Finite Deformation Algorithms for Elastoplasticity with isotropic Elasticity:

Isotropic Elasticity :

$$\underline{\sigma} = \underline{F}\,\hat{\underline{G}}(\underline{C})\,\underline{F}^T \rightarrow \text{Frame-Indifferent.}$$

Material Symmetry requires that $\underline{F} \rightarrow \underline{F}\underline{Q}$.

$$\underline{F}\underline{Q}\,\hat{\underline{G}}(\underline{Q}^T\underline{C}\underline{Q})\underline{Q}^T\underline{F}^T = \underline{F}\,\hat{\underline{G}}(\underline{C})\underline{F}^T \quad \forall\, \underline{Q} \in \text{orth}^+.$$

$$\underline{R}\,\underline{U}\underline{Q}\,\hat{\underline{G}}(\underline{Q}^T\underline{C}\underline{Q})\underline{Q}^T\underline{U}\underline{R}^T = \underline{R}\,\underline{U}\hat{\underline{G}}(\underline{C})\underline{U}\underline{R}^T \quad \forall\, \underline{Q} \in \text{orth}^+.$$

Since $\underline{R}$, $\underline{U}$ are invertible

$$\Rightarrow \underline{Q}\,\hat{\underline{G}}(\underline{Q}^T\underline{C}\underline{Q})\underline{Q}^T = \hat{\underline{G}}(\underline{C}) \quad \forall\, \underline{Q} \in \text{orth}^+.$$

$\Rightarrow$ $\hat{\underline{G}}$ is an isotropic tensor valued function of a symmetric second order tensor.

$\Rightarrow$ $\hat{\underline{G}}$ admits the representation

$$\hat{\underline{G}}(\underline{C}) = \varphi_0(I_1(\underline{C}), I_2(\underline{C}), I_3(\underline{C}))\,\underline{I} + \varphi_1(\text{---})\,\underline{C} + \varphi_2(\text{---})\,\underline{C}^2 \qquad (\star)$$

i.e. $\exists$ fn $\varphi_0, \varphi_1, \varphi_2$ s.t. $(\star)$ holds.

So.

$$\underline{\sigma} = \underline{F}\{\varphi_0 \underline{I} + \varphi_1 \underline{C} + \varphi_2 \underline{C}^2\}\underline{F}^T.$$

$$\boxed{\underline{\sigma} = \varphi_0 \underline{B} + \varphi_1 \underline{B}^2 + \varphi_2 \underline{B}^3.}$$

In spectral representation

$$\underline{\sigma} = \sum \sigma^A \, \underset{\sim}{n}^A \otimes \underset{\sim}{n}^A$$

$(\underset{\sim}{n}^A) \equiv (\underset{\sim}{p}^A)$
$(\underset{\sim}{N}^A) \equiv (\underset{\sim}{P}^A).$

where

$$\boxed{\sigma^A = \varphi_0 \lambda_A^2 + \varphi_1 \lambda_A^4 + \varphi_2 \lambda_A^6}$$

Typical Material Model:

$$\underline{\sigma} = \underline{F}^e \frac{\partial \psi}{\partial \underline{C}}(\underline{C}^e)\underline{F}^{e^T}.$$

Multiplicative Decomposition:

$\underline{F} = \underline{F}^e \underline{F}^p.$ $\det F^p = 1.$ $\det F^e = \det F$

(Explain Physical Basis).

$$\underline{F}^e \dot{\underline{F}}^p \underline{F}^{p^{-1}} \underline{F}^{e^{-1}} = \dot{\gamma} \frac{\underline{\sigma}}{|\underline{\sigma}|}$$

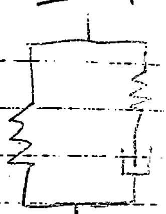
$\ln \lambda_A^e =: \varepsilon_A^e.$

$$\dot{\gamma} = f(\gamma, \underline{\sigma}, \underline{F}^p).$$

$\underline{\tau} = J\underline{\sigma}.$

Candidate for Stress Response of Metals:

$$\tau^A = \kappa(\varepsilon_1^e + \varepsilon_2^e + \varepsilon_3^e) + 2\mu\,\varepsilon_A^e.$$



