9. Orthogonal curvilinear coordinates

Motivation. As became clear in the preceding chapter, the *field eqs*. of the classical theory of elasticity may be simplified considerably through appropriate changes in the *dependent variables*, i.e. by means of suitable displacement potentials (stress functions). This simplifica is accompanied by an increase in the complexity of the *boundary conditions*. To simplify the latter for a regi with curved boundaries, it is essential to subject the *independent variables*, i.e. the cartesian coordinates, to a convenient transformation by introducing suita curvilinear coordinates with a view toward embeddin the boundary among the coord. surfaces.

The approach we adopt is a compromise between a *general* tensorial treatment and an ad-hoc treatment of special orthogonal curvilinear coordinate systems.

Example: Circular cylindrical coordinates

Coordinate transformation:

$$\left.\begin{aligned} &x_1 = \rho\cos\phi\ ,\ x_2 = \rho\sin\phi\ ,\ x_3 = z \\ &0 \le \rho < \infty\ ,\ 0 \le \phi < 2\pi\ ,\ -\infty < z < \infty \end{aligned}\right\}\ (a)$$

This mapping is one-to-one, except at $\rho = 0$. Indeed the Jacobian determinant associated with (a) is given b

$$\Delta(\rho,\phi,z) = \det\begin{bmatrix} \frac{\partial x_1}{\partial \rho} & \frac{\partial x_1}{\partial \phi} & \frac{\partial x_1}{\partial z} \\ \vdots & \vdots & \vdots \end{bmatrix} = \rho \ge 0 \quad (b)$$

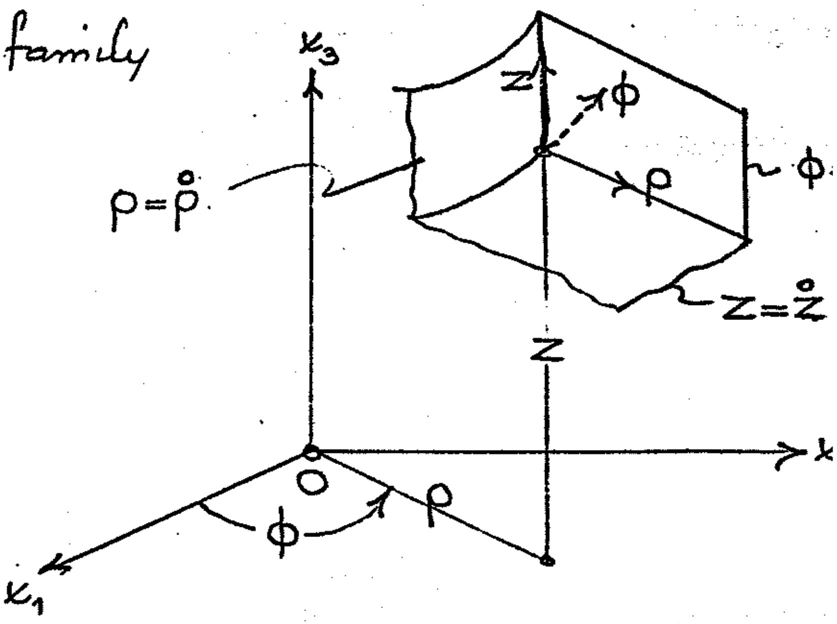
Inverse mapping:

$$\rho = \sqrt{x_1^2 + x_2^2}\ ,\ \cos\phi = \frac{x_1}{\rho}\ ,\ \sin\phi = \frac{x_2}{\rho}\ ,\ z = x_3\ (c)$$

Coordinate surfaces: $\rho = \text{const}$, $\phi = \text{const.}$, $z = \text{const.}$

Describe triply orthog. family

Define coord. lines.



General orthogonal curvilinear coordinates

$$\mathbb{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}, \quad x_i = \underset{\sim}{x} \cdot \underset{\sim}{e}_i$$

Coordinate transformation

$$\mathcal{M} : x_i = x_i(\hat{x}_1, \hat{x}_2, \hat{x}_3) \; \forall \; (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3 = \hat{\mathcal{R}} \quad ($$

Here $\mathcal{L}_i$ are linear intervals (not necessarily closed or open). We assume: $\mathcal{M}$ has the range E; $\mathcal{M}$ is twice continuously differentiable on $\hat{\mathcal{R}}$; $\mathcal{M}$ is one-to-one except possibly at a finite number of singular points in $\hat{\mathcal{R}}$; $\mathcal{M}^{-1}$ is continuously differentiable at all non-singular points of $\hat{\mathcal{R}}$.

Inverse transformation

$$\mathcal{M}^{-1} : \hat{x}_i = \hat{x}_i(x_1, x_2, x_3) \; \forall \text{ regular point } \underset{\sim}{x} \in E \quad (9$$

