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On Compatibility Conditions for the Left Cauchy–Green Deformation Field in Three Dimensions

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Abstract. A fairly general sufficient condition for compatibility of the left Cauchy–Green deformation field in three dimensions has been derived. A related necessary condition is also indicated. The kinematical problem is phrased as a suitable problem in Riemannian geometry, whence the method of solution emerges naturally. The main result of the paper is general in scope and provides conditions for the existence of solutions to certain types of overdetermined systems of first-order, quasilinear partial differential equations with algebraic constraints.

Key words: compatibility, left Cauchy–Green deformation, three dimensions or (3-D).

1. Introduction

The aim of this paper is to discuss the issue of necessary and sufficient conditions for the existence of a local deformation of a simply-connected reference configuration whose left Cauchy–Green deformation field matches a prescribed symmetric, positive-definite tensor field on the same reference. In this paper, the term deformation refers to a map $\mathbf{f}$ with an invertible deformation gradient ($\det \mathbf{F} \neq 0$; $\mathbf{F} = D\mathbf{f}$), and the left Cauchy–Green deformation tensor is generically represented by the symbol $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$. To the best of the author's knowledge, this problem remains open in three dimensions, having been solved for the plane case by Blume [1]. Related results, for the plane case, are also presented in Duda and Martins [2]. A fairly general sufficient condition for compatibility in three dimensions is provided in this paper.

The problem dealt with in this paper is a fundamental issue in continuum kinematics, apart from being of mathematical interest. As pointed out by Blume [1], the problem of deriving sufficient conditions for the existence of a deformation compatible with a given left Cauchy–Green field is fairly difficult, especially when compared with the same question for the right Cauchy–Green field ($\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$) or the even simpler question of 'deformation-gradient' compatibility. While in the latter cases the compatibility conditions come out neatly as the vanishing of the

Riemann–Christoffel curvature tensor and the 'curl', respectively, of the prescribed fields, the conditions in the present case are not as explicit.

As in the case of the proof of compatibility for the right Cauchy–Green deformation field, we phrase the compatibility question for the left Cauchy–Green deformation field as a problem in Riemannian geometry. The problem is posed as a special case of the determination of conditions for which two pre-assigned positive-definite, symmetric matrix fields may be considered as components of a Riemannian metric on a manifold. The speciality of the problem lies in the fact that the metric is known to be Euclidean (its components in a *given* parametrization are, uniformly, the identity matrix) – the question is to construct another coordinate system (parametrization) on which the other pre-assigned matrix field (rectangular Cartesian components of the **B** tensor field in question) can be considered as contravariant components of the same metric. It should be carefully noted that such a question is quite different in its details than the one asked in the case of the proof of right Cauchy–Green compatibility – a symmetric positive-definite matrix field on a *given* parametrization is available (interpreted as the covariant components of a metric tensor), and another coordinate system has to be determined such that on that system the covariant metric components are, uniformly, the components of the identity matrix.

The answer to the geometric version of the **B**-compatibility problem rests on being able to determine necessary and sufficient conditions for the existence of a solution to an overdetermined system of algebraic and partial differential equations which, typically, is not completely integrable. Such systems have been considered in the differential geometry literature in connection with the determination of various kinds of invariants of metric and affinely connected manifolds (Veblen and Thomas [9]; Thomas and Michal [8]; Schouten [5]) and, apparently, were first considered by Christoffel (Eisenhart [3, footnotes p. 17 and p. 77]) whose concern seems to be very closely related to the geometric version of the issue of **B**-compatibility, as phrased in this paper.

In Section 1, we examine the issue of **B**-compatibility and phrase the question in terms of suitably defined coordinates and components. The coordinate-components form of the problem motivates the geometric formulation, which we discuss in Section 2. This approach differs significantly from the one in Blume [1], which is based on the left polar decomposition of the deformation gradient. In Section 3 a solution to the problem posed in Section 2 is provided with a sketch of the proof of the main result.