Hyperelasticity with Creep

Formulation for Network B (Inelastic)

$$\boxed{\underset{\sim}{\sigma} = \sum \sigma^A \, \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A} \longrightarrow \text{(Implication of Isotropy)}$$

$$\underset{\sim}{F}^e \cdot \dot{\underset{\sim}{F}}^p \cdot \underset{\sim}{F}^{p^{-1}} \cdot \underset{\sim}{F}^{e^{-1}} = \dot{\gamma} \frac{\underset{\sim}{\sigma}'}{\sqrt{\underset{\sim}{\sigma}' : \underset{\sim}{\sigma}'}} = \frac{\dot{\gamma}}{\sqrt{2}} \frac{\underset{\sim}{\sigma}'}{\sigma'}$$

where $\sigma' = \sqrt{\frac{1}{2} \underset{\sim}{\sigma}' : \underset{\sim}{\sigma}'}$

$$\boxed{\dot{\gamma} = \hat{C}_1 \left[\sqrt{\frac{I_1^p}{3}} - 1 \right]^{C_2} \sigma'^{\,m}}$$

Now $\underset{\sim}{\sigma}' = \sum \left(\sigma^A - \frac{I_1^\sigma}{3} \right) \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A$.

where $I_1^\sigma = \sum \sigma^A$

$$\therefore \quad \underset{\sim}{F}^e \cdot \dot{\underset{\sim}{F}}^p \cdot \underset{\sim}{F}^{p^{-1}} \cdot \underset{\sim}{F}^{e^{-1}} = \frac{\dot{\gamma}}{\sqrt{2}\,\sigma'} \sum \left(\sigma^A - \frac{I_1^\sigma}{3} \right) \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A$$

Now $\underset{\sim}{F}^e \cdot \underset{\sim}{N}_e^A = \lambda_e^A \, \underset{\sim}{n}_e^A$

Hence $\dot{\underset{\sim}{F}}^p \cdot \underset{\sim}{F}^{p^{-1}} = \dfrac{\dot{\gamma}}{\sqrt{2}\,\sigma'} \sum \left(\sigma^A - \dfrac{I_1^\sigma}{3} \right) \dfrac{1}{(\lambda_e^A)^2} \left(\underset{\sim}{N}_e^A \otimes \underset{\sim}{N}_e^A \right) \cdot \underset{\sim}{C}^e$

$$\boxed{\dot{\underset{\sim}{F}}^p = \left[\frac{\dot{\gamma}}{\sqrt{2}\,\sigma'} \sum \left(\sigma^A - \frac{I_1^\sigma}{3} \right) \left(\underset{\sim}{N}_e^A \otimes \underset{\sim}{N}_e^A \right) \right] \cdot \underset{\sim}{F}^p}$$

$$\underline{F}^p = \left[\sum \frac{\lambda_*^A}{\lambda_e^A} \underset{\sim *}{N}^A \otimes \underset{\sim *}{N}^A\right] \cdot \underline{F}_n^p$$

②

Introduce update formula: (Based on soln of Const. Coeff linear system)

$$\underline{F}^p = \exp\left[\sum_A \frac{\Delta\gamma}{\sqrt{2}\sigma'}\left(\sigma^A - \frac{I_1^\sigma}{3}\right) \underset{\sim e}{N}^A \otimes \underset{\sim e}{N}^A\right] \cdot \underline{F}_n^p.$$

where $\Delta\gamma \equiv \dot{\gamma}\,\Delta t$

Note that if $\underline{F}_0^p = \mathbb{1}$ then $\det(\underline{F}^p) = 1$

since $\det[\exp(\cdot)] = \exp[\mathrm{tr}(\cdot)]$.

Now, $\underline{F}^e = \underline{F} \cdot \underline{F}^{p^{-1}}$

$$= \underline{F} \cdot \underline{F}_n^{p^{-1}} \exp\left[\sum_A -\frac{\Delta\gamma}{\sqrt{2}\sigma'}\left(\sigma^A - \frac{I_1^\sigma}{3}\right) \underset{\sim e}{N}^A \otimes \underset{\sim e}{N}^A\right]$$

Denote $\underline{F} \cdot \underline{F}_n^{p^{-1}} = \underline{F}^* \longrightarrow$ elastic predictor for the defmn gradient.