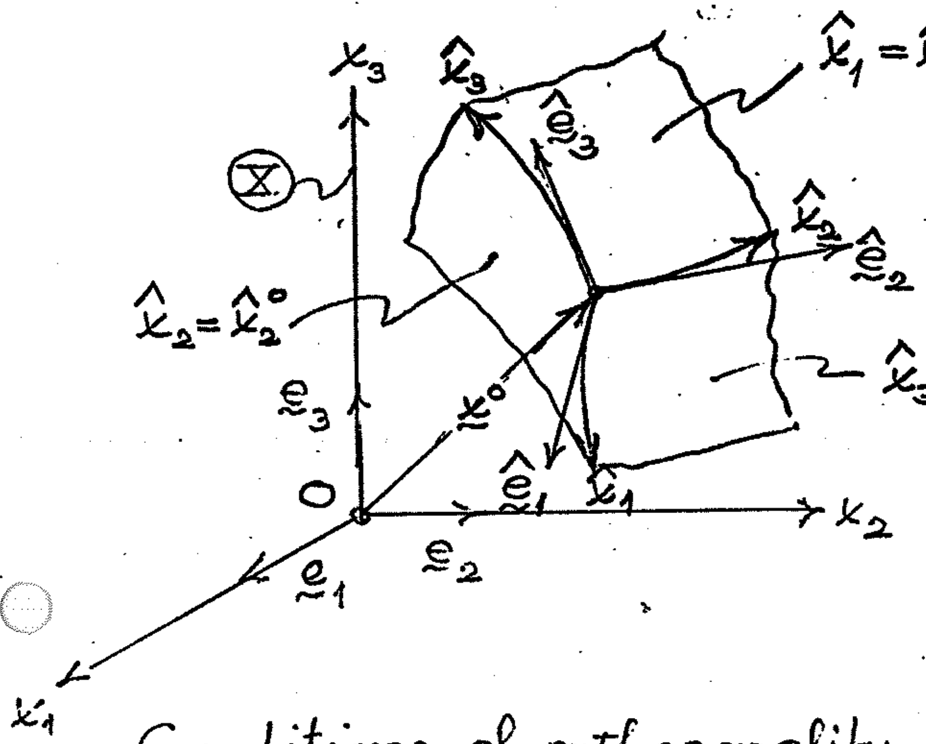
Jacobian determinant of $\mathcal{M}$

$$\Delta(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \det[\partial x_i / \partial \hat{x}_j]$$

$$= \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} \frac{\partial x_i}{\partial \hat{x}_p} \frac{\partial x_j}{\partial \hat{x}_q} \frac{\partial x_k}{\partial \hat{x}_r} \neq 0 \; \forall \text{ regular } \hat{\underset{\sim}{x}} \in \hat{\mathcal{R}} \quad (9.$$

<u>Coordinate surfaces</u>: $\hat{x}_i(x_1,x_2,x_3) = \hat{x}_i^0 = \text{const}$.

<u>Coordinate lines</u>: pairwise intersections of coordinate ~~lines~~ surfaces



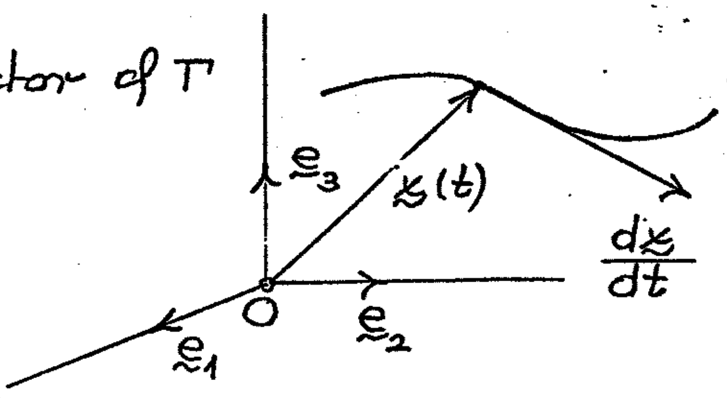
$$\underset{\sim}{x}^0 = \underset{\sim}{x}(\hat{x}_1^0, \hat{x}_2^0, \hat{x}_3^0)$$

<u>Conditions of orthogonality, scale moduli</u>

Consider a regular curve $\Gamma \subset E$.

$$(9.4)\left\{\begin{array}{l} \Gamma : \underset{\sim}{x} = \underset{\sim}{x}(t) \quad (\alpha \le t \le \beta)\ ,\ \underset{\sim}{x} \in \mathcal{C}^1([\alpha,\beta]) \\ \dfrac{d\underset{\sim}{x}}{dt} = \underset{\sim}{e}_i \dfrac{dx_i}{dt} \ \ldots \text{ tang. vector of } \Gamma \\ \left|\dfrac{d\underset{\sim}{x}}{dt}\right| = \sqrt{\dfrac{dx_i}{dt}\dfrac{dx_i}{dt}} = \left|\dfrac{ds}{dt}\right| \end{array}\right.$$

$s(t)$ ... arc-length of Γ



In particular, let Γ be an $\hat{x}_1$-coord. line, whence $\Gamma : \underset{\sim}{x} = \underset{\sim}{x}(\hat{x}_1, \hat{x}_2^{\circ}, \hat{x}_3^{\circ})$ $\forall\, \hat{x}_1 \in \mathcal{L}_1$. Then $\partial \underset{\sim}{x} / \partial \hat{x}$ is a tangent vector of Γ. Thus,

$$\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} \ \ldots \ \text{tangent vector of } \hat{x}_i\text{-lines}$$

Accordingly, the curvil. coord. system defined by (9.1) is <u>orthogonal</u> if and only if

$$\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} \cdot \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_j} = 0 \ (i \neq j) \ \text{or} \ \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial x_k}{\partial \hat{x}_j} = 0 \ (i \neq j) \quad (9.$$

Return to general regular curve Γ (See (9.4)).

$$\left(\frac{d\underset{\sim}{x}}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} \frac{d\hat{x}_i}{dt} \cdot \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_j} \frac{d\hat{x}_j}{dt} \ , \ \text{so that}$$

$$\left(\frac{ds}{dt}\right)^2 = g_{ij} \frac{d\hat{x}_i}{dt} \frac{d\hat{x}_j}{dt} \ , \ g_{ij} = \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} \cdot \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_j} = \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial x_k}{\partial \hat{x}_j} \quad (9.6$$

Define

$$h_i = \sqrt{g_{i\underline{i}}} = \left| \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} \right| = \sqrt{\left(\frac{\partial x_1}{\partial \hat{x}_i}\right)^2 + \left(\frac{\partial x_2}{\partial \hat{x}_i}\right)^2 + \left(\frac{\partial x_3}{\partial \hat{x}_i}\right)^2} > 0 \quad (9.$$

<u>Note that $h_i = 0$ is incompatible with $\Delta \neq 0$.</u>

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* Suspend summation w.r.t. a repeated index by underlining.