With respect to notation used in the paper, we use standard notational conventions of the tensor calculus when dealing with coordinates and components. The symbol D represents a derivative operator, and the same symbol with a subscript will represent a partial derivative on a suitably defined product space. The symbol ∇ will represent the gradient of a scalar field. All tensorial quantities from the first order onwards will be represented in boldface and scalar quantities will be represented in lower case italics. In representing a list as function arguments, we shall often use just the kernel letter to represent the entire list, e.g., x to represent the n-tuple $(x^1, x^2, \dots, x^n)$. Also, associated tensors will be freely used in the following.

2. The problem of B-Compatibility

The compatibility problem for the Left Cauchy–Green deformation field may be stated as follows:

Given a pre-assigned positive-definite symmetric second order tensor field $\mathbf{B}$ on a reference configuration R, find necessary and sufficient conditions for the existence of a regular deformation $\mathbf{y}$ of the reference such that

$$D\mathbf{y}(\mathbf{x})\big(D\mathbf{y}(\mathbf{x})\big)^{\mathrm{T}} = \mathbf{B}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathsf{R}. \tag{1}$$

Consider a rectangular Cartesian parametrization of $\mathbb{E}_3$, the ambient three dimensional Euclidean point space. Let the reference configuration R be represented by coordinates $\{x^\alpha\}$ in this coordinate system. It is now easy to see that the aforementioned $\mathbf{B}$-compatibility question is equivalent to the following problem:

Given the rectangular Cartesian components of the tensor $\mathbf{B}$ (B^{km}) as functions of $\{x^\alpha\}$, find necessary and sufficient conditions on the field (B^{km}) for the existence of functions y^i of $\{x^\alpha\}$ satisfying

$$\frac{\partial y^i}{\partial x^\rho}\frac{\partial y^j}{\partial x^\rho} = B^{ij}. \tag{2}$$

The equivalence of the two questions follows from the equivalence of the existence assertions defined by Equations (1) and (2). In fact, if functions y^i exist such that (2) holds, then, choosing the region of $\mathbb{E}_3$, say C, that corresponds to $\{y^i(x)\}$ for all $\{x^\alpha\}$ corresponding to R (where the $\{y^i\}$ are coordinates in the same rectangular Cartesian parametrization of $\mathbb{E}_3$ chosen to define $\{x^\alpha\}$), as a deformed image of R, we see that we have defined a regular deformation $\mathbf{y}$: $\mathsf{R} \to \mathsf{C}$ through the association, represented symbolically as

$$\mathbf{x}(x) \to \mathbf{y}\big(y(x)\big) \quad \text{for all } \mathbf{x} \in \mathsf{R}$$

with

$$\mathbf{d}^i \cdot D\mathbf{y}(\mathbf{x})\mathbf{e}_\rho = \frac{\partial y^i}{\partial x^\rho},$$

where

$$\mathbf{d}^i = \nabla y^i(\mathbf{y}); \qquad \mathbf{e}_\rho = D_\rho \mathbf{x}(x) \quad [1].$$

Since we are working with a rectangular Cartesian parametrization, $\mathbf{d}^i = \mathbf{d}_i = \mathbf{e}_i = \mathbf{e}^i$ and, hence,

$$B^{ij} = \frac{\partial y^i}{\partial x^\rho}\frac{\partial y^j}{\partial x^\rho} = \mathbf{e}^i \cdot \big[D\mathbf{y}(D\mathbf{y})^{\mathrm{T}}\big]\mathbf{e}^j \Rightarrow D\mathbf{y}(D\mathbf{y})^{\mathrm{T}} = \mathbf{B}.$$

The converse assertion, of course, is even simpler.

3. Geometric Formulation

The issue of existence posed in (2) suggests the following question in Riemannian geometry:

Given a real, positive-definite symmetric matrix field B^{ij} on a coordinate patch $\{x^\alpha\}$ of some manifold, what are the necessary and sufficient conditions for the existence of another coordinate patch $\{y^i\}$ such that, if B^{ij} are viewed as the components of a metric tensor on $\{y^i\}$, then the corresponding components on the $\{x^\alpha\}$ system are, uniformly, the components of the identity matrix. That is, we seek to find conditions for the existence of the coordinate patch $\{y^i\}$ such that $\delta^{\alpha\beta}$ and B^{ij} may be viewed as the contravariant components of a metric tensor on the coordinate patches $\{x^\alpha\}$ and $\{y^i\}$, respectively. The corresponding covariant components are represented by the symbols $\delta_{\alpha\beta}$ and B_{ij}.