Now $\underline{F}^e = \underline{R}^e \cdot \underline{U}^e$

$\underline{F}^* = \underline{R}^* \cdot \underline{U}^*$

Then $\underline{R}^e \cdot \underline{U}^e = \underline{R}^* \cdot \underline{U}^* \cdot \exp\left[\sum_A -\frac{\Delta\gamma}{\sqrt{2}\sigma'}\left(\sigma^A - \frac{I_1^\sigma}{3}\right) \underset{\sim e}{N}^A \otimes \underset{\sim e}{N}^A\right]$

$$\Rightarrow \underline{R}^e \cdot \underline{U}^e \cdot \exp\left[\sum \frac{\Delta\gamma}{\sqrt{2}\sigma'}\left(\sigma^A - \frac{I_1^\sigma}{3}\right) \underset{\sim e}{N}^A \otimes \underset{\sim e}{N}^A\right] = \underline{R}^* \cdot \underline{U}^*.$$

Now U^e is coaxial with $\exp[\cdot\ ]$ & both are symmetric.

Consequently their product is symmetric.

Then, by uniqueness of Polar Decomposition.

of $\overset{*}{\underline{F}}$ we have that

$$\underline{U}^{e} \cdot \exp\left[\sum_{A} \frac{\Delta\gamma}{\sqrt{2}\,\sigma'}\left(\sigma^{A} - \frac{I_1^{\sigma}}{3}\right) \underset{\sim}{N}_e^{A} \otimes \underset{\sim}{N}_e^{A}\right] = \overset{*}{\underline{U}}$$

and $\underline{R}^{e} = \overset{*}{\underline{R}}$.

Let $\underline{U}^{*} = \sum_{A} \lambda_*^{A}\, \underset{\sim}{N}_*^{A} \otimes \underset{\sim}{N}_*^{A}$

By the uniqueness of the eigenvalues of a tensor

$$\lambda_*^{A} = \lambda_e^{A} \exp\left[\frac{\Delta\gamma}{\sqrt{2}\,\sigma'}\left(\sigma^{A} - \frac{I_1^{\sigma}}{3}\right)\right] \quad \text{for } A = 1,2,3.$$

and without loss of generality we can assign

$$\underset{\sim}{N}_e^{A} = \underset{\sim}{N}_*^{A} \qquad \text{for } A = 1,2,3.$$

Consequently,

$$\underline{F}^{p} = \exp\left[\sum_{A} \frac{\Delta\gamma}{\sqrt{2}\,\sigma'}\left(\sigma^{A} - \frac{I_1^{\sigma}}{3}\right) \underset{\sim}{N}_*^{A} \otimes \underset{\sim}{N}_*^{A}\right] \cdot \underline{F}_n^{p}.$$

$\therefore$ ☆ and

$$\Delta\gamma = \hat{C}_1 \left[\sqrt{\frac{I_1^{p}(\underset{\sim}{\lambda}_e, \Delta\gamma)}{3}} - 1\right]^{C_2} \sigma'^{\,m}$$

are the 4 eqns to be solved for

$\underset{\sim}{\lambda}_e = \{\lambda_e^1, \lambda_e^2, \lambda_e^3\}$ and $\Delta\gamma$.

Stress Constitutive Equns: Alt. 1

Compressible Case

$$U = U(\bar{I}_1, \bar{I}_2, J). \quad \text{Assume Isotropy at Outset}$$

$$\bar{I}_1 = \frac{I_1}{J^{2/3}} \qquad 2J\frac{\partial J}{\partial \underline{C}} = \frac{dJ^2}{d\underline{C}} = J^2 \underline{C}^{-1}$$