(9.5), (9.6), (9.7) $\Rightarrow$

$$g_{ij}=0\ (i\neq j)\,,\quad g_{\underline{ii}}=h_i^2\,,\quad g_{ij}=\delta_{ij}h_{\underline{i}}h_{\underline{j}} \qquad (9.8)$$

(9.6), (9.7), (9.8) $\Rightarrow$

$$(ds)^2=\hat{g}_{ij}\,d\hat{x}_i\,d\hat{x}_j=(h_1 d\hat{x}_1)^2+(h_2 d\hat{x}_2)^2+(h_3 d\hat{x}_3)^2 \qquad (9.9$$

(9.9) reveals the geom. signif. of the scale moduli h_i:

$$h_i=\frac{ds}{d\hat{x}_i}\ \text{along}\ \hat{x}_i\text{-lines}$$

<u>Caution</u>: some authors (e.g. Love) define h_i as $1/\sqrt{g_{\underline{ii}}}$.

<u>Example</u>: Circular cyl. coords. Here one finds

$$h_1=1\,,\ h_2=\rho\,,\ h_3=1\,,$$

as should be the case.

Local base, direction cosines of curvil. coord. line

Let $\underset{\sim}{\hat{e}}_i$ be the unit tang. vector of $\hat{x}_i$-lines in the direction of increasing $\hat{x}_i$.

Then from signif. of $\partial \underset{\sim}{x}/\partial \hat{x}_i$ and (9.7), (9.5),

$$\hat{\underset{\sim}{e}}_i = \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} \Big/ \left|\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i}\right| = \frac{1}{h_{\underline{i}}}\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i}, \quad \hat{\underset{\sim}{e}}_i \cdot \hat{\underset{\sim}{e}}_j = \delta_{ij}. \qquad (9.10)$$

$$A_{ij} = \cos(\hat{\underset{\sim}{e}}_i, \underset{\sim}{e}_j) = \hat{\underset{\sim}{e}}_i \cdot \underset{\sim}{e}_j = \frac{1}{h_{\underline{i}}}\frac{\partial x_j}{\partial \hat{x}_i}. \qquad (9.11)$$

Jacobian determinant of $\mathcal{M}$ in terms of scale m

By (9.3), (9.11),

$$\Delta = \det[\partial x_i/\partial \hat{x}_j] = \det[h_{\underline{j}} A_{ji}] = h_1 h_2 h_3 \det[A_{ij}]$$

; $\det[A_{ij}] = (\hat{\underset{\sim}{e}}_1 \wedge \hat{\underset{\sim}{e}}_2)\cdot\hat{\underset{\sim}{e}}_3 = 1$ if $(\hat{e}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)$ are a right-handed triad, as we assume from here on. Accordingly,

$$\Delta = h_1 h_2 h_3 > 0 \text{ at all regular pts.} \qquad (9.$$

Inverse partial derivatives

Consider the identity:

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* See picture on p.51.

$$x_i = x_i(\hat{x}_1(\underset{\sim}{x}), \hat{x}_2(\underset{\sim}{x}), \hat{x}_3(\underset{\sim}{x}))$$

Thus from chain-rule,

$$\frac{\partial x_i}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_j} = \delta_{ij}$$

"Multiply" by $\partial x_i/\partial \hat{x}_m$ and use (9.5), (9.8) ∴S

$$\frac{\partial x_j}{\partial \hat{x}_m} = \underbrace{\frac{\partial x_i}{\partial \hat{x}_k}\frac{\partial x_i}{\partial \hat{x}_m}}_{g_{km}}\frac{\partial \hat{x}_k}{\partial x_j} = g_{km}\frac{\partial \hat{x}_k}{\partial x_j} = h^2_{\underline{m}}\frac{\partial \hat{x}_m}{\partial x_j},$$

whence

$$\frac{\partial \hat{x}_i}{\partial x_j} = \frac{1}{h^2_{\underline{i}}}\frac{\partial x_j}{\partial \hat{x}_i} \qquad (9.13)$$

Now (9.13), (9.11) ⇒

$$A_{ij} = \hat{\underset{\sim}{e}}_i \cdot \underset{\sim}{e}_j = h_{\underline{i}}\frac{\partial \hat{x}_i}{\partial x_j} \qquad (9.14)$$

Also, (9.7), (9.13) ⇒

$$h_i^2 = \frac{\partial x_k}{\partial \hat{x}_{\underline{i}}}\frac{\partial x_k}{\partial \hat{x}_i} = h^4_{\underline{i}}\frac{\partial \hat{x}_i}{\partial x_k}\frac{\partial \hat{x}_i}{\partial x_k}, \text{ from which}$$

$$\frac{1}{h_i^2} = \frac{\partial \hat{x}_{\underline{i}}}{\partial x_k}\frac{\partial \hat{x}_i}{\partial x_k}, \quad h_i = 1\Big/\sqrt{\left(\frac{\partial \hat{x}_i}{\partial x_1}\right)^2 + \left(\frac{\partial \hat{x}_i}{\partial x_2}\right)^2 + \left(\frac{\partial \hat{x}_i}{\partial x_3}\right)^2} \qquad (9.15)$$

Differentiation of a scalar field with respect to x_i,

Let $R \in E$ be a domain of regularity of $\mathcal{M}$ and $f \in \mathcal{C}^1(R)$ be a scalar field. Let

$$\bar{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = f(\underset{\sim}{x}(\hat{x}_1, \hat{x}_2, \hat{x}_3))$$

Then, from chain-rule and (9.14),

$$\frac{\partial f}{\partial x_i} = \frac{\partial \bar{f}}{\partial \hat{x}_k} \underbrace{\boxed{\frac{\partial \hat{x}_k}{\partial x_i}}}_{\frac{1}{h_{\underline{k}}} A_{ki}}, \text{ from which follows}$$

$$\frac{\partial f}{\partial x_i} = \sum_k \frac{1}{h_k} A_{ki} \frac{\partial \bar{f}}{\partial \hat{x}_k}. \tag{9.16}$$

