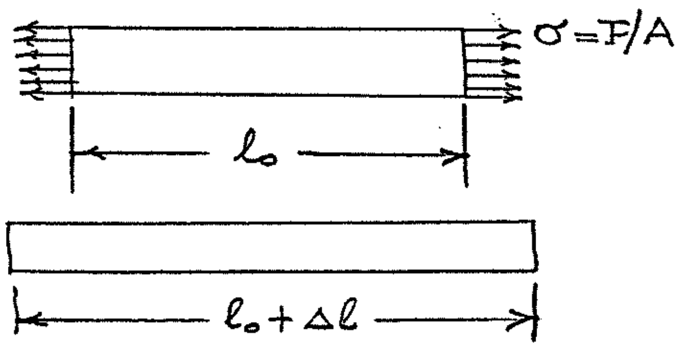
4. The constitutive law: stress-strain relations for infinitesimal deformations of simple linear solids

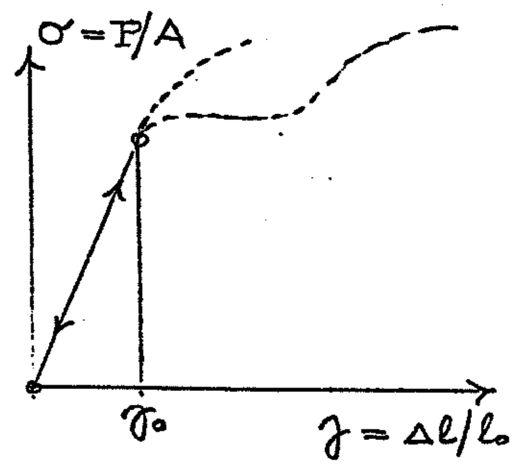
References: Sokolnikoff, Love, Gurtin.

Introductory remarks. The analysis of deformations in Chapter 2 and the anal. of stress in Chapter 3 belong to the general kinematics and Newtonian kinetics of continuous media respectively. The question as to the connection between stresses and deformations is ultimately empirical — the answer depending on the nature of the material to be considered. By postulating a particular law relating stresses and deformations one in fact defines a class of ideal materials (e.g. "rigid", "elastic", "viscoelastic", "plastic" body, "ideal fluid" etc.) the relevance of which to actual physical materials is founded on experimental verification of the resulting predictions.

A fully satisfactory approach to the constitutive law for ideally elastic bodies in the presence of infinitesimal deformation would proceed via the appropriate linearization of the constitutive law governing <u>finite</u> deformations of ideally elastic solids. Since this approach would lead us too far afield we limit our attention directly to inf. deformations and in this chapter introduce a linear stress-deformation law that includes the linearized constitutive law for elastic solids as a special case.

We begin by recalling Hooke's law for sufficiently small deformations of an elastic body in uni-axial tension or compression —





$$\sigma = \eta\gamma \quad (0 \leq \gamma \leq \gamma_0)$$

We now generalize this one-dimensional constitutive law.

From here on adopt the notation:

$\mathcal{L}$ ... set of all second-order tensors

$\mathcal{L}_s$ ... set of all symmetric second-order tensors

Definition A body $\mathcal{B}$ is composed of an (infinitesimally) linear simple solid (l.s.s.) if there exists a closed region $\mathcal{R} \subset \mathcal{E}$ & a reference configuration of $\mathcal{B}$ in $\mathcal{R}$, called an undeformed configuration of $\mathcal{B}$, as well as a linear function $\underset{\sim}{g}(\cdot, \underset{\sim}{x})$ defined on $\mathcal{L}_s$ $\forall\, \underset{\sim}{x} \in \mathcal{R}$ with values in $\mathcal{L}_s$ $\ni$ for every admissible motion $\hat{\underset{\sim}{y}}(\cdot, t) : \mathcal{R} \to \mathcal{R}_t$ $(t_0 \leq t \leq t_1)$,

$$\boxed{\underset{\sim}{\sigma}(\underset{\sim}{x}, t) = \underset{\sim}{g}(\underset{\sim}{\gamma}(\underset{\sim}{x}, t), \underset{\sim}{x}) \;\forall\; (\underset{\sim}{x}, t) \in \mathcal{R} \times \mathcal{T},\; \mathcal{T} = [t_0, t_1],} \quad (\S)$$

where $\underset{\sim}{\gamma} = \text{sym}\nabla\underset{\sim}{u}$ is the infinitesimal strain field and $\underset{\sim}{\sigma}$ the stress field associated with the motion.