We also note that the aforementioned geometric problem is equivalent to a local version of the compatibility problem for the left Cauchy–Green deformation field as represented by the existence problem (1).

In order to solve the geometric problem, we first consider necessity. Suppose there exists $\{y^i\}$ such that (2) holds. This implies that the mapping $\{x^\alpha\} \to \{y^i\}$ is locally invertible which further implies that $\partial B_{rp}/\partial y^s$ is well defined and given by

$$\frac{\partial B_{rp}}{\partial y^s} = \frac{\partial B_{rp}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^s}. \tag{3}$$

Now, if $\{z^i\}$ is some coordinate patch, then the Christoffel symbols of the second kind with respect to $\{z^i\}$ and their transformation rules are given by

$${}_{(z)}\Gamma^i_{rs} = \frac{B^{ip}}{2}\left[\frac{\partial B_{rp}}{\partial z^s} + \frac{\partial B_{sp}}{\partial z^r} - \frac{\partial B_{rs}}{\partial z^p}\right] \tag{4}$$

and

$$\frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = {}_{(x)}\Gamma^\rho_{\alpha\beta}\frac{\partial y^i}{\partial x^\rho} - {}_{(y)}\Gamma^i_{rs}\frac{\partial y^r}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta}, \tag{5}$$

respectively (Sokolnikoff [6]). Since the metric components are constant on $\{x^i\}$, ${}_{(x)}\Gamma^\rho_{\alpha\beta} = 0$, and hence, (3)–(5) imply

$$\frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = -\frac{B^{im}}{2}\left[\frac{\partial B_{rm}}{\partial x^\beta}\frac{\partial y^r}{\partial x^\alpha} + \frac{\partial B_{sm}}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta} - \frac{\partial B_{rs}}{\partial x^\rho}\frac{\partial x^\rho}{\partial y^m}\frac{\partial y^r}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta}\right].$$

Hence, our hypothesis implies that there exists a solution to the system of equations

$$u^i_\rho u^j_\rho = B^{ij}, \tag{6}$$

$$\frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha, \tag{7}$$

$$\frac{\partial u^i_\alpha}{\partial x^\beta} = -\frac{B^{im}}{2}\left[\frac{\partial B_{rm}}{\partial x^\beta}u^r_\alpha + \frac{\partial B_{sm}}{\partial x^\alpha}u^s_\beta - \frac{\partial B_{rs}}{\partial x^\rho}f^\rho_m(u)u^r_\alpha u^s_\beta\right], \tag{8}$$

where f^{ρ}_{m} represents the matrix inverse function. Conversely, suppose the fields B^{ij} are such that there exists a solution to (6)–(8). Then, it is clear from (6) and (7) that there exist functions $\{y^i\}$, defined on the coordinate patch $\{x^\alpha\}$, that satisfy (2). Hence, our problem reduces to finding necessary and sufficient conditions for the existence of solutions to the system (6)–(8).

We also note here that if an invertible matrix-valued solution to (8) exists, then a solution to (7) necessarily exists because of the symmetry, in α and β, of the right-hand side of (8) (Thomas [7]), and the solution is locally invertible as a function of $\{x^\alpha\}$.

In the next section we provide a sketch of the proof of the result that provides a sufficient condition for the existence of solutions to (6) and (8) and, consequently, (7).

