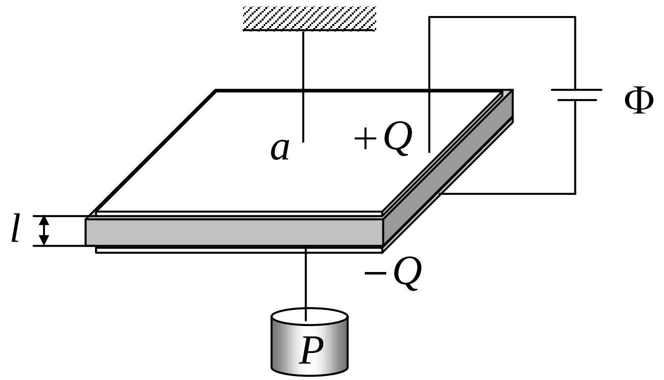
Construct a specific material model: $W\left(\mathbf{F}, \tilde{\mathbf{D}}\right)$

The nominal vs. the true

Reference State



Current State



$$\tilde{E} = \Phi / L \qquad (E = \Phi / l)$$

$$\tilde{D} = Q / A \qquad (D = Q / a)$$

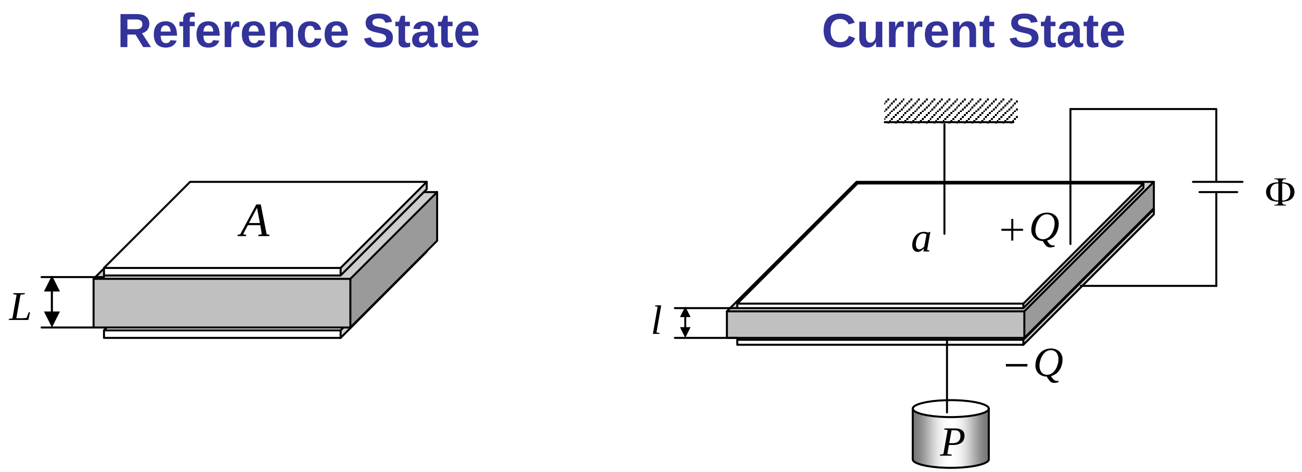
Nominal electric field and nominal electric displacement are work-conjugate

Battery does work $\quad \Phi \delta Q = \left(\tilde{E}L\right)\delta\left(\tilde{D}A\right) = \left(AL\right)\tilde{E}\,\delta\tilde{D}$

True electric field and true electric displacement are NOT work-conjugate

Battery does work $\quad \Phi \delta Q = \left(El\right)\delta\left(Da\right) = \left(la\right)E\,\delta D + EDl\,\delta a$

The nominal vs the true



$s = P / A$	$\sigma = P / a$	$\sigma_{ij} = \dfrac{F_{jK}}{\det(\mathbf{F})} s_{iK}$
$\tilde{E} = \Phi / L$	$E = \Phi / l$	$F_{iK} E_i = \tilde{E}_K$
$\tilde{D} = Q / A$	$D = Q / a$	$D_i = \dfrac{F_{iK}}{\det(\mathbf{F})} \tilde{D}_K$