One calls $\underset{\sim}{g}$ the tensorial stress-response function. The l.s.s. is said to be homogeneous (as to its stress response) if $\underset{\sim}{g}(\cdot, \underset{\sim}{x})$ is independent of $\underset{\sim}{x}$.

Emphasize exclusion of history (rate) dependence: $\underset{\sim}{\sigma}(\underset{\sim}{x}, t)$ is completely determined by the corresponding local instantaneous $\underset{\sim}{\gamma}(\underset{\sim}{x}, t)$.

Clearly, $(\S) \Rightarrow$

$$\overset{X}{\sigma}_{ij}(\underset{\sim}{x},t) = \overset{X}{c}_{ijkl}(\underset{\sim}{x})\,\overset{X}{\gamma}_{kl}(\underset{\sim}{x},t) \;\forall\; (\underset{\sim}{x},t)\in \mathcal{R}\times\mathcal{T},\ \forall X\in\mathcal{F} \tag{4.1}$$

<u>Remarks</u>. For fixed $X\in\mathcal{F}$ the coefficients $\overset{X}{c}_{ijkl}$ are $4^3=81$ scalar material response functions.

(a) $\mathcal{B}$ is homogeneous if and only if $\overset{X}{c}_{ijkl}=$ constant on $\mathcal{R}$.

(b) $\underset{\sim}{\gamma}(\underset{\sim}{x},\bar{t})=\underset{\sim}{0}$ for some $(\underset{\sim}{x},t)\in\mathcal{R}\times t \Rightarrow \underset{\sim}{\sigma}(\underset{\sim}{x},\bar{t})=\underset{\sim}{0}$. In particular for any infinitesimally rigid motion of $\mathcal{B}$ one has $\underset{\sim}{\sigma}=\underset{\sim}{0}$ on $\mathcal{R}\times\mathcal{T}$.

Clearly, since $\underset{\sim}{\sigma}$ is symmetric

$$\overset{X}{c}_{ijkl} = \overset{X}{c}_{jikl} \text{ on } \mathcal{R}\ \forall X\in\mathcal{F} \quad (*)$$

We now show that one may without loss in generality assume

$$\overset{X}{c}_{ijkl} = \overset{X}{c}_{ijlk} \text{ on } \mathcal{R}\ \forall X\in\mathcal{F} \quad (**)$$

To see this fix $X\in\mathcal{F}$, drop frame label and define

$$\bar{c}_{ijkl} = c_{ij(kl)} = \tfrac{1}{2}(c_{ijkl}+c_{ijlk}) \text{ on } \mathcal{R}$$

$$\bar{\bar{c}}_{ijkl} = c_{ij[kl]} = \tfrac{1}{2}(c_{ijkl}-c_{ijlk}) \text{ on } \mathcal{R}$$

Then from (4.1) and the symmetry of $\underset{\sim}{\gamma}$ one has

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} = (\bar{c}_{ijkl} + \bar{\bar{c}}_{ijkl})\gamma_{kl} = \bar{c}_{ijkl}\gamma_{kl}$$

Drop bar to confirm (**). Thus assume henceforth

$$\overset{X}{c}_{ijkl} = \overset{X}{c}_{jikl} = \overset{X}{c}_{ijlk} \text{ on } R \ \forall\ X \in \mathcal{F} \qquad (4.2)$$

The symm. relations (4.2) reduce the number of response functions from $4^3 = 81$ to 36.

<u>Definition</u> A linear simple solid is <u>isotropic</u> (as to its ~~mechanical~~ response) if $\overset{X}{c}_{ijkl}(\underset{\sim}{x})$ is independent of of X $\forall$ $\underset{\sim}{x} \in R$.

<u>Remarks</u>. Note that mechanical homogeneity and isotropy are independent notions (illustrate). Note form-invariance of (4.1) for isotropic l.s.s.