$$\Rightarrow \frac{dJ}{d\underline{C}} = \frac{1}{2} J \underline{C}^{-1}$$

$$\bar{I}_2 = \frac{I_2}{J^{4/3}}$$

$$\tau = 2\underline{F}\cdot\left[\frac{\partial U}{\partial \bar{I}_1}\left(\frac{\partial \bar{I}_1}{\partial I_1}\frac{\partial I_1}{\partial \underline{C}} + \frac{\partial \bar{I}_1}{\partial J}\frac{\partial J}{\partial \underline{C}}\right) + \frac{\partial U}{\partial \bar{I}_2}\left(\frac{\partial \bar{I}_2}{\partial I_2}\frac{\partial I_2}{\partial \underline{C}} + \frac{\partial \bar{I}_2}{\partial J}\frac{\partial J}{\partial \underline{C}}\right) + \frac{\partial U}{\partial J}\frac{\partial J}{\partial \underline{C}}\right]\cdot \underline{F}^T.$$

$$= 2\underline{F}\cdot\left[\frac{\partial U}{\partial \bar{I}_1}\cdot\frac{1}{J^{2/3}}\mathbb{1} + \frac{\partial U}{\partial \bar{I}_2}\frac{1}{J^{4/3}}\left(I_1\mathbb{1} - \underline{C}\right) + \left\{\frac{\partial U}{\partial \bar{I}_1} I_1 \frac{1}{J^{5/3}}\left(-\frac{2}{3}\right) + \frac{\partial U}{\partial \bar{I}_2}\frac{I_2}{J^{7/3}}\left(-\frac{4}{3}\right) + \frac{\partial U}{\partial J}\right\}\frac{1}{2} J \underline{C}^{-1}\right]\cdot \underline{F}^T$$

$$= 2\underline{F}\cdot\left[\frac{1}{J^{2/3}}\left(\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial U}{\partial \bar{I}_2}\right)\mathbb{1} - \frac{1}{J^{2/3}}\frac{\partial U}{\partial \bar{I}_2}\bar{C} + \frac{1}{2}\left\{\frac{\partial U}{\partial \bar{I}_1}\bar{I}_1\left(-\frac{2}{3}\right) + \frac{\partial U}{\partial \bar{I}_2}\bar{I}_2\left(-\frac{4}{3}\right) + J\frac{\partial U}{\partial J}\right\}\underline{C}^{-1}\right]\cdot \underline{F}^T.$$

$$= 2\left[\left(\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial U}{\partial \bar{I}_2}\right)\bar{\underline{B}} - \frac{\partial U}{\partial \bar{I}_2}\bar{\underline{B}}^2 + \frac{1}{2}\left\{J\frac{\partial U}{\partial J} - \frac{2}{3}\frac{\partial U}{\partial \bar{I}_1}\bar{I}_1 - \frac{4}{3}\frac{\partial U}{\partial \bar{I}_2}\bar{I}_2\right\}\mathbb{1}\right]$$

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Now $\mathrm{tr}\left(\left[\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial U}{\partial \bar{I}_2}\right]\bar{B} - \frac{\partial U}{\partial \bar{I}_2}\bar{B}^2\right)$

$$= \left[\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial U}{\partial \bar{I}_2}\right]\bar{I}_1 - \frac{\partial U}{\partial \bar{I}_2}\mathrm{tr}\left(\bar{B}^2\right)$$

$$\mathrm{tr}\left(\bar{B}^2\right) = \bar{\lambda}_1^4 + \bar{\lambda}_2^4 + \bar{\lambda}_3^4$$
$$= \left(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2\right)^2$$
$$- 2\bar{\lambda}_1^2\bar{\lambda}_2^2 - 2\bar{\lambda}_1^2\bar{\lambda}_3^2 - 2\bar{\lambda}_3^2\bar{\lambda}_2^2$$
$$= \bar{I}_1^2 - 2\bar{I}_2$$

$$= \left[\frac{\partial U}{\partial \bar{I}_1}\bar{I}_1 + \cancel{\frac{\partial U}{\partial \bar{I}_2}\bar{I}_1^2} - \cancel{\frac{\partial U}{\partial \bar{I}_2}\bar{I}_1^2} + 2\frac{\partial U}{\partial \bar{I}_2}\bar{I}_2\right]$$