Dielectric constant is insensitive to stretch

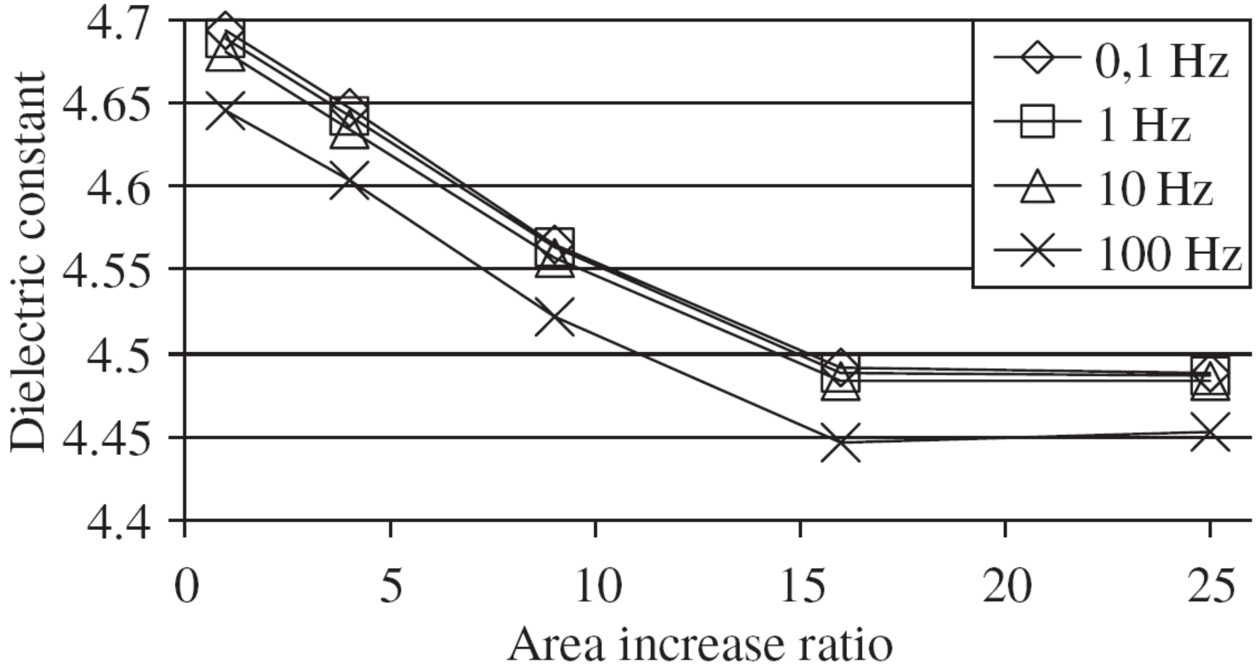


Figure 5. *The relative dielectric constant of VHB^{TM} 4910 drops, when it is stretched.*

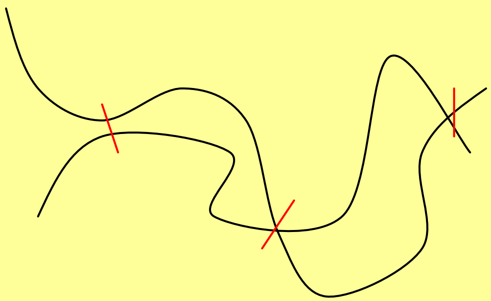
Kofod, Sommer-Larsen, Kornbluh, Pelrine
Journal of Intelligent Material Systems and Structures **14**, 787-793 (2003).

Ideal dielectric elastomers

Dielectric behavior of an elastomer is liquid-like, unaffected by deformation

$$W\left(\mathbf{F},\tilde{\mathbf{D}}\right)=W_s\left(\mathbf{F}\right)+\frac{D^2}{2\varepsilon}$$

Stretch Polarization



Ideal electromechanical coupling is purely a geometric effect:

$$D_i=\frac{F_{iK}}{\det\left(\mathbf{F}\right)}\tilde{D}_K$$

$$\tilde{E}_K\left(\mathbf{F},\tilde{\mathbf{D}}\right)=\frac{\partial W\left(\mathbf{F},\tilde{\mathbf{D}}\right)}{\partial \tilde{D}_K}$$

$$s_{iK}\left(\mathbf{F},\tilde{\mathbf{D}}\right)=\frac{\partial W\left(\mathbf{F},\tilde{\mathbf{D}}\right)}{\partial F_{iK}}$$