<u>Theorem</u> 4.1. The response functions $\overset{X}{c}_{ijkl}$ of a linear simple solid are the components in X of a fourth-order tensor field $\underset{\sim}{c}$ on R. The solid is homogeneous iff $\underset{\sim}{c}$ = const on R; it is isotropic iff $\underset{\sim}{c}(\underset{\sim}{x})$ is an isotropic tensor $\forall$ $\underset{\sim}{x} \in R$.

$\underset{\sim}{c}$ ... "response tensor field"

<u>Proof</u>. Consider $[\underset{\sim}{A}] = [A_{ij}] : X \to X'$. Since $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ are two-tensor fields, one has

$$\overset{X'}{\sigma}_{pq} = A_{pi} A_{qj} \overset{X}{\sigma}_{ij} \quad , \quad \overset{X}{\gamma}_{kl} = A_{mk} A_{nl} \overset{X'}{\gamma}_{mn} \qquad (*)$$

Now (*), (4.1) $\Rightarrow$

$$\overset{X'}{\sigma}_{pq} = \underbrace{A_{pi} A_{qj} A_{mk} A_{nl} \overset{X}{c}_{ijkl}}_{\overset{X'}{c}_{pqmn}} \overset{X'}{\gamma}_{mn} \text{ on } R \times T, \text{ whence}$$

$$\overset{X'}{c}_{pqmn} = A_{pi} A_{qj} A_{mk} A_{nl} \overset{X}{c}_{ijkl} \text{ on } R \qquad (4.3)$$

This confirms main claim. The remaining assertions are true ~~by~~ def. of homogeneity and isotropy. qed.

The const. law (4.1), (4.2) when written out becomes

$$\left.\begin{array}{l} \overset{X}{\sigma}_{11} = \overset{X}{c}_{1111} \overset{X}{\gamma}_{11} + \overset{X}{c}_{1122} \overset{X}{\gamma}_{22} + \overset{X}{c}_{1133} \overset{X}{\gamma}_{33} + 2\overset{X}{c}_{1123} \overset{X}{\gamma}_{23} + 2\overset{X}{c}_{1131} \overset{X}{\gamma}_{31} + 2\overset{X}{c}_{1112} \overset{X}{\gamma}_{12} \\ \text{etc.} \end{array}\right\} (4.4)$$

The 36 response functs. $\overset{X}{c}_{ijkl}$ may be regarded as elements of 6×6 matrix $[\overset{X}{\underset{\sim}{C}}]$ defined by

$$[\overset{X}{\underset{\sim}{C}}] = \begin{bmatrix} \overset{X}{c}_{1111} & \overset{X}{c}_{1122} & \cdot & \cdot & \cdot & \overset{X}{c}_{1112} \\ \overset{X}{c}_{2211} & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \overset{X}{c}_{3111} & & & & & \\ \overset{X}{c}_{1211} & \cdot & \cdot & \cdot & \cdot & \overset{X}{c}_{1212} \end{bmatrix} \qquad (4.5)$$

We assume $[\overset{X}{\underset{\sim}{C}}]$ invertible, i.e.

$$\det[\overset{X}{\underset{\sim}{C}}] \neq 0 \text{ on } R, \tag{4.6}$$

so that (4.1) may be inverted. Doing so one obtains in view of the symmetry relations (4.2),

$$\left.\begin{aligned} \overset{X}{\gamma}_{ij}(x,t) &= \overset{X}{\varkappa}_{ijkl}(x)\,\overset{X}{\sigma}_{kl}(x,t) \;\forall\; (x,t)\in R\times J \text{ and } \forall\; X\in F \\ \overset{X}{\varkappa}_{ijkl} &= \overset{X}{\varkappa}_{jikl} = \overset{X}{\varkappa}_{ijlk} \text{ on } R \end{aligned}\right\} \tag{4.7}$$