4. Algebraic Conditions for the Existence of Solutions

Consider the system of differential equations

$$\frac{\partial w^i}{\partial x^\alpha}(x) = \psi^i_\alpha\big(w(x), x\big), \quad i = 1, 2, \ldots, R; \alpha = 1, 2, \ldots, n, \tag{9}$$

in which the w^i are functions of independent variables $\{x^\alpha\}$ and ψ^i_α are differentiable as many times as required. The domain of ψ^i_α, also referred to as (z, x) space in the following, is assumed to be an open, connected set of $\mathfrak{R}^R \times \mathfrak{R}^n$. For application to the main problem of the paper, we identify w^i with the functions u^k_μ, $k, \mu = 1, 2, 3$, so that $R = 9$ and $n = 3$. Also, we identify (9) with system (8). We seek solutions

$$w^i(x) \tag{10}$$

of (9) which satisfy a system of equations

$$F^{(0)}\big(w(x), x\big) = 0, \tag{11}$$

where the domain of $F^{(0)}$ is the (z, x) space. For the compatibility problem at hand, we identify $F^{(0)}(z, x) = 0$ as the nine equations appearing in (6), i.e., $u^i_\alpha u^j_\alpha - B^{ij}(x) = 0$, of which only six are independent.

Define $F^{(1)}(z, x) = 0$ to be the set of equations consisting of the equations of integrability of (9), i.e., the equations obtained by formally differentiating (9) with respect to x^α (assuming w^i to be functions of $\{x\}$) and then eliminating the second derivative of w^i using the relations

$$\frac{\partial w^i}{\partial x^\alpha \partial x^\beta} = \frac{\partial w^i}{\partial x^\beta \partial x^\alpha}$$

and eliminating the first derivative of w^i using (9).

In searching for solutions to (9), the equations in the set $F^{(1)}(z, x) = 0$ are either identically satisfied, or not. We first consider the former case.

If the equations in the set $F^{(1)} = 0$ are identically satisfied in the (z, x) space, then it is well known that there exists a unique solution to (9) (Thomas [7]) with aribitrarily assignable initial data at $\{x_0\}$. Let the initial data for $w^i(x_0)$ be chosen such that the conditions $u^i_\alpha(x_0)u^j_\alpha(x_0) = B^{ij}(x_0)$ are satisfied. We now show that such a local solution of (9), which corresponds to a solution $\{u^i_\alpha\}$ to (8), is sufficient for the existence of a solution to (6), (7).

First we note that since the solution to (8) is an invertible matrix at $\{x_0\}$, it is invertible in a neighborhood around $\{x_0\}$ due to the continuity of the solution. Also, as pointed out in the previous section, a solution to (7) necessarily exists, and the solution is locally invertible as a function of $\{x^\alpha\}$.

We now assume that such a local solution to (7) has been constructed. With the solutions of (8) and (7) in hand, we shall be able to prove that $\{u^i_\alpha\}$ that satisfy (8) also satisfy (6) by using a general property of the Riemannian geometry – that of preservation of angles between vector fields under parallel transport.

We denote the solution of (7) based on the solution $\{u^i_\alpha\}$ of (8) as $\{y^i\}$. Define

$$ {}_{(y)}\Gamma^i_{rs} := \frac{B^{ip}}{2}\left[\frac{\partial B_{rp}}{\partial y^s} + \frac{\partial B_{sp}}{\partial y^r} - \frac{\partial B_{rs}}{\partial y^p}\right]. $$

Clearly, the following relations hold

$$ \frac{\partial u^i_\alpha}{\partial x^\beta} = -{}_{(y)}\Gamma^i_{rs} u^r_\alpha u^s_\beta. $$

We now consider the expression

$$ \frac{\partial}{\partial y^m}\left(B^{ij} u^\alpha_i u^\beta_j\right), $$

where u^α_i represent the components of the inverse of u^i_α. If it can be shown that the above expression is identically zero in a local neighborhood of $\{y(x_0)\}$, then the chain rule and the choice of the initial data imply that (6) is indeed satisfied locally around $\{x_0\}$. Now,

$$ B^{ij}\frac{\partial u^\alpha_i}{\partial y^m} u^\beta_j = -B^{ij} u^\alpha_k \frac{\partial u^k_\gamma}{\partial x^\rho} u^\rho_m u^\gamma_i u^\beta_j = B^{kj}\, {}_{(y)}\Gamma^i_{km} u^\alpha_i u^\beta_j, $$