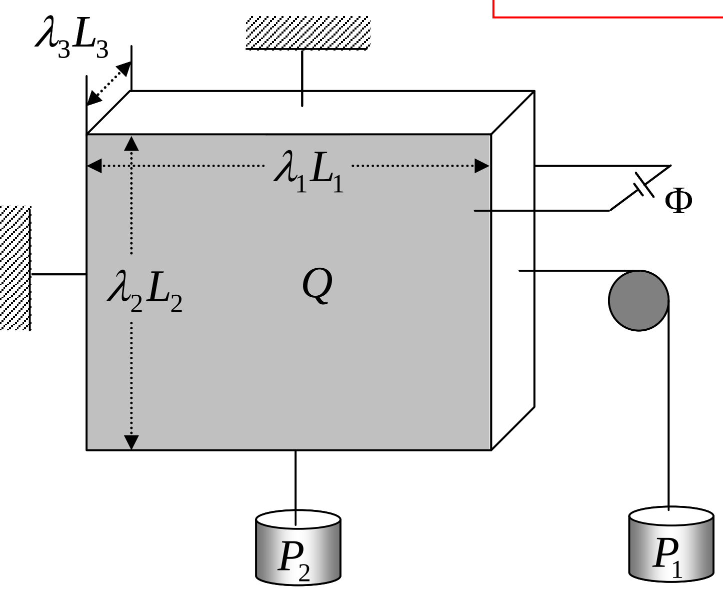
$$D_i=\varepsilon E_i$$

$$\sigma_{ij}=\frac{F_{iK}}{\det\left(\mathbf{F}\right)}\frac{\partial W_s\left(\mathbf{F}\right)}{\partial F_{jK}}+\varepsilon\left(E_iE_j-\frac{1}{2}E_kE_k\delta_{ij}\right)$$

Zhao, Hong, Suo, Physical Review B 76, 134113 (2007)

Neo-Hookean, incompressible dielectric elastomers

$$W\left(\lambda_1,\lambda_2,\tilde{D}\right)=\frac{\mu}{2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)+\frac{\tilde{D}^2}{2\varepsilon\lambda_1^2\lambda_2^2} \qquad \lambda_1\lambda_2\lambda_3=1$$



$$s_1=\frac{\partial W}{\partial \lambda_1}=\mu\left(\lambda_1-\lambda_1^{-3}\lambda_2^{-2}\right)-\frac{\tilde{D}^2}{\varepsilon}\lambda_1^{-3}\lambda_2^{-2}$$

$$s_2=\frac{\partial W}{\partial \lambda_2}=\mu\left(\lambda_2-\lambda_2^{-3}\lambda_1^{-2}\right)-\frac{\tilde{D}^2}{\varepsilon}\lambda_2^{-3}\lambda_1^{-2}$$

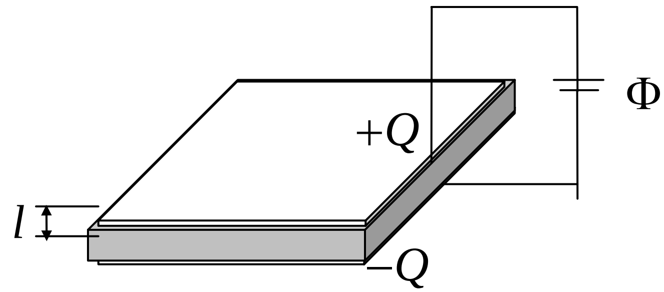
$$\tilde{E}=\frac{\partial W}{\partial \tilde{D}}=\frac{\tilde{D}}{\varepsilon}\lambda_1^{-2}\lambda_2^{-2}$$

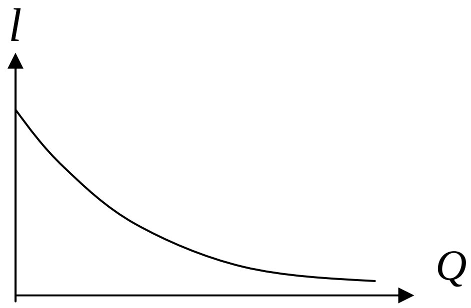
$$\sigma_1=\mu\left(\lambda_1^2-\lambda_3^2\right)-\varepsilon E^2 \qquad D=\varepsilon E$$