Components of $\partial \hat{\underset{\sim}{e}}_i / \partial \hat{x}_j$ relative to the local ba

(9.10) $\Rightarrow$

$$\frac{\partial \hat{\underset{\sim}{e}}_i}{\partial \hat{x}_j} \cdot \hat{\underset{\sim}{e}}_k = \left\{ -\frac{1}{h_{\underline{i}}^2} \frac{\partial h_i}{\partial \hat{x}_j} \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i} + \frac{1}{h_{\underline{i}}} \frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_i \partial \hat{x}_j} \right\} \cdot \frac{1}{h_{\underline{k}}} \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_k}$$

or, in view of (9.6), (9.8),

$$\frac{\partial \hat{\underset{\sim}{e}}_i}{\partial \hat{x}_j} \cdot \hat{\underset{\sim}{e}}_k \overset{\checkmark}{=} -\frac{1}{h_{\underline{i}}} \frac{\partial h_i}{\partial \hat{x}_j} \delta_{ik} + \frac{1}{h_{\underline{i}} h_{\underline{k}}} \frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_i \partial \hat{x}_j} \cdot \frac{\partial \underset{\sim}{x}}{\partial \hat{x}_k} \tag{§}$$

Recall from (9.6), (9.8) that

$$\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_j} = h_{\underline{i}}\, h_{\underline{j}}\, \delta_{ij}$$

Differentiate this identity with respect to $\hat{x}_k$ to reac

$$\underset{\checkmark}{\frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_i \partial \hat{x}_k}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_j}} + \underset{\checkmark\checkmark}{\frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_j \partial \hat{x}_k}\cdot\frac{\partial \hat{\underset{\sim}{x}}}{\partial \hat{x}_i}} = \frac{\partial}{\partial \hat{x}_k}(h_{\underline{i}} h_{\underline{j}})\,\delta_{ij}\,, \quad (a)$$

whence

$$\underline{\frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_j \partial \hat{x}_i}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_k}} + \underset{\checkmark}{\frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_k \partial \hat{x}_i}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_j}} = \frac{\partial}{\partial \hat{x}_i}(h_{\underline{j}} h_{\underline{k}})\,\delta_{jk} \quad (b)$$

$$\underset{\checkmark\checkmark}{\frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_k \partial \hat{x}_j}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_i}} + \underline{\frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_i \partial \hat{x}_j}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_k}} = \frac{\partial}{\partial \hat{x}_j}(h_{\underline{k}} h_{\underline{i}})\,\delta_{ki} \quad (c)$$

Form $\frac{1}{2}\{(b)+(c)-(a)\}$ to conclude that

$$\checkmark\ \frac{\partial^2 \underset{\sim}{x}}{\partial \hat{x}_i \partial \hat{x}_j}\cdot\frac{\partial \underset{\sim}{x}}{\partial \hat{x}_k} = \frac{1}{2}\Big\{\delta_{\underline{jk}}\frac{\partial}{\partial \hat{x}_i}(h_j h_k) + \delta_{\underline{ki}}\frac{\partial}{\partial \hat{x}_j}(h_k h_i) - \delta_{\underline{ij}}\frac{\partial}{\partial \hat{x}_k}(h_i h_j)\Big\} \quad (\S$$

Substitution from (§§) into (§) yields

$$\frac{\partial \hat{\underset{\sim}{e}}_i}{\partial \hat{x}_j}\cdot \hat{\underset{\sim}{e}}_k \overset{\checkmark}{=} -\frac{1}{h_{\underline{i}}}\frac{\partial h_i}{\partial \hat{x}_j}\,\delta_{ik} + \frac{1}{2h_{\underline{i}}h_{\underline{k}}}\Big\{\delta_{\underline{j}k}\frac{\partial}{\partial \hat{x}_i}(h_j h_k) + \delta_{ki}\frac{\partial}{\partial \hat{x}_j}(h_k h_i) - \delta_{\underline{ij}}\frac{\partial}{\partial \hat{x}_k}(h_i h_j)\Big\}$$

(9.1

(9.17) $\Rightarrow$

$$\boxed{\begin{array}{l} \dfrac{\partial \underset{\sim}{\hat{e}}_i}{\partial \hat{x}_j} \cdot \underset{\sim}{\hat{e}}_k \overset{\checkmark}{=} 0 \quad \text{if } (i,j,k) \text{ distinct or } k=i \\ \dfrac{\partial \underset{\sim}{\hat{e}}_i}{\partial \hat{x}_{\underline{i}}} \cdot \underset{\sim}{\hat{e}}_k \overset{\checkmark}{=} -\dfrac{1}{h_{\underline{k}}}\dfrac{\partial h_i}{\partial \hat{x}_k}, \quad \dfrac{\partial \underset{\sim}{\hat{e}}_i}{\partial \hat{x}_{\underline{k}}} \cdot \underset{\sim}{\hat{e}}_k \overset{\checkmark}{=} \dfrac{1}{h_{\underline{i}}}\dfrac{\partial h_k}{\partial \hat{x}_i} \quad \text{if } k \neq i \end{array}} \quad (9$$

Note that (9.18) exhausts all possibilities as to the values of (i,j,k). See matrix table on p.5

Transformation of basic tensor field quantitie

If $\underset{\sim}{T}$ is a cartesian tensor of order N, the curvilinear components $\hat{T}_{ij\ldots n}$ of $\underset{\sim}{T}$ in an orthog. curvil. coordinate system are the components of $\underset{\sim}{T}$ in the local base $(\underset{\sim}{\hat{e}}_1, \underset{\sim}{\hat{e}}_2, \underset{\sim}{\hat{e}}_3$