where $u^\alpha_i u^i_\beta = \delta^\alpha_\beta$ and the fact that u^i_α satisfy (8) have been used. Consequently,

$$ \frac{\partial}{\partial y^m}\left(B^{ij} u^\alpha_i u^\beta_j\right) = \left[\frac{\partial B^{ij}}{\partial y^m} + B^{kj}\, {}_{(y)}\Gamma^i_{km} + B^{ik}\, {}_{(y)}\Gamma^j_{km}\right] u^\alpha_i u^\beta_j = 0, $$

which follows from Ricci's theorem (Sokolnikoff [6, p. 77 and p. 86]) – the term in the square parenthesis vanishes (merely by the smoothness and definition of the ${}_{(y)}\Gamma$ field) since it is the covariant derivative of the contravariant metric tensor and the fundamental tensors are "covariantly constant".

Next we consider the case where the equations of the set $F^{(1)}(z,x) = 0$ are *not* identically satisfied in the (z,x) space. In such a case, we define the equations $\overline{F}^{(1)}(z,x) = 0$ as a system of equations consisting of $F^{(1)}(z,x) = 0$ and $F^{(0)}(z,x) = 0$. We also define $\overline{F}^{(j+1)}(z,x) = 0$ $(j \geqslant 1)$ to be the set of equations obtained by formally differentiating $\overline{F}^{(j)}(z,x) = 0$ with respect to x^α and eliminating the derivative of z^i by means of (9).

We now consider two integers N and M, with $N \geqslant 1$ and $1 \leqslant M \leqslant R$. We assume that

(1) *there exist M equations in the sets $\overline{F}^{(1)} = 0$ through $\overline{F}^{(N)} = 0$ denoted by $\widetilde{G}_\lambda = 0$, $\lambda = 1$ to M, and M of the variables z^i (from the list z^i, $i = 1, 2, \ldots, R$) denoted by $\bar{z}^i$, $i = 1, 2, \ldots, M$ which satisfy*

$$\det\left[\frac{\partial \widetilde{G}_\lambda}{\partial \bar{z}^i}(z,x)\right] \neq 0, \quad i, \lambda = 1, \ldots, M, \tag{12}$$

in the (z,x) space.

Let the remaining $R - M =: P$ variables z^i, obtained by ignoring $\bar{z}^i$, $i = 1, 2, \ldots, M$, from the list z^i, $i = 1, 2, \ldots, R$, be denoted by $\hat{z}^j$, $j = 1, 2, \ldots, P$. We also assume that, given a point $(\hat{z}_0^1, \ldots, \hat{z}_0^P, x_0^1, \ldots, x_0^n)$

(2) *the unique local solution $\bar{z}^i = \varphi^i(\hat{z}^1, \ldots, \hat{z}^P, x)$, $i = 1, \ldots, M$, of $\widetilde{G}_\lambda = 0$, $\lambda = 1, \ldots, M$, (which exists because of (12)) satisfies all the equations of the sets $\overline{F}^{(1)} = 0$ through $\overline{F}^{(N+1)} = 0$ identically in a local neighborhood of $(\hat{z}_0^1, \ldots, \hat{z}_0^P, x_0^1, \ldots, x_0^n)$.*

Under these assumptions, we show that there exists a local solution to (9) and (11) of the form (10) that is determined by P constants.

We denote the functions ψ_α^i, $i = 1, \ldots, M$ by $\widetilde{\overline{\psi}}{}_\alpha^i$. Similarly, we denote the functions ψ_α^j, $j = 1, \ldots, P$, by $\widetilde{\widehat{\psi}}{}_\alpha^j$. For each of the functions $\widetilde{G}_\lambda$, $\widetilde{\overline{\psi}}{}_\alpha^i$, and $\widetilde{\widehat{\psi}}{}_\alpha^j$ of (z,x) we define the corresponding functions of $(\bar{z}, \hat{z}, x)$ by the rules given below. Let Π be a function that delivers the list (z,x) corresponding to the list $(\bar{z}, \hat{z}, x)$. We now define