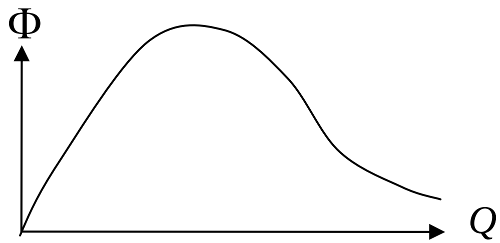
$$\sigma_2=\mu\left(\lambda_2^2-\lambda_3^2\right)-\varepsilon E^2 \qquad \lambda_1\lambda_2\lambda_3=1$$

An application: Electromechanical instability

Electromechanical instability







$$W\left(\lambda,\tilde{D}\right)=\frac{\mu}{2}\left(\lambda^2+2\lambda^{-2}-3\right)+\frac{\lambda^2\tilde{D}^2}{2\varepsilon}$$

$$s=\frac{\partial W\left(\lambda,\tilde{D}\right)}{\partial\lambda}=0 \qquad \lambda=\left(1+\frac{\tilde{D}^2}{\varepsilon\mu}\right)^{-1/3}$$

$$\tilde{E}=\frac{\partial W\left(\lambda,\tilde{D}\right)}{\partial\tilde{D}} \qquad \tilde{E}=\frac{\tilde{D}\lambda^2}{\varepsilon}$$

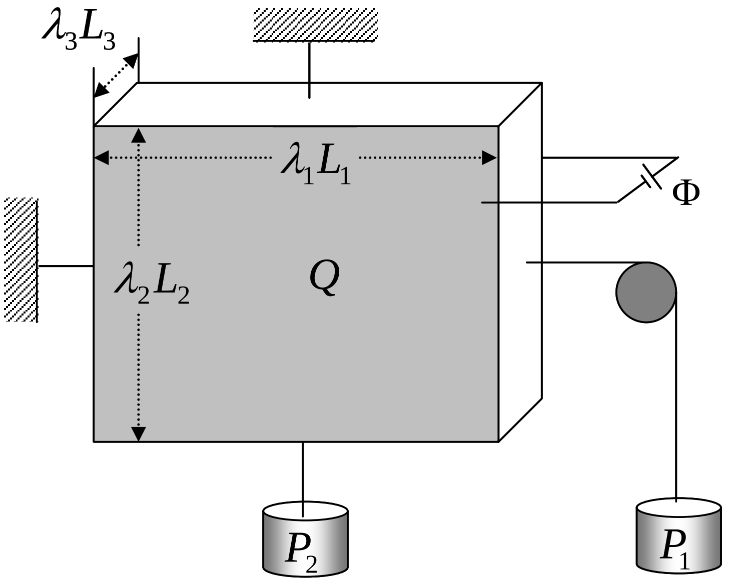
$$\frac{\tilde{E}}{\sqrt{\mu/\varepsilon}}=\frac{\tilde{D}}{\sqrt{\varepsilon\mu}}\left(1+\frac{\tilde{D}^2}{\varepsilon\mu}\right)^{-2/3}$$

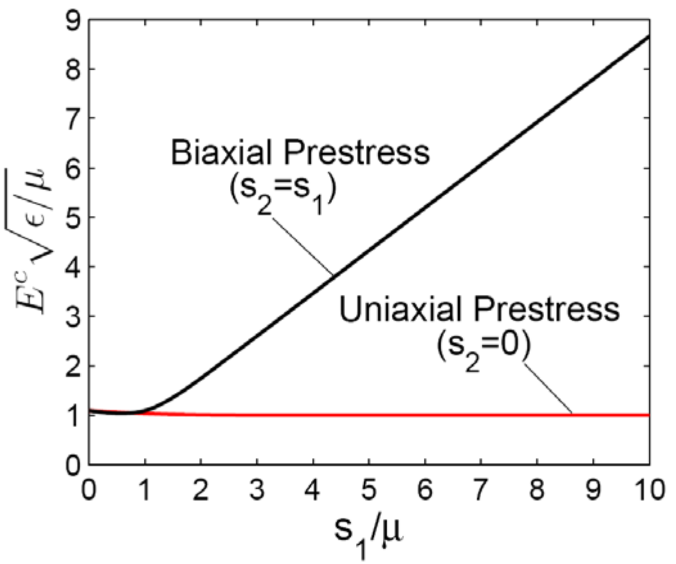
$$\tilde{E}_c\sim\sqrt{\frac{\mu}{\varepsilon}}\sim\sqrt{\frac{10^6\,N/m}{10^{-10}\,F/m}}=10^8\,V/m \qquad \lambda_c\approx 0.63$$