Thus,

$$\hat{T}_{ij\ldots n} = A_{ip} A_{jq} \ldots A_{nr} T_{pq\ldots r}, \qquad (9.19$$

where $A_{ip} = \underset{\sim}{\hat{e}}_i \cdot \underset{\sim}{e}_p$ (direction cosines)

$$\boxed{\beta_{jk}^{(i)} = \frac{\partial \hat{\underset{\sim}{e}}_i}{\partial \hat{x}_j} \cdot \hat{\underset{\sim}{e}}_k}$$

$$[\beta_{jk}^{(1)}] = \begin{bmatrix} 0 & -\frac{1}{h_2}\frac{\partial h_1}{\partial \hat{x}_2} & -\frac{1}{h_3}\frac{\partial h_1}{\partial \hat{x}_3} \\ 0 & \frac{1}{h_1}\frac{\partial h_2}{\partial \hat{x}_1} & 0 \\ 0 & 0 & \frac{1}{h_1}\frac{\partial h_3}{\partial \hat{x}_1} \end{bmatrix}$$

$$[\beta_{jk}^{(2)}] = \begin{bmatrix} \frac{1}{h_2}\frac{\partial h_1}{\partial \hat{x}_2} & 0 & 0 \\ -\frac{1}{h_1}\frac{\partial h_2}{\partial \hat{x}_1} & 0 & -\frac{1}{h_3}\frac{\partial h_2}{\partial \hat{x}_3} \\ 0 & 0 & \frac{1}{h_2}\frac{\partial h_3}{\partial \hat{x}_2} \end{bmatrix}$$

$$[\beta_{jk}^{(3)}] = \begin{bmatrix} \frac{1}{h_3}\frac{\partial h_1}{\partial \hat{x}_3} & 0 & 0 \\ 0 & \frac{1}{h_3}\frac{\partial h_2}{\partial \hat{x}_3} & 0 \\ -\frac{1}{h_1}\frac{\partial h_3}{\partial \hat{x}_1} & -\frac{1}{h_2}\frac{\partial h_3}{\partial \hat{x}_2} & 0 \end{bmatrix}$$

(A) Gradient of a scalar field

$$\underset{\sim}{v} = \nabla f = \underset{\sim}{e}_i \frac{\partial f}{\partial x_i}$$

Set $\bar{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = f(\underset{\sim}{x}(\hat{x}_1, \hat{x}_2, \hat{x}_3))$. From above, (9.19), (9.11) one has

$$\hat{v}_k = A_{ki} v_i = A_{ki} \frac{\partial f}{\partial x_i} = \frac{1}{h_{\underline{k}}} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \hat{x}_k} \quad , \text{ whence}$$

$$\boxed{\hat{v}_k = \frac{1}{h_{\underline{k}}} \frac{\partial \bar{f}}{\partial \hat{x}_k} \quad , \quad \underset{\sim}{v} = \sum_k \hat{\underset{\sim}{e}}_k \frac{1}{h_{\underline{k}}} \frac{\partial \bar{f}}{\partial \hat{x}_k}} \quad (9.20)$$

(B) Gradient of a vector field

$$\underset{\sim}{W} = \nabla \underset{\sim}{v} \quad \text{or} \quad W_{ij} = \frac{\partial v_i}{\partial x_j}$$

$$\hat{W}_{ij} = A_{ip} A_{jq} \frac{\partial}{\partial x_q} \underbrace{(A_{np} \hat{v}_n)}_{v_p} \quad \text{or , because of (9.16),}$$

$$\hat{W}_{ij} = A_{ip} A_{jq} \sum_m \frac{1}{h_m} A_{mq} \frac{\partial}{\partial \hat{x}_m} (A_{np} \hat{v}_n) \; , \; A_{jq} A_{mq} =$$

$$\hat{W}_{ij} = A_{ip} \frac{1}{h_{\underline{j}}} \frac{\partial}{\partial \hat{x}_j} (A_{np} \hat{v}_n)$$

$$\hat{W}_{ij} = \frac{1}{h_{\underline{j}}} \left\{ \frac{\partial \hat{v}_i}{\partial \hat{x}_j} + A_{ip} \frac{\partial A_{np}}{\partial \hat{x}_j} \hat{v}_n \right\}$$

$$A_{ip}\frac{\partial A_{np}}{\partial \hat{x}_j}\hat{v}_n = A_{ip}\frac{\partial}{\partial \hat{x}_j}(\hat{\underset{\sim}{e}}_n \cdot \underset{\sim}{e}_p)\hat{v}_n$$

$$= A_{ip}\underset{\sim}{e}_p \cdot \frac{\partial \hat{\underset{\sim}{e}}_n}{\partial \hat{x}_j}\hat{v}_n = \hat{\underset{\sim}{e}}_i \cdot \frac{\partial \hat{v}_n^{\,\hat{e}_n}}{\partial \hat{x}_j}\hat{v}_n \text{ , so that}$$

✓ $$\boxed{\hat{W}_{ij} = \frac{1}{h_{\underline{j}}}\left\{\frac{\partial \hat{v}_i}{\partial \hat{x}_j} + \hat{\underset{\sim}{e}}_i \cdot \frac{\partial \hat{\underset{\sim}{e}}_n}{\partial \hat{x}_j}\hat{v}_n\right\};} \quad (9.21)$$

in which the coefficient of $\hat{v}_n$ is available in intrinsic form from (9.17).