$$\begin{aligned}
G_\lambda(\bar{z}, \hat{z}, x) &= \widetilde{G}_\lambda(\Pi(\bar{z}, \hat{z}, x)), \quad \lambda = 1, \ldots, M, \\
\overline{\psi}{}_\alpha^i(\bar{z}, \hat{z}, x) &= \widetilde{\overline{\psi}}{}_\alpha^i(\Pi(\bar{z}, \hat{z}, x)), \quad i = 1, \ldots, M, \alpha = 1, \ldots, n, \\
\widehat{\psi}{}_\alpha^j(\bar{z}, \hat{z}, x) &= \widetilde{\widehat{\psi}}{}_\alpha^j(\Pi(\bar{z}, \hat{z}, x)), \quad j = 1, \ldots, P, \alpha = 1, \ldots, n.
\end{aligned}$$

Since the equations of the set $\overline{F}^{(N+1)} = 0$ are satisfied, the relations

$$\begin{aligned}
&\frac{\partial G_\lambda}{\partial \bar{z}^i}(\varphi(\hat{z}, x), \hat{z}, x)\overline{\psi}{}_\alpha^i(\varphi(\hat{z}, x), \hat{z}, x) + \frac{\partial G_\lambda}{\partial \hat{z}^j}(\varphi(\hat{z}, x), \hat{z}, x)\widehat{\psi}{}_\alpha^j(\varphi(\hat{z}, x), \hat{z}, x) \\
&\quad + \frac{\partial G_\lambda}{\partial x^\alpha}(\varphi(\hat{z}, x), \hat{z}, x) = 0, \quad i = 1, \ldots, M, j = 1, \ldots, P
\end{aligned} \tag{13}$$

are satisfied in the variables $(\hat{z}^1, \ldots, \hat{z}^P, x)$. Choose an arbitrary point x_0^α and $\hat{z}_0^j$, $j = 1, \ldots, P$. Let $f^\alpha(t)$ be a path in the $\{x\}$ space with $f^\alpha(0) = x_0^\alpha$. Along this path we seek the solution of the system

$$\begin{aligned} \frac{\mathrm{d}g^j}{\mathrm{d}t}(t) &= \widehat{\Psi}_\alpha^j\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t), \quad j = 1, \ldots, P, \\ g^j(0) &= \hat{z}_0^j. \end{aligned} \tag{14}$$

From the existence theorem of ordinary differential equations, a local solution to such a system exists. With such a solution in hand, we note that

$$G_\lambda\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big) = 0 \tag{15}$$

is identically satisfied in t, and differentiating the above expression yields

$$\begin{aligned} &\frac{\partial G_\lambda}{\partial \bar{z}^i}\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\left\{\frac{\partial \varphi^i}{\partial \hat{z}^j}\big(g(t), f(t)\big)\frac{\mathrm{d}g^j}{\mathrm{d}t}(t)\right. \\ &\quad \left. + \frac{\partial \varphi^i}{\partial x^\alpha}\big(g(t), f(t)\big)\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\right\} + \frac{\partial G_\lambda}{\partial \hat{z}^j}\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\frac{\mathrm{d}g^j}{\mathrm{d}t}(t) \\ &\quad + \frac{\partial G_\lambda}{\partial x^\alpha}\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t) = 0. \end{aligned} \tag{16}$$

Combining (13) and (16) yields

$$\begin{aligned} &\frac{\partial G_\lambda}{\partial \bar{z}^i}\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\left\{\overline{\Psi}_\alpha^i\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\right. \\ &\quad \left. - \frac{\partial \varphi^i}{\partial \hat{z}^j}\big(g(t), f(t)\big)\frac{\mathrm{d}g^j}{\mathrm{d}t}(t) - \frac{\partial \varphi^i}{\partial x^\alpha}\big(g(t), f(t)\big)\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\right\} \\ &\quad + \frac{\partial G_\lambda}{\partial \hat{z}^j}\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big) \\ &\quad \times \left\{\widehat{\Psi}_\alpha^j\big(\varphi\big(g(t), f(t)\big), g(t), f(t)\big)\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t) - \frac{\mathrm{d}g^j}{\mathrm{d}t}(t)\right\} = 0 \end{aligned}$$

