7. Minimum-energy and variational principles in linear elastostatics

Introduction. The principles about to be discussed here furnish alternative characterizations of the solutions to the fundamental bdy-value problems of equil. theory. Apart from their intrinsic theoretical interest, they are the basis of important approx. methods of solution of such probs. (Rayleigh-Ritz, Galerkin).
Mention also role in finite-element methods.

Def. (Kinematically admissible state). Let R be a regular region and $\partial_1 R$ a subset of ∂R. We say that the ordered array $\mathcal{S} = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a kinematically admissible state on $\bar{R}$, corresponding to the elasticity tensor field $\mathbf{C}$ and the surface displacements $\overset{*}{\mathbf{u}}$ on $\partial_1 R$ and write

$$\mathcal{S} = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \in \mathcal{A}_1(\mathbf{C}, \overset{*}{\mathbf{u}}; \bar{R}, \partial_1 R)$$

provided:

(a) $\underset{\sim}{C} \in \mathcal{C}^1(\bar{R})$, $\underset{\sim}{C}$ is "symmetric" & invertible on $\bar{R}$

$\underset{\sim}{u} \in \mathcal{C}^2(R) \cap \mathcal{C}^1(\bar{R})$

(b) Equations (6.1), (6.2) hold on R

(c) $\underset{\sim}{u} = \overset{*}{\underset{\sim}{u}}$ on $\partial_1 R$

<u>Thm.</u> 7.1 (Principle of minimum potential energy, Green)
Let R be a bounded regular region & let $\partial_1 R, \partial_2 R$ be integrable complementary subsets of ∂R. Suppose:

(a) $s = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}] \in \mathcal{E}(\underset{\sim}{C}, \underset{\sim}{f}; \bar{R})$

(b) $\underset{\sim}{u} = \overset{*}{\underset{\sim}{u}}$ on $\partial_1 R$, $\underset{\sim}{s} = \overset{*}{\underset{\sim}{s}}$ on $\partial_2 R$

(c) $\underset{\sim}{C}$ is positive definite on R

Let $\mathcal{A} \equiv \mathcal{A}(\underset{\sim}{C}, \overset{*}{\underset{\sim}{u}}; \bar{R}, \partial_1 R)$ and let Φ be the functional defined on $\mathcal{A}$ by means of

$$\left.\begin{aligned} \Phi\{\hat{s}\} &= U_{\underset{\sim}{C}}\{\hat{\underset{\sim}{E}}\} - \int_R \underset{\sim}{f} \cdot \hat{\underset{\sim}{u}}\, dV - \int_{\partial_2 R} \overset{*}{\underset{\sim}{s}} \cdot \hat{\underset{\sim}{u}}\, dA \quad \forall \hat{s} \in \mathcal{A} \\ U_{\underset{\sim}{C}}\{\hat{\underset{\sim}{E}}\} &= \frac{1}{2}\int_R (\underset{\sim}{C}\hat{\underset{\sim}{E}}) \cdot \hat{\underset{\sim}{E}}\, dV . \end{aligned}\right\} \quad (7.1)$$

Then,
$$\Phi\{s\} = \min_{\hat{s} \in \mathcal{A}} \Phi\{\hat{s}\}$$

and this minimum is assumed by $\Phi\{\hat{s}\}$ only if $\hat{\underset{\sim}{E}} = \underset{\sim}{E}$, $\hat{\underset{\sim}{S}} = \underset{\sim}{S}$ on $\bar{R}$.

$\Phi\{\hat{s}\}$ ... "potential energy" of $\hat{s}$.

Remark. A rough statement of Thm. 7.1 asserts: Among all kinematically admissible states associated with a given mixed boundary-value problem of elastostatics, the actual solution of the problem is uniquely characterized — as far as the stresses and strains are concerned — by the fact that it renders the potential energy Φ a minimum.

Proof.

Observe that (a), (b), (c) uniquely characterize $\underset{\sim}{\gamma}$ and $\underset{\sim}{\sigma}$ according to Thm. 6.4. This fact will, however, not be used in the subsequent proof.

Clearly, $s \in \mathcal{A}(\underset{\sim}{c}, \overset{*}{\underset{\sim}{u}}, \overline{R}, \partial_1 R)$ and it suffices to show

$$\Phi\{\hat{s}\} \geq \Phi\{s\} \;\forall\, \hat{s} \in \mathcal{A} \text{ and } \Phi\{\hat{s}\} = \Phi\{s\},\; \hat{s} \in \mathcal{A} \Rightarrow \hat{\underset{\sim}{\gamma}} = \underset{\sim}{\gamma} \text{ on } \overline{R} \;($$

Choose $\hat{s} = [\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{\gamma}}, \hat{\underset{\sim}{\sigma}}] \in \mathcal{A}$ and let

$$s' = [\underset{\sim}{u}', \underset{\sim}{\gamma}', \underset{\sim}{\sigma}'] = \hat{s} - s \text{ on } \overline{R} \tag{2}$$