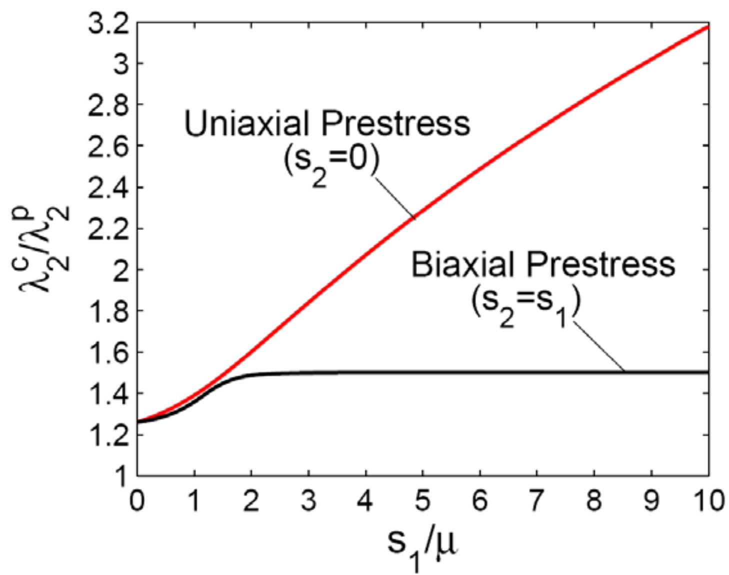
Stark & Garton, Nature 176, 1225 (1955)

Zhao, Suo, APL 91, 061921 (2007)

Pre-stretch increases actuation strain



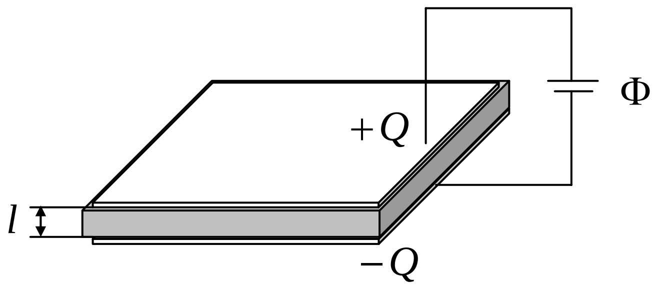


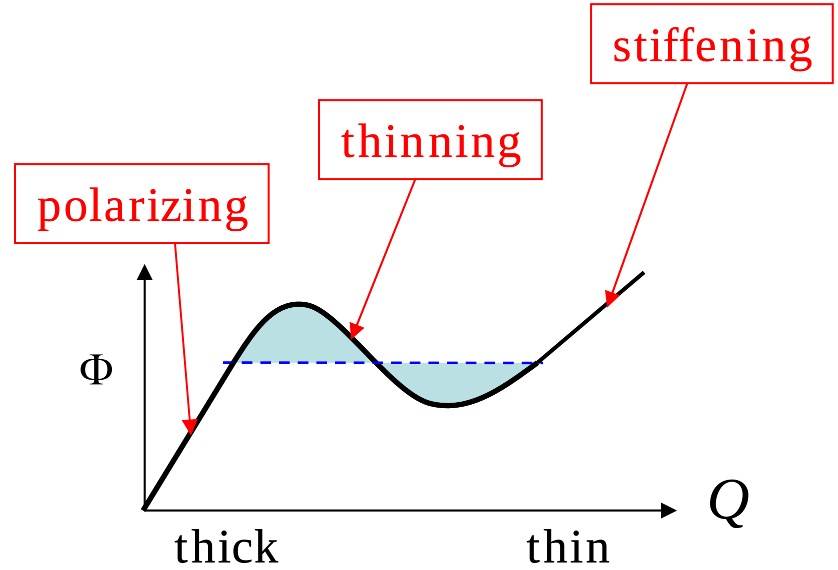


Experiment: Pelrine, Kornbluh, Pei, Joseph
Science 287, 836 (2000).