and, noting the definition of g^j and the fact that the matrix $(\partial G_\lambda/\partial \bar{z}^i)(z, x)$ is invertible at all (z, x), we find that

$$\begin{aligned} &\left\{\overline{\Psi}_\alpha^i\big(\varphi\big(g(0), f(0)\big), g(0), f(0)\big) - \frac{\partial \varphi^i}{\partial \hat{z}^j}\big(g(0), f(0)\big)\right. \\ &\quad \left. \times \widehat{\Psi}_\alpha^j\big(\varphi\big(g(0), f(0)\big), g(0), f(0)\big) - \frac{\partial \varphi^i}{\partial x^\alpha}\big(g(0), f(0)\big)\right\}\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(0) = 0 \end{aligned} \tag{17}$$

holds. Since $\{\hat{z}_0^j, x_0^\alpha\}$ and the path were chosen arbitrarily, (17) implies that

$$\overline{\Psi}_\alpha^i\big(\varphi(\hat{z}, x), \hat{z}, x\big) - \frac{\partial \varphi^i}{\partial \hat{z}^j}(\hat{z}, x)\widehat{\Psi}_\alpha^j\big(\varphi(\hat{z}, x), \hat{z}, x\big) - \frac{\partial \varphi^i}{\partial x^\alpha}(\hat{z}, x) = 0 \tag{18}$$

is satisfied identically in the variables $(\hat{z}^1, \dots, \hat{z}^P, x)$.

Now choose, arbitrarily, two distinct points x_0^α and x^α. Join these points by a path f^α, parametrized by the variable t. Along any such path we seek the solution of system (14) of ordinary differential equations with arbitrarily chosen initial conditions at x_0^α, where the existence of a local solution is guaranteed.

According to the result of Thomas [7], g^j defined by (14) are path-independent and satisfy

$$\frac{\partial g^j}{\partial x^\alpha}(x) = \widehat{\psi}_\alpha^j\big(\varphi\big(g(x), x\big)g(x), x\big) \tag{20}$$

if the conditions

$$\begin{aligned}
&\frac{\partial \widehat{\psi}_\alpha^k}{\partial \hat{z}^i}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\bigg(\frac{\partial \varphi^i}{\partial \hat{z}^j}(\hat{z}, x)\widehat{\psi}_\beta^j\big(\varphi(\hat{z}, x), \hat{z}, x\big) + \frac{\partial \varphi^i}{\partial x^\beta}(\hat{z}, x)\bigg) \\
&\quad + \frac{\partial \widehat{\psi}_\alpha^k}{\partial x^\beta}\big(\varphi(\hat{z}, x), \hat{z}, x\big) + \frac{\partial \widehat{\psi}_\alpha^k}{\partial \hat{z}^j}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\widehat{\psi}_\beta^j\big(\varphi(\hat{z}, x), \hat{z}, x\big) \\
&= \frac{\partial \widehat{\psi}_\beta^k}{\partial \hat{z}^i}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\bigg(\frac{\partial \varphi^i}{\partial \hat{z}^j}(\hat{z}, x)\widehat{\psi}_\alpha^j\big(\varphi(\hat{z}, x), \hat{z}, x\big) + \frac{\partial \varphi^i}{\partial x^\alpha}(\hat{z}, x)\bigg) \\
&\quad + \frac{\partial \widehat{\psi}_\beta^k}{\partial x^\alpha}\big(\varphi(\hat{z}, x), \hat{z}, x\big) + \frac{\partial \widehat{\psi}_\beta^k}{\partial \hat{z}^j}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\widehat{\psi}_\alpha^j\big(\varphi(\hat{z}, x), \hat{z}, x\big),
\end{aligned} \tag{21}$$