(C) Divergence of a vector field

$$\nabla\cdot\underset{\sim}{v} = \mathrm{tr}\,\nabla\underset{\sim}{v} = v_{i,i}$$

Hence from (B) and (9.18),

$$\nabla\cdot\underset{\sim}{v} = \mathrm{tr}\,\underset{\sim}{W} = \hat{W}_{ii} = \sum_i \frac{1}{h_i}\left\{\frac{\partial \hat{v}_i}{\partial \hat{x}_i} + \sum_{n\neq i}\frac{1}{h_n}\frac{\partial h_i}{\partial \hat{x}_n}\hat{v}_n\right\}$$

Collect terms involving $\hat{v}_1$, etc. Thus,

$$\nabla\cdot\underset{\sim}{v} = \frac{1}{h_1}\frac{\partial \hat{v}_1}{\partial \hat{x}_1} + \frac{\hat{v}_1}{h_2 h_1}\frac{\partial h_2}{\partial \hat{x}_1} + \frac{\hat{v}_1}{h_3 h_1}\frac{\partial h_3}{\partial \hat{x}_1} + \dots + \dots$$

✓ $$\boxed{\nabla\cdot\underset{\sim}{v} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial \hat{x}_1}(h_2 h_3 \hat{v}_1) + \frac{\partial}{\partial \hat{x}_2}(h_3 h_1 \hat{v}_2) + \frac{\partial}{\partial \hat{x}_3}(h_1 h_2 \hat{v}_3)\right\}}$$

(9.22)

(D) Laplacian of a scalar field

$$\nabla^2 f = \nabla \cdot \nabla f = f_{,kk}$$

Set $f(\underset{\sim}{x}) = f(\underset{\sim}{x}(\hat{x}_1, \hat{x}_2, \hat{x}_3)) = \bar{f}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$

and use (A), (C) above to infer that

$$\boxed{\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial \hat{x}_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \bar{f}}{\partial \hat{x}_1}\right) + \frac{\partial}{\partial \hat{x}_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \bar{f}}{\partial \hat{x}_2}\right) + \frac{\partial}{\partial \hat{x}_3}\left(\frac{h_1 h_2}{h_3}\partial\right.\right.}$$

(9.2

(E) Curl of a vector field

$$\underset{\sim}{W} = \nabla \wedge \underset{\sim}{V} \quad \text{or} \quad w_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

So,

$$w_i = \varepsilon_{ijk} W_{kj}\ , \quad W_{kj} = \frac{\partial v_k}{\partial x_j}\ , \quad \hat{w}_i = \varepsilon_{ijk}\hat{W}_{kj}$$

Therefore from (B) above one has

$$\hat{w}_i = \sum_j \frac{1}{h_j}\varepsilon_{ijk}\left\{\frac{\partial \hat{v}_k}{\partial \hat{x}_j} + \underset{\sim}{\hat{e}}_k \cdot \frac{\partial \underset{\sim}{\hat{e}}_n}{\partial \hat{x}_j}\hat{v}_n\right\}.$$

But (9.18) $\Rightarrow$ second term sums out to zero unless $n = j$. Hence using (9.18) one has

$$\hat{w}_i = \sum_{j,k}\frac{1}{h_j}\varepsilon_{ijk}\left\{\frac{\partial \hat{v}_k}{\partial \hat{x}_j} - \frac{1}{h_k}\frac{\partial h_j}{\partial \hat{x}_k}\hat{v}_j\right\}$$

This yields easily

$$\boxed{\hat{w}_1 = \frac{1}{h_2h_3}\left\{\frac{\partial}{\partial \hat{x}_2}(h_3\hat{v}_3) - \frac{\partial}{\partial \hat{x}_3}(h_2\hat{v}_2)\right\}, \dots, \dots} \quad (9.24)$$

(F) Divergence of a symm. two-tensor field

$$\underset{\sim}{v} = \nabla \cdot \underset{\approx}{S}, \quad \underset{\approx}{S}^T = \underset{\approx}{S} \quad \text{or} \quad v_i = \frac{\partial S_{ij}}{\partial x_j}, \quad S_{ji} = S_{ij}$$

Using $\hat{v}_i = A_{ip} v_p$, $S_{ij} = A_{mp} A_{nj} \hat{S}_{mn}$, and invoking (9.16), the orthogonality of $[A_{ij}]$, as well as (9.18), and $\hat{S}_{ji} = \hat{S}_{ij}$, one arrives at

$$\hat{v}_1 = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial \hat{x}_1}(h_2h_3\hat{S}_{11}) + \frac{\partial}{\partial \hat{x}_2}(h_3h_1\hat{S}_{12}) + \frac{\partial}{\partial \hat{x}_3}(h_1h_2\hat{S}_{1}\right.$$

$$+ \frac{1}{h_1h_2}\frac{\partial h_1}{\partial \hat{x}_2}\hat{S}_{12} + \frac{1}{h_1h_3}\frac{\partial h_1}{\partial \hat{x}_3}\hat{S}_{13} - \frac{1}{h_1h_2}\frac{\partial h_2}{\partial \hat{x}_1}\hat{S}_{22} - \frac{1}{h_1h_3}\frac{\partial h_3}{\partial \hat{x}_1}\hat{S}$$

$$\dots, \dots$$

(9.25

Remark. Eqs. (9.20) – (9.25) reduce to their Cartesian analogues if $\mathcal{M}$ is the identity tran

Exercise. Deduce (9.25)

Transformation of the fundamental field eqs. of linear elasticity theory

$$\hat{u}_i = A_{ip} u_p \,, \quad \hat{\gamma}_{ij} = A_{ip} A_{jq} \gamma_{pq} \,, \quad \hat{\sigma}_{ij} = A_{ip} A_{jq} \sigma_{pq}$$

(a) Displacement-strain relations

$$\underset{\sim}{\gamma} = \operatorname{sym} \nabla \underset{\sim}{u} \quad \text{or} \quad \gamma_{pq} = \frac{1}{2}\left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\right)$$

From (B) and the formulas for $\frac{\partial \hat{\underset{\sim}{e}}_i}{\partial \hat{x}_j} \cdot \hat{\underset{\sim}{e}}_k$ one finds after elementary simplifications:

$$\boxed{\begin{aligned} \hat{\gamma}_{11} &= \frac{1}{h_1}\frac{\partial \hat{u}_1}{\partial \hat{x}_1} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \hat{x}_2}\hat{u}_2 + \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial \hat{x}_3}\hat{u}_3 \,, \cdots, \cdots \\ \hat{\gamma}_{12} = \hat{\gamma}_{21} &= \frac{1}{2}\left\{ \frac{h_1}{h_2}\frac{\partial}{\partial \hat{x}_2}\left(\frac{\hat{u}_1}{h_1}\right) + \frac{h_2}{h_1}\frac{\partial}{\partial \hat{x}_1}\left(\frac{\hat{u}_2}{h_2}\right)\right\}, \cdots, \cdots \end{aligned}}$$

(9.