$$= \frac{\partial U}{\partial \bar{I}_1}\bar{I}_1 + 2\frac{\partial U}{\partial \bar{I}_2}\bar{I}_2.$$

$$\therefore \boxed{\begin{aligned}\underline{\tau} = {} & 2\,\mathrm{dev}\left[\left(\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial U}{\partial \bar{I}_2}\right)\bar{B} - \frac{\partial U}{\partial \bar{I}_2}\bar{B}^2\right] + 2\left[\frac{1}{3}\frac{\partial U}{\partial \bar{I}_1}\bar{I}_1 + \frac{2}{3}\frac{\partial U}{\partial \bar{I}_2}\bar{I}_2\right]\mathbb{1} \\ & + 2\left[\frac{J}{2}\frac{\partial U}{\partial J} - \frac{1}{3}\frac{\partial U}{\partial \bar{I}_1}\bar{I}_1 - \frac{2}{3}\frac{\partial U}{\partial \bar{I}_2}\bar{I}_2\right]\mathbb{1}\end{aligned}}$$

$$\Rightarrow \underline{\tau} = 2\,\mathrm{dev}\left[\left(\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial U}{\partial \bar{I}_2}\right)\bar{B} - \frac{\partial U}{\partial \bar{I}_2}\bar{B}^2\right] + J\frac{\partial U}{\partial J}\mathbb{1}$$

Compressible Case Derivation

(3c)

$$\bar{C} = \frac{C}{J^{2/3}}$$

Att 2 (with tangential gradient) (1)

$$\frac{\partial \bar{C}}{\partial C} = \frac{\mathbb{I}}{J^{2/3}} + C\left(\frac{-2}{3}\right)\frac{1}{J^{2/3}} C^{-1} \cdot \frac{1}{2}$$

More general than Att. 1.

$$U = U(\bar{C}, J)$$

$$\therefore\ \underline{\tau} = F \cdot 2\,\nabla_t U(\bar{C}) : \left[\frac{\mathbb{I}}{J^{2/3}} - \frac{1}{J^{2/3}\,3} \bar{C}\, C^{-1}\right] F^T + \cancel{2} F \cdot \frac{\partial V}{\partial J \cancel{2}} J C^{-1} \cdot F^T$$

$$= \frac{2}{J^{2/3}} F \cdot \nabla_t U(\bar{C}) : \left[\mathbb{I} - \frac{1}{3} C\, C^{-1}\right] F^T + \mathbf{1}\, J \frac{\partial V}{\partial J} \mathbb{I}$$

$$= \frac{2}{J^{2/3}} F \cdot \left[\left\{\frac{\partial U}{\partial \bar{C}} - \left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right)\frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right\} : \left\{\mathbb{I} - \frac{1}{3} C\, C^{-1}\right\}\right] \cdot F^T + \mathbf{1} J \frac{\partial V}{\partial J}$$

$$= 2\bar{F} \cdot \left[\frac{\partial U}{\partial \bar{C}} - \frac{1}{3}\frac{\partial U}{\partial \bar{C}} : C\, C^{-1} - \left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right)\frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|} : \mathbb{I}\right.$$
$$\left. + \frac{1}{3}\left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right)\frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|} : C\, C^{-1}\right] \cdot \bar{F}^T + \mathbf{1} J \frac{\partial V}{\partial J} \mathbb{I}$$

$$= 2\bar{F} \cdot \left[\frac{\partial U}{\partial \bar{C}} - \left(\frac{1}{3}\frac{\partial U}{\partial \bar{C}} : C\right) C^{-1} - \left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right)\frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right.$$
$$\left. + \frac{1}{3}\left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right) \frac{J^{2/3}\, 3}{\|\bar{C}^{-1}\|} C^{-1}\right] \cdot \bar{F}^T + \mathbf{1} J \frac{\partial V}{\partial J} \mathbb{I}$$

$$= 2\bar{F} \cdot \frac{\partial U}{\partial \bar{C}} \cdot \bar{F}^T - \left(\frac{2}{3}\frac{\partial U}{\partial \bar{C}} : C\right)\frac{\mathbb{1}}{J^{2/3}} \underbrace{\cancel{- 2\left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right)\frac{\mathbb{1}}{\|\bar{C}^{-1}\|} + 2 \cdot \frac{1}{3}\left(\frac{\partial U}{\partial \bar{C}} : \frac{\bar{C}^{-1}}{\|\bar{C}^{-1}\|}\right)\frac{J^{2/3}\, 3}{\|\bar{C}^{-1}\|}\frac{\mathbb{1}}{J^{2/3}}}}_{0} + J\frac{\partial V}{\partial J}\mathbb{1}$$