$k = 1, \dots, P$, are satisfied identically in the variables $(\hat{z}^1, \dots, \hat{z}^P, x)$. Since the equations of the set $F^{(1)} = 0$ are identically satisfied, we find that

$$\begin{aligned}
&\frac{\partial \widehat{\psi}_\alpha^k}{\partial \hat{z}^i}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\overline{\psi}_\beta^i\big(\varphi(\hat{z}, x), \hat{z}, x\big) + \frac{\partial \widehat{\psi}_\alpha^k}{\partial x^\beta}\big(\varphi(\hat{z}, x), \hat{z}, x\big) \\
&\quad + \frac{\partial \widehat{\psi}_\alpha^k}{\partial \hat{z}^j}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\widehat{\psi}_\beta^j\big(\varphi(\hat{z}, x), \hat{z}, x\big) \\
&= \frac{\partial \widehat{\psi}_\beta^k}{\partial \hat{z}^i}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\overline{\psi}_\alpha^i\big(\varphi(\hat{z}, x), \hat{z}, x\big) + \frac{\partial \widehat{\psi}_\beta^k}{\partial x^\alpha}\big(\varphi(\hat{z}, x), \hat{z}, x\big) \\
&\quad + \frac{\partial \widehat{\psi}_\beta^k}{\partial \hat{z}^j}\big(\varphi(\hat{z}, x), \hat{z}, x\big)\widehat{\psi}_\alpha^j\big(\varphi(\hat{z}, x), \hat{z}, x\big),
\end{aligned} \tag{22}$$

$i = 1, \dots, M$; $j, k = 1, \dots, P$, holds. Consequently, (18) and (22) together imply that (21) holds.

We now consider the functions of $\{x\}$ defined by

$$h^i(x) = \varphi^i\big(g(x), x\big), \quad i = 1, \dots, M, \tag{23}$$

and note that the relations

$$G_\lambda\big(h(x), g(x), x\big) = 0, \quad \lambda = 1, \dots, M, \tag{24}$$

are satisfied for all $\{x\}$. Differentiating (24) with respect to x^α, we obtain

$$\frac{\partial G_\lambda}{\partial \bar{z}^i}\big(h(x), g(x), x\big)\frac{\partial h^i}{\partial x^\alpha}(x) + \frac{\partial G_\lambda}{\partial \hat{z}^j}\big(h(x), g(x), x\big)\frac{\partial g^j}{\partial x^\alpha}(x) + \frac{\partial G_\lambda}{\partial x^\alpha}\big(h(x), g(x), x\big) = 0. \tag{25}$$

Now, (13) yields

$$\frac{\partial G_\lambda}{\partial \bar{z}^i}\big(h(x), g(x), x\big)\overline{\psi}^i_\alpha\big(h(x), g(x), x\big) + \frac{\partial G_\lambda}{\partial \hat{z}^j}\big(h(x), g(x), x\big)\widehat{\psi}^j_\alpha\big(h(x), g(x), x\big) + \frac{\partial G_\lambda}{\partial x^\alpha}\big(h(x), g(x), x\big) = 0. \tag{26}$$

Subtracting (26) from (25), and noting (20) and the invertibility of

$$\frac{\partial G_\lambda}{\partial \bar{z}^i}\big(h(x), g(x), x\big),$$

we now find that the set of functions h and g defined by (23) and the solution of (20) indeed satisfy (9). Noting the fact that (11) are some of the equations $F^{(1)} = 0$, it is clear that we have determined a solution to (9) and (11). The general solution has P arbitrary constants that enter as initial conditions in defining the functions $g^j, j = 1, \ldots, P$.

The above proof is an expanded version of the proofs of sufficiency in Eisenhart [3] and Veblen and Thomas [9]. The sufficient condition above, in the main, is claimed as necessary as well, in earlier works [3, 5, 8, 9]; their line of argument could not be followed to produce a proof of necessity of the sufficient condition above, for the existence of solutions. The main issue dealt with in this paper also appears to be related to Cartan's Method of Equivalence (Gardner [4]).

Finally, an immediate necessary condition for the existence of a solution to the differential-algebraic system of Equations (9) and (11) is that there exist functions of the form (10) which satisfy the purely algebraic system of equations

$$\overline{F}^{(1)}\big(w(x), x\big) = 0, \quad \overline{F}^{(2)}\big(w(x), x\big) = 0, \ldots$$

identically in the variables $(x^1, x^2, \ldots, x^n)$.

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