Remarks: One sees easily that $\overset{X}{\varkappa}_{ijkl}$ are the compts. in X of a fourth-order tensor field $\underset{\sim}{\varkappa}$, called "compliance tensor field". Clearly, a l.s.s. is homogeneous iff $\underset{\sim}{\varkappa}$ = const. on R and isotropic iff $\underset{\sim}{\varkappa}(x)$ is an isotropic 4-tensor $\forall\; x\in R$. Also, evidently,

$$\dim \underset{\sim}{c} = [FL^{-2}], \quad \dim \underset{\sim}{\varkappa} = [F^{-1}L^{2}] \tag{4.8}$$

Theorem 4.2. A necessary and sufficient condition that a linear simple solid be isotropic is that for every $X\in F$,

$$\overset{X}{c}_{ijkl}(x) = \lambda(x)\,\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \;\forall\; x\in R, \tag{4.9}$$

where λ and μ are scalar fields on $\mathcal{R}$. This cond. is equivalent to

$$[\underset{\sim}{C}^{\mathbb{X}}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \quad \text{on } \mathcal{R}\ \forall\ \mathbb{X} \in \mathcal{F}. \tag{4.10}$$

The stress-strain relations (4.1) now become

$$\left.\begin{aligned} &\sigma_{ij}(\underset{\sim}{x},t) = \lambda(\underset{\sim}{x})\,\delta_{ij}\,\gamma_{kk}(\underset{\sim}{x},t) + 2\mu(\underset{\sim}{x})\,\gamma_{ij}(\underset{\sim}{x},t)\ \forall\ (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T} \\ \text{or}\quad &\underset{\sim}{\sigma}(\underset{\sim}{x},t) = \lambda(\underset{\sim}{x})\,\mathrm{tr}\,\underset{\sim}{\gamma}(\underset{\sim}{x},t)\,\underset{\sim}{1} + 2\mu(\underset{\sim}{x})\,\underset{\sim}{\gamma}(\underset{\sim}{x},t)\ \forall\ (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T} \\ \text{or}\quad &\sigma_{11} = \lambda\vartheta + 2\mu\gamma_{11},\ \cdots,\ \cdots,\ \sigma_{12} = 2\mu\gamma_{12},\ \cdots,\ \cdots \\ &\qquad \vartheta = I_1(\underset{\sim}{\gamma}) = \gamma_{kk} = \gamma_{11}+\gamma_{22}+\gamma_{33} \end{aligned}\right\} \tag{4.11}$$

One calls λ and μ respectively "Lamé's modulus" and the "shear modulus*" of the isotropic, linear simple solid; these two moduli are consts. if the solid is homogeneous.

<u>Proof.</u> The equivalence of (4.9), (4.10), (4.11) is immediate from (4.1) and (4.5). Explain.

<u>Re sufficiency</u>. Obvious from Thms. 4.1, 1.1, 1.2: clearly $\underset{\sim}{C}$ given by (4.9) is an isotropic 4-tensor field with the symmetries (4.2). To confirm the form-invariance of (4.11) <u>directly</u>, proceed as follows.

* [illegible] "G"

(4.11) $\Rightarrow$

$$\sigma_{ij}^{X} = \lambda\delta_{ij}\gamma_{kk}^{X} + 2\mu\gamma_{ij}^{X} \quad (*)$$

Consider $[\underset{\sim}{A}] = [A_{ij}] : X \to X'$. From $(*)$ and since $\underset{\sim}{\sigma}, \underset{\sim}{\gamma}$ are two-tensors,

$$\sigma_{pq}^{X'} = A_{pi}A_{qj}(\lambda\delta_{ij}\gamma_{kk}^{X} + 2\mu\gamma_{ij}^{X}) =$$

$$\underbrace{A_{pi}A_{qi}}_{\delta_{pq}}\lambda\underbrace{\gamma_{kk}^{X}}_{\gamma_{kk}^{X'}} + 2\mu\gamma_{pq}^{X'}, \text{ whence}$$

$$\sigma_{pq}^{X'} = \lambda\delta_{pq}\gamma_{kk}^{X'} + 2\mu\gamma_{pq}^{X'}, \text{ as was to be shown.}$$

Note: remains true if $[\underset{\sim}{A}]$ is orth. but not proper orthog.