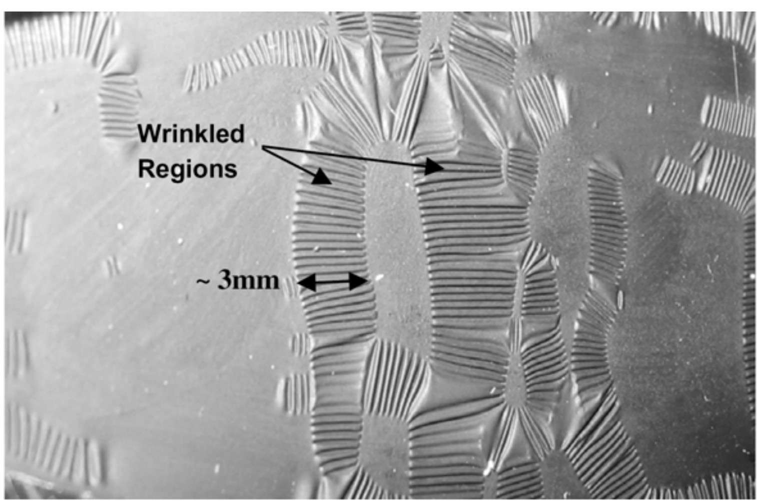
Theory: Zhao, Suo
APL 91, 061921 (2007)

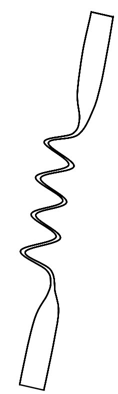
Coexistent states





Top view





Cross section

Coexistent states: flat and wrinkled

Experiment: Plante, Dubowsky,
Int. J. Solids and Structures **43**, 7727 (2006)

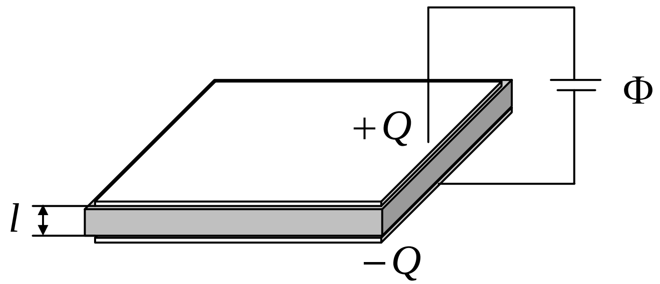
Theory: Zhao, Hong, Suo
Physical Review B 76, 134113 (2007)

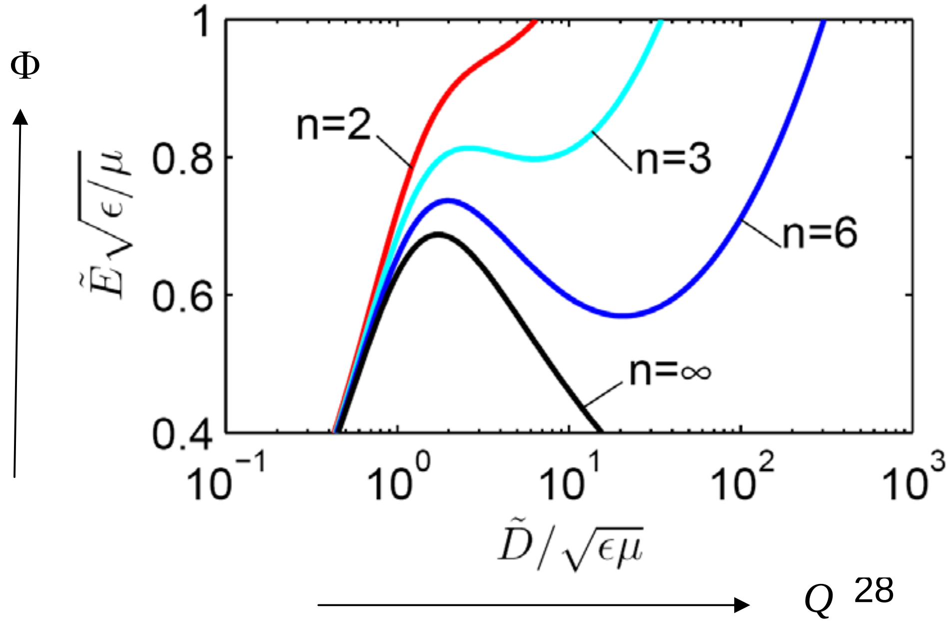
Stiffening due to extension limit

Non Gaussian statistics (e.g., Arruda-Boyce model): $W_s = \mu\left[\frac{1}{2}(I-3)+\frac{1}{20n}\left(I^2-9\right)+\ldots\right]$