(b) Displacement-dilation & displacement-rotati relations

$$\vartheta = \nabla \cdot \underset{\sim}{u} = \gamma_{kk} \,, \quad \underset{\sim}{w} = \frac{1}{2} \nabla \wedge \underset{\sim}{u}$$

Hence from (C) and (E):

$$\boxed{\begin{aligned}&\vartheta = \hat{\gamma}_{kk} = \\ &\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\hat{x}_1}(h_2h_3\hat{u}_1)+\frac{\partial}{\partial\hat{x}_2}(h_3h_1\hat{u}_2)+\frac{\partial}{\partial\hat{x}_3}(h_1h_2\hat{u}_3)\right\}\end{aligned}}$$

$$\boxed{\hat{w}_1 = \frac{1}{2h_2h_3}\left\{\frac{\partial}{\partial\hat{x}_2}(h_3\hat{u}_3)-\frac{\partial}{\partial\hat{x}_3}(h_2\hat{u}_2)\right\}, \dots, \dots} \quad (9.2$$

(c) Stress-strain relations for isotropic elastic soli

$$\sigma_{ij} = 2\mu\left(\frac{\nu}{1-2\nu}\delta_{ij}\gamma_{kk}+\gamma_{ij}\right),$$

$$\gamma_{ij} = \frac{1}{\eta}[(1+\nu)\sigma_{ij}-\delta_{ij}\nu\sigma_{kk}]$$

$$\boxed{\begin{aligned}\hat{\sigma}_{ij} &= 2\mu\left(\frac{\nu}{1-2\nu}\delta_{ij}\hat{\gamma}_{kk}+\hat{\gamma}_{ij}\right),\\ \hat{\gamma}_{ij} &= \frac{1}{\eta}[(1+\nu)\hat{\sigma}_{ij}-\delta_{ij}\nu\hat{\sigma}_{kk}]\end{aligned}} \quad (9.29)$$

(d) Stress equations of motion and equilibrium

$$\nabla\cdot\underset{\sim}{\sigma}+\underline{f}=\rho\ddot{\underset{\sim}{u}} \quad \text{or} \quad \sigma_{ij,j}+f_i=\rho\frac{\partial^2u_i}{\partial t^2}$$

Hence from (F):

$$\boxed{\begin{aligned}
&\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\hat{x}_1}(h_2h_3\hat{\sigma}_{11})+\frac{\partial}{\partial\hat{x}_2}(h_3h_1\hat{\sigma}_{12})+\frac{\partial}{\partial\hat{x}_3}(h_1h_2\hat{\sigma}_{13})\right\}\\
&+\frac{1}{h_1h_2}\frac{\partial h_1}{\partial\hat{x}_2}\hat{\sigma}_{12}+\frac{1}{h_1h_3}\frac{\partial h_1}{\partial\hat{x}_3}\hat{\sigma}_{13}-\frac{1}{h_1h_2}\frac{\partial h_2}{\partial\hat{x}_1}\hat{\sigma}_{22}-\frac{1}{h_1h_3}\frac{\partial h_3}{\partial\hat{x}_1}\hat{\sigma}_{3}\\
&+\hat{f}_1=\rho\frac{\partial^2\hat{u}_1}{\partial t^2}\quad,\;\cdots\;,\;\cdots
\end{aligned}}$$

Transformation of the Popkovich-Neuber sol. into orthogonal curvil. coordinates

Recall the PN-solution:

$$\underset{\sim}{u} = \frac{1}{2\mu}\{\nabla(\varphi + \underset{\sim}{x}\cdot\underset{\sim}{\psi}) - 4(1-\nu)\underset{\sim}{\psi}\} \text{ on } R \quad (a)$$

$$\nabla^2\varphi = -\frac{\underset{\sim}{x}\cdot\underset{\sim}{f}}{2(1-\nu)}\ , \quad \nabla^2\underset{\sim}{\psi} = \frac{\underset{\sim}{f}}{2(1-\nu)} \text{ on } R \quad (b)$$

Note that the $\hat{\psi}_i$ do <u>not</u> satisfy Poisson equations. For this reason we adhere to the Cartesian components of $\underset{\sim}{\psi}$ and set

$$\varphi(\underset{\sim}{x}) = \overline{\varphi}(\hat{x}_1, \hat{x}_2, \hat{x}_3), \quad \psi_i(\underset{\sim}{x}) = \overline{\Psi}_i(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

Our objective is to express $\hat{u}_i, \hat{\sigma}_{ij}$ in terms of $\overline{\varphi}, \overline{\Psi}_i$ and intrinsic partial derivatives of the four scalar potentials. Recalling (A), as well as

$$\hat{\psi}_i = A_{ij}\overline{\Psi}_j = \frac{1}{h_{\underline{i}}}\frac{\partial x_j}{\partial \hat{x}_i}\overline{\Psi}_j\ ,$$

one infers from (a) that

$$\hat{u}_i = \frac{1}{2\mu h_{\underline{i}}}\left\{\frac{\partial}{\partial \hat{x}_i}(\overline{\varphi}+x_j\overline{\Psi}_j) - 4(1-\nu)\overline{\Psi}_j\frac{\partial x_j}{\partial \hat{x}_i}\right\} \quad \text{or}$$

$$✓ \quad \boxed{\hat{u}_i = \frac{1}{2\mu h_{\underline{i}}}\left\{\frac{\partial\overline{\varphi}}{\partial\hat{x}_i} + x_j\frac{\partial\overline{\Psi}_j}{\partial\hat{x}_i} - (3-4\nu)\overline{\Psi}_j\frac{\partial x_j}{\partial \hat{x}_i}\right\}} \quad (1)$$