Re necessity

Assume isotropy. Then, for every $[\underset{\sim}{A}] = [A_{ij}] : X \to X'$, one has, on writing $\sigma_{ij} \equiv \sigma_{ij}^{X}$, $\sigma'_{ij} \equiv \sigma_{ij}^{X'}$, etc.,

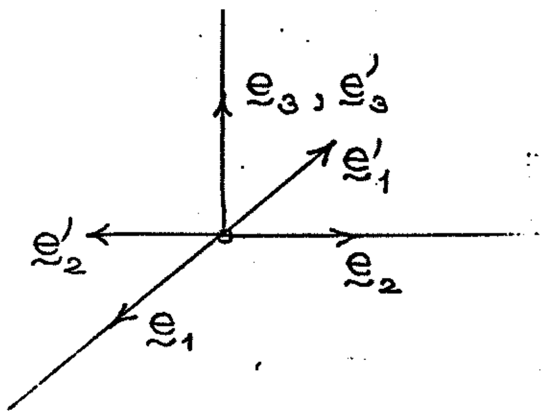
$$\sigma_{ij} = c_{ijkl}\gamma_{kl}, \quad \sigma'_{ij} = c_{ijkl}\gamma'_{kl}, \qquad (1)$$

while

$$\sigma'_{pq} = A_{pi}A_{qj}\sigma_{ij}, \quad \gamma'_{pq} = A_{pi}A_{qj}\gamma_{ij} \quad (2)$$

We now apply (1), (2) to various special choices of $[\underset{\sim}{A}]$:

(a) Rotation about $\underset{\sim}{e}_3$-axis through π



$$[\underset{\sim}{A}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here (2), or – more conveniently (3.20) – furnishes

$$\sigma'_{ii} = \sigma_{ii} \text{ (no sum)}, \quad \sigma'_{23} = -\sigma_{23}, \quad \sigma'_{31} = -\sigma_{31}, \quad \sigma_{12} = \sigma'_{12}$$

$$\gamma'_{ii} = \gamma_{ii} \text{ (no sum)}, \quad \gamma'_{23} = -\gamma_{23}, \quad \gamma'_{31} = -\gamma_{31}, \quad \gamma_{12} = \gamma'_{12}$$

Hence, from (1),

$$\sigma_{11} = c_{1111}\gamma_{11} + c_{1122}\gamma_{22} + c_{1133}\gamma_{33} + 2c_{1123}\gamma_{23} + 2c_{1131}\gamma_{31} + 2c_{1112}\gamma_{12}$$

$$\sigma_{11} = \sigma'_{11} = c_{1111}\gamma_{11} + c_{1122}\gamma_{22} + c_{1133}\gamma_{33} - 2c_{1123}\gamma_{23} - 2c_{1131}\gamma_{31} + 2c_{1112}\gamma_{12}$$

etc.

Since above must hold for every γ, one draws

$c_{1123} = c_{1131} = 0$. Thus and similarly,

$$\left.\begin{aligned} &c_{1123} = c_{1131} = 0, \quad c_{2223} = c_{2231} = 0, \quad c_{3323} = c_{3331} = 0, \\ &c_{2311} = c_{2322} = c_{2333} = c_{2312} = 0, \\ &c_{3111} = c_{3122} = c_{3133} = c_{3112} = 0, \end{aligned}\right\} \quad (\alpha)$$

(b) Rotation about $\underset{\sim}{e}_1$-axis through π

Yields

$$c_{1112} = c_{2212} = c_{3312} = c_{2331} = c_{3123} = c_{1211} = c_{1222} = c_{1233} = 0 \quad (\beta)$$

(c) Rotation about $\underset{\sim}{e}_1$-axis through $\pi/2$

Yields

$$\left.\begin{array}{c} c_{1122} = c_{1133}\,,\; c_{2211} = c_{3311}\,,\; c_{2233} = c_{3322}\,, \\ c_{2222} = c_{3333}\,,\; c_{3131} = c_{1212} \end{array}\right\} \quad (\gamma)$$

(d) Rotation about $\underset{\sim}{e}_3$-axis through $\pi/2$

Yields

$$\left.\begin{array}{c} c_{1122} = c_{2211}\,,\; c_{1111} = c_{2222}\,,\; c_{1133} = c_{2233}\,, \\ c_{3311} = c_{3322}\,,\; c_{2323} = c_{3131} \end{array}\right\} \quad (\delta)$$

(e) Rotation about $\underset{\sim}{e}_3$-axis through $\pi/4$

Yields

$$c_{2323} = \frac{1}{2}(c_{1111} - c_{1122}) \quad (\varepsilon)$$

Now set $c_{1122} \equiv \lambda$, $c_{2323} \equiv \mu$. Each of the remaining c_{ijkl} is either zero or expressible in terms of λ, μ. Collecting (α) through (ε), we reach (4.10). qed.