(2), Thm. 5.5 (addition thm. for the strain-energy density)

$$U_{\underset{\sim}{c}}\{\hat{\underset{\sim}{\gamma}}\} = U_{\underset{\sim}{c}}\{\underset{\sim}{\gamma}\} + U_{\underset{\sim}{c}}\{\underset{\sim}{\gamma}'\} + \int_R \underset{\sim}{\sigma} \cdot \underset{\sim}{\gamma}' \, dV \tag{3}$$

(3), (7.1) $\Rightarrow$

$$\Phi\{\hat{s}\} - \Phi\{s\} = U_{\underset{\sim}{C}}\{\underset{\sim}{E}'\} + \int_R \underset{\sim}{\sigma}\cdot\underset{\sim}{E}'\,dV - \int_R \underset{\sim}{f}\cdot\underset{\sim}{u}'\,dV - \int_{\partial_2 R} \overset{*}{\underset{\sim}{s}}\cdot\underset{\sim}{u}'\,dV \tag{4}$$

By hyp., the def. of $\mathcal{C}$ and (2),

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{E}' = \sigma_{ij}\tfrac{1}{2}(u'_{i,j}+u'_{j,i}) = \sigma_{ij}u'_{i,j} = (\sigma_{ij}u'_i)_{,j} - \sigma_{ij,j}u'_i$$

so that from (a),

$$\underbrace{\underset{\sim}{\sigma}\cdot\underset{\sim}{E}'}_{C(\bar{R})} = \nabla\cdot(\underset{\sim}{\sigma}\underset{\sim}{u}') + \underbrace{\underset{\sim}{f}\cdot\underset{\sim}{u}'}_{C(\bar{R})} \quad \text{on } R \tag{5}$$

Thus $\nabla\cdot(\underset{\sim}{\sigma}\underset{\sim}{u}')\in C(\bar{R})$. Hence (5), Thm. 1.15 (divergence thm) and (1.23) (reciprocal formula) give

$$\int_R \underset{\sim}{\sigma}\cdot\underset{\sim}{E}'\,dV = \int_{\partial R} \underbrace{\underset{\sim}{\sigma}\underset{\sim}{u}'\cdot\underset{\sim}{n}}_{\underset{\sim}{\sigma}\underset{\sim}{n}\cdot\underset{\sim}{u}'}\,dA + \int_R \underset{\sim}{f}\cdot\underset{\sim}{u}'\,dV = \int_{\partial R}\underset{\sim}{s}\cdot\underset{\sim}{u}'\,dA + \int_R \underset{\sim}{f}\cdot\underset{\sim}{u}'\,dV \tag{6}$$

But (b), (2) $\Rightarrow$ $\underset{\sim}{u}' = \underset{\sim}{0}$ on $\partial_1 R$, $\underset{\sim}{s} = \overset{*}{\underset{\sim}{s}}$ on $\partial_2 R$. Therefore,

$$\int_R \underset{\sim}{\sigma}\cdot\underset{\sim}{E}'\,dV = \int_{\partial_2 R}\overset{*}{\underset{\sim}{s}}\cdot\underset{\sim}{u}'\,dA + \int_R \underset{\sim}{f}\cdot\underset{\sim}{u}'\,dV \tag{7}$$

Finally, (7), (4) $\Rightarrow$

$$\Phi\{\hat{s}\} - \Phi\{s\} = U_{\underset{\sim}{C}}\{\underset{\sim}{E}'\} = \frac{1}{2}\int_R (\underset{\sim}{C}\underset{\sim}{E}')\cdot\underset{\sim}{E}'\,dV \tag{8}$$

(8), (2), (c) $\Rightarrow$ (1). qed.

Remarks. Note generality of Thm. 7.1 (inhomog., anisotropic media). Mention extension to mixed-mixed problem and unbounded regular regions.

Corollary. Thm. 7.1 implies Thm. 6.4 (uniqueness) provided $\partial_1 R$, $\partial_2 R$ in Thm. 6.4 are restricted to be integrable.

Proof. Suppose $\mathcal{S} = [\underline{u}, \underline{E}, \underline{S}]$ and $\bar{\mathcal{S}} = [\underline{\bar{u}}, \underline{\bar{E}}, \underline{\bar{S}}]$ both satisfy the hyp. (a), (b) of Thm. 7.1. Now take first $\hat{\mathcal{S}} = \mathcal{S}$ and then $\hat{\mathcal{S}} = \bar{\mathcal{S}}$ on $\bar{R}$ (both $\mathcal{S}$ and $\bar{\mathcal{S}}$ are kinematically admissible). Thus, by Thm. 7.1,

$$\Phi\{\bar{\mathcal{S}}\} \geq \Phi\{\mathcal{S}\}, \quad \Phi\{\mathcal{S}\} \geq \Phi\{\bar{\mathcal{S}}\},$$

whence $\Phi\{\mathcal{S}\} = \Phi\{\bar{\mathcal{S}}\}$ and thus $\underline{E} = \underline{\bar{E}}$, $\underline{S} = \underline{\bar{S}}$ on $\bar{R}$.