Also, on setting $f_i(\underset{\sim}{x}) = \overline{f}_i(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ and invoking (D), one draws from (b) that

$$\nabla^2\overline{\varphi} = -\frac{x_j\overline{f}_j}{2(1-\nu)}, \quad \nabla^2\overline{\Psi}_i = \frac{\overline{f}_i}{2(1-\nu)}$$

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\hat{x}_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial\hat{x}_1}\right) + \frac{\partial}{\partial\hat{x}_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial\hat{x}_2}\right) + \frac{\partial}{\partial\hat{x}_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial\hat{x}_3}\right)\right.$$

Recall that (a), (b) imply

$$\vartheta = I_1(\underset{\sim}{\gamma}) = \gamma_{kk} = \nabla\cdot\underset{\sim}{u} = -\frac{1-2\nu}{\mu}\Psi_{k,k}, \quad \text{whence}$$

$$\gamma_{kk} = \hat{\gamma}_{kk} = -\frac{1-2\nu}{\mu}\frac{\partial\overline{\Psi}_k}{\partial\hat{x}_j}\frac{\partial\hat{x}_j}{\partial x_k}.$$

Remembering that

$$\frac{\partial\hat{x}_j}{\partial x_k} = \frac{1}{h_{\underline{j}}^2}\frac{\partial x_k}{\partial\hat{x}_j},$$

one thus arrives at

$$\boxed{\vartheta = \hat{\partial}_{kk} = -\frac{1-2\nu}{\mu} \sum_{j} \frac{1}{h_j^2} \frac{\partial x_k}{\partial \hat{x}_j} \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_j}} \qquad (3)$$

Now compute $\hat{\partial}_{ij}$ from (1) and the curvilinear version of the displacement-strain relations. Subsequently use the resulting formulas in conjunction with (3) and the stress-strain relations

$$\hat{\sigma}_{ij} = 2\mu \left(\frac{\nu}{1-2\nu} \delta_{ij} \hat{\partial}_{kk} + \hat{\partial}_{ij} \right)$$

to reach the desired representation of $\hat{\sigma}_{ij}$. In this manner one finds after simplifications

$$\begin{aligned}
\hat{\sigma}_{11} = {} & \frac{1}{h_1^2} \left\{ \frac{\partial^2 \overline{\varphi}}{\partial \hat{x}_1^2} - \frac{1}{h_1} \frac{\partial h_1}{\partial \hat{x}_1} \frac{\partial \overline{\varphi}}{\partial \hat{x}_1} + \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial \hat{x}_2} \frac{\partial \overline{\varphi}}{\partial \hat{x}_2} + \frac{h_1}{h_3^2} \frac{\partial h_1}{\partial \hat{x}_3} \frac{\partial \overline{\varphi}}{\partial \hat{x}_3} \right\} \\
& + \frac{1}{h_1^2} \left\{ x_k \frac{\partial^2 \overline{\Psi}_k}{\partial \hat{x}_1^2} - \left(\frac{1}{h_1} \frac{\partial h_1}{\partial \hat{x}_1} x_k + 2 \frac{\partial x_k}{\partial \hat{x}_1} \right) \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_1} \right\} \\
& + \frac{1}{h_1} \left\{ \frac{1}{h_2^2} \frac{\partial h_1}{\partial \hat{x}_2} x_k \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_2} + \frac{1}{h_3^2} \frac{\partial h_1}{\partial \hat{x}_3} x_k \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_3} \right\} \\
& + 2\nu \left\{ \frac{1}{h_1^2} \frac{\partial x_k}{\partial \hat{x}_1} \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_1} - \frac{1}{h_2^2} \frac{\partial x_k}{\partial \hat{x}_2} \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_2} - \frac{1}{h_3^2} \frac{\partial x_k}{\partial \hat{x}_3} \frac{\partial \overline{\Psi}_k}{\partial \hat{x}_3} \right\}, \ldots, \ldots
\end{aligned}$$

$$
\begin{aligned}
\sigma_{12} = \frac{1}{h_1 h_2}\left\{\frac{\partial^2\overline{\varphi}}{\partial\hat{x}_1\partial\hat{x}_2} - \frac{1}{h_1}\frac{\partial h_1}{\partial\hat{x}_2}\frac{\partial\overline{\varphi}}{\partial\hat{x}_1} - \frac{1}{h_2}\frac{\partial h_2}{\partial\hat{x}_1}\frac{\partial\overline{\varphi}}{\partial\hat{x}_2}\right\} \\
+ \frac{1}{h_1 h_2}\left\{x_k\frac{\partial^2\overline{\psi}_k}{\partial\hat{x}_1\partial\hat{x}_2} - \frac{1}{h_2}\frac{\partial h_2}{\partial\hat{x}_1}x_k\frac{\partial\overline{\psi}_k}{\partial\hat{x}_2} - \frac{1}{h_1}\frac{\partial h_1}{\partial\hat{x}_2}x_k\frac{\partial\overline{\psi}_k}{\partial\hat{x}_1}\right. \\
\left. - (1-2\nu)\left(\frac{\partial x_k}{\partial\hat{x}_2}\frac{\partial\overline{\psi}_k}{\partial\hat{x}_1} + \frac{\partial x_k}{\partial\hat{x}_1}\frac{\partial\overline{\psi}_k}{\partial\hat{x}_2}\right)\right\}, \cdots, \cdots
\end{aligned}
$$