Exercise 16. Fill in the omitted details in the proof of Thm. 4.2.

Discussion. We have in fact shown that every isotropic 4-tensor $\underset{\sim}{c}$ that obeys the symm. relations (4.2) admits the representation (4.9). Mention general representation thm on isotropic tensors of arbitrary order. It is now clear that "Hooke's law" (4.11) is the most general stress-strain law of the form (4.1) consistent with isotropy. Mention partial anisotropy, eg. transverse isotropy.

Corollary. For an isotropic linear simple solid

$$\overset{X}{c}_{ijkl} = \overset{X}{c}_{klij} \text{ on } \mathcal{R} \ \forall \ X \in \mathcal{F}. \tag{4.12}$$

Further, for such a solid the principal axes of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ coincide.

Proof. Note that $[\overset{X}{\underset{\sim}{C}}]$ given by (4.10) is symmetric or inspect (4.9). Observe from (4.11), (4.6) that $\sigma_{ij} = 0 \Leftrightarrow \gamma_{ij} = 0 \ (i \neq j)$.

Remark. It will emerge later on that (4.12) are nec. & suff. cond for a linear simple solid to be "elastic". Thus every isotropic s.l.s. is automatically elastic.

Study of constitutive law for isotropic linear simple solids

Recall that for such a solid

$$\sigma_{ij} = \lambda \delta_{ij} \gamma_{kk} + 2\mu \gamma_{ij} \qquad (4.11)\text{ bis}$$

In view of (4.10) the invertibility condition (4.6) here becomes

$$\mu(3\lambda + 2\mu) \neq 0 \text{ on } R \qquad (4.13)$$

To invert (4.11) note first that

$$\left.\begin{aligned} &\sigma_{ii} = (3\lambda+2\mu)\gamma_{ii} \quad \text{or} \quad \Theta = (3\lambda+2\mu)\vartheta \\ &\Theta = \sigma_{ii} = \operatorname{tr} \underset{\sim}{\sigma} = I_1(\underset{\sim}{\sigma}), \quad \vartheta = \gamma_{ii} = \operatorname{tr} \underset{\sim}{\gamma} = I_1(\underset{\sim}{\gamma}) \end{aligned}\right\} (4.14)$$

(4.11), (4.13), (4.14) $\Rightarrow$

$$\gamma_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} \qquad (*)$$

Assume

$$\lambda + \mu \neq 0 \text{ on } R \qquad (4.15)$$

so that (*) may be written as

$$\gamma_{ij} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left\{\left[1 + \frac{\lambda}{2(\lambda+\mu)}\right]\sigma_{ij} - \frac{\lambda}{2(\lambda+\mu)}\delta_{ij}\sigma_{kk}\right\} \qquad (**)$$

Define two new scalar response functions in terms of μ, λ:

$$\left.\begin{aligned}\eta &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \ \ldots \text{Young's modulus (usually "E")} \\ \nu &= \frac{\lambda}{2(\lambda+\mu)} \ \ldots \text{Poisson's ratio}\end{aligned}\right\} \quad (4.16)$$

Clearly, (4.16) $\Rightarrow$

$$\eta = 2\mu(1+\nu) \tag{4.17}$$

In view of (4.16) one may now write (**) as

$$\left.\begin{aligned}&\gamma_{ij} = \frac{1}{\eta}[(1+\nu)\sigma_{ij} - \nu\,\delta_{ij}\sigma_{kk}] \ \text{ or } \ \underset{\sim}{\gamma} = \frac{1}{\eta}[(1+\nu)\underset{\sim}{\sigma} - \nu\,\mathrm{tr}\,\underset{\sim}{\sigma}\,\underset{\sim}{1}] \\ &\text{or } \ \gamma_{11} = \frac{1}{\eta}[\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})], \cdots, \cdots, \gamma_{12} = \frac{1+\nu}{\eta}\sigma_{12} = \frac{\sigma_{12}}{2\mu}, \cdots, \cdots\end{aligned}\right\} \quad (4.1$$