Def. (Statically admissible stress field). Let R be a regular region and $\partial_2 R$ be a subset of ∂R. We say that $\underline{S}$ is a statically admissible stress field on $\bar{R}$, corresponding to the ~~elastic compliance tensor field~~

the body-force density $\underset{\sim}{f}$, and the surface tractions $\overset{*}{\underset{\sim}{s}}$ on $\partial_2 R$ and write

$$\underset{\sim}{\sigma} \in \mathcal{A}(\underset{\sim}{f}, \overset{*}{\underset{\sim}{s}}; \bar{R}, \partial_2 R)$$

provided

(a) $\underset{\sim}{\sigma} \in \mathcal{C}^1(R) \cap \mathcal{C}(\bar{R})$ (b) (6.3′) holds on R, $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$ on R (c) $\underset{\sim}{\sigma}\underset{\sim}{n} = \overset{*}{\underset{\sim}{s}}$ on ∂_2

Thm. 7.2 (Principle of minimum complementary potential energy, Castigliano). Let $R, \partial_1 R, \partial_2 R$ be as in Thm. 7.1 and suppose hypotheses (a), (b), (c) of Thm. 7.1 hold true. Let $\mathcal{A} \equiv \mathcal{A}(\underset{\sim}{f}, \overset{*}{\underset{\sim}{s}}; \bar{R}, \partial_2 R)$ and let Ψ be the functional defined on $\mathcal{A}$ by means of

$$\Psi\{\hat{\underset{\sim}{\sigma}}\} = U_{\underset{\sim}{\varkappa}}\{\hat{\underset{\sim}{\sigma}}\} - \int_{\partial_1 R} \hat{\underset{\sim}{s}} \cdot \overset{*}{\underset{\sim}{u}} \, dA \;\; \forall\, \hat{\underset{\sim}{\sigma}} \in \mathcal{A}, \quad U_{\underset{\sim}{\varkappa}}(\hat{\underset{\sim}{\sigma}}) = \frac{1}{2} \int_R (\underset{\sim}{\varkappa} \hat{\underset{\sim}{\sigma}}) \cdot \hat{\underset{\sim}{\sigma}} \, dV \quad (7.2$$

where $\underset{\sim}{\varkappa}$ is the elastic compliance field corresponding to $\underset{\sim}{c}$ and $\hat{\underset{\sim}{s}} = \hat{\underset{\sim}{\sigma}}\underset{\sim}{n}$ on ∂R. Then,

$$\Psi\{\underset{\sim}{\sigma}\} = \min_{\hat{\underset{\sim}{\sigma}} \in \mathcal{A}} \Psi\{\hat{\underset{\sim}{\sigma}}\}$$

and this minimum is assumed by $\Psi\{\hat{\underset{\sim}{\sigma}}\}$ only if $\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}$ on $\bar{R}$.

$\Psi\{\underset{\sim}{\sigma}\}$ ... "complementary potential energy" of $\hat{\underset{\sim}{\sigma}}$

Exercise 21. Prove Thm. 7.2

Remark. A rough statement of Thm. 7.2 asserts: Among all statically admissible stress fields associated with a given Problem BIII, the actual stress field is uniquely characterized by the fact that it renders the complementary potential energy a minimum.

Remarks. Mention extension to unbounded regions [Gurtin & Co., A.R.M.A., 8 (1961)], extension to mixed-mixed prob. [Knowles & Co., A.R.M.A. 21 (1966)]. Thm. 6 (uniqueness) is also an immediate corollary of Thm. 7.2

The duality of the two foregoing minimum principles is apparent from the table below.

Thm.	The admissible fields satisfy: Field equats.	Bdy. Conds.	Minimum condition
7.1	(6.1), (6.2)	on $\partial_1 R$	$\Phi = \Phi_{min}$
7.2	(6.3′)	on $\partial_2 R$	$\Psi = \Psi_{min}$

Remarks on variational principles

(1) Mention special circumstances in which the functional Φ and Ψ reduce to U. Extension to mixed-mixed pb.

(2) Sketch use of Thm. 7.2 in connection with direct variational methods.

(3) Mention variational principles associated with Thms. 7.1, 7.2 which assert stationarity of appropriate functional at the actual solution of Prob. BIII. These no longer require $\underset{\sim}{C}$ pos. def.

(4) Mention generalization of classical variational principles of elastostatics [Hellinger (1915), Reissner (1950), Hu (1955), Washizu (1955)]. Describe.

(5) Comment on variational principles in elastodynamics. Hamilton's principle, Gurtin's variational work.

Refer to Gurtin (Enc. of Phys.)