$$I = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

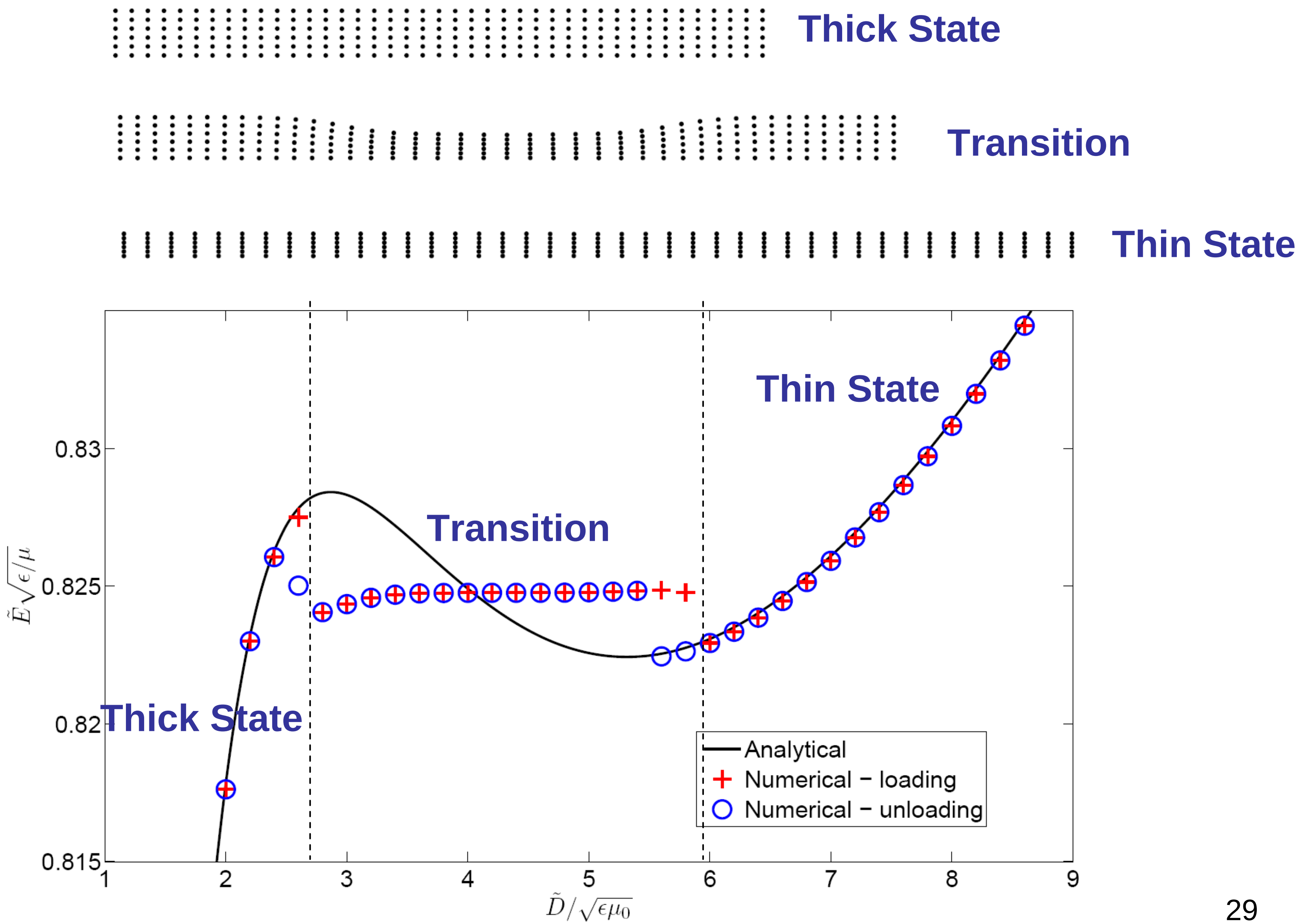
μ: small-strain shear modulus
n: number of monomers per chain





Zhao, Hong, Suo
Physical Review B 76, 134113 (2007)

Finite element method



Zhou, Hong, Zhao, Zhang, Suo, Int. J. Solids and Structures 45, 3739 (2008)

Summary

- A field theory. No Maxwell stress. No electric body force.
- Effect of electric field on deformation is a part of material model.
- Ideal dielectric elastomers: Maxwell stress emerges.
- Electromechanical instability.
- Add other effects (solvent, ions, enzymes…)