Define another auxiliary response function κ through

$$\kappa = \lambda + \frac{2\mu}{3} \ \ldots \text{bulk modulus} \tag{4.19}$$

(4.14), (4.19) $\Rightarrow$

$$\sigma_{ii} = 3\kappa\gamma_{ii} \ \text{ or } \ \mathrm{tr}\,\underset{\sim}{\sigma} = 3\kappa\,\mathrm{tr}\,\underset{\sim}{\gamma} \ \text{ or } \ \Theta = 3\kappa\vartheta \quad (4.20$$

From (4.11), (4.16), (4.19) follows

$$\dim\lambda = \dim\mu = \dim\eta = \dim\kappa = [FL^{-2}], \ \dim\nu = [0] \quad (4.21$$

(4.18), (4.7) give for an isotropic linear simple solid:

$$\varkappa_{ijkl} = \frac{-\nu}{\eta}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2\eta}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ \text{ on } \mathcal{R} \quad (4.2$$

Physical interpretation of the response parameters μ, η, ν, κ

Remark on lack of direct physical interpretation of λ.

(a) (4.11) $\Rightarrow \sigma_{ij} = 2\mu\gamma_{ij}$ $(i \neq j)$, interpret μ and note that $\mu > 0$ for actual material (explain).

(b) Consider a uni-axial $\underset{\sim}{\sigma}$, say

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \sigma_{11} \neq 0$$

Here (4.18) gives

$$\sigma_{11} = \eta\gamma_{11}, \quad \frac{\gamma_{22}}{\gamma_{11}} = \frac{\gamma_{33}}{\gamma_{11}} = -\nu, \text{ interpret } \eta \text{ and } \nu.$$

Note that $\eta > 0$, $\nu \geq 0$ for actual materials. Mention cork, rubber.

(c) Consider an isotropic compressive $\underset{\sim}{\sigma}$, i.e.

$$\sigma_{ij} = -p\,\delta_{ij} \text{ or } \underset{\sim}{\sigma} = -p\mathbf{1}, \; p > 0.$$

Here from (4.20),

$$p = -\kappa\vartheta, \text{ interpret } \kappa \text{ and note that}$$

$\kappa \geq 0$ for actual materials.

TABLE OF RELATIONS AMONG ELASTIC CONSTANTS

	LAMÉ'S MODULUS λ	SHEAR MODULUS μ	YOUNG'S MODULUS η	POISSON'S RATIO ν	BULK MODULUS κ
λ, μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ, η		irrational		irrational	irrational
λ, ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ, κ		$\frac{3(\kappa-\lambda)}{2}$	$\frac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda}$	$\frac{\lambda}{3\kappa-\lambda}$	
μ, η	$\frac{(2\mu-\eta)\mu}{\eta-3\mu}$			$\frac{\eta-2\mu}{2\mu}$	$\frac{\mu\eta}{3(3\mu-\eta)}$
μ, ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ, κ	$\frac{3\kappa-2\mu}{3}$		$\frac{9\kappa\mu}{3\kappa+\mu}$	$\frac{3\kappa-2\mu}{2(3\kappa+\mu)}$	
η, ν	$\frac{\nu\eta}{(1+\nu)(1-2\nu)}$	$\frac{\eta}{2(1+\nu)}$			$\frac{\eta}{3(1-2\nu)}$
η, κ	$\frac{3\kappa(3\kappa-\eta)}{9\kappa-\eta}$	$\frac{3\eta\kappa}{9\kappa-\eta}$		$\frac{3\kappa-\eta}{6\kappa}$	
ν, κ	$\frac{3\kappa\nu}{1+\nu}$	$\frac{3\kappa(1-2\nu)}{2(1+\nu)}$	$3\kappa(1-2\nu)$		