3d

2

Now $\mathrm{tr}\left(2\bar{\underline{F}}\cdot\frac{\partial U}{\partial \bar{\underline{C}}}\cdot\bar{\underline{F}}^T\right) = 2\frac{\partial U}{\partial \bar{\underline{C}}} : \bar{\underline{C}}$

$$\therefore\ \underline{\tau} = \mathrm{dev}\left(2\bar{\underline{F}}\cdot\frac{\partial U}{\partial \bar{\underline{C}}}\cdot\bar{\underline{F}}^T\right) + \cancel{\frac{2}{3}\left(\frac{\partial U}{\partial \bar{\underline{C}}}:\bar{\underline{C}}\right)\mathbb{1}} - \cancel{\frac{2}{3}\left(\frac{\partial U}{\partial \bar{\underline{C}}}:\bar{\underline{C}}\right)}\mathbb{1} + J\frac{\partial V}{\partial J}\mathbb{1}$$

$$\boxed{\underline{\tau} = \mathrm{dev}\left(2\bar{\underline{F}}\cdot\frac{\partial U}{\partial \bar{\underline{C}}}\cdot\bar{\underline{F}}^T\right) + J\frac{\partial V}{\partial J}\mathbb{1}}.$$

For solving material update eqns. on pg. 3, we phrase them in terms of logs, i.e

$$\log \lambda_*^A = \log \lambda_e^A + \Delta\gamma \frac{1}{\sqrt{2}\,\sigma'} s^A$$

we now use the notation

$$\log[\lambda] = \varepsilon$$

& $s^A = \sigma^A - \frac{I_1^\sigma}{3}$ = principal values of Cauchy deviator.

Then

$$\varepsilon_*^1 = \varepsilon_e^1 + \Delta\gamma \frac{s^1}{\sqrt{2}\,\sigma'} \qquad \sigma' = \left\{\frac{1}{2}\sum_B s^{B2}\right\}^{\frac{1}{2}}$$

$$\varepsilon_*^2 = \varepsilon_e^2 + \Delta\gamma \frac{s^2}{\sqrt{2}\,\sigma'}$$

$$\varepsilon_*^3 = \varepsilon_e^3 + \frac{\Delta\gamma\, s^3}{\sqrt{2}\,\sigma'}$$

$$\Delta\gamma = \left[\hat{C}_1 \left[\sqrt{\frac{I_1^p}{3}(\varepsilon_e^i, \Delta\gamma)} - 1\right]^{c_2} \sigma'^{\,m}\right]\Delta t$$

Let $\underset{\sim}{f}(\varepsilon_e^B, \Delta\gamma) := f^A(\varepsilon_e^B, \Delta\gamma) = \varepsilon_*^A - \frac{\Delta\gamma\, s^A}{\sqrt{2}\,\sigma'} - \varepsilon_e^A$

$$g(\varepsilon_e^B, \Delta\gamma) = \Delta\gamma - \left[\hat{C}_1 \left[\sqrt{\frac{I_1^p}{3}(\varepsilon_e^i, \Delta\gamma)} - 1\right]^{c_2} \sigma'^{\,m}\right]\Delta t$$

N-R :

$$\left[\begin{array}{c|c} \frac{\partial f^A}{\partial \varepsilon_e^B} & \frac{\partial f^A}{\partial \Delta\gamma} \\ \hline \frac{\partial g}{\partial \varepsilon_e^B} & \frac{\partial g}{\partial \Delta\gamma} \end{array}\right]^{-1} \left[\begin{array}{c} -f^A \\ \hline -g \end{array}\right] = \left[\begin{array}{c} \delta\varepsilon_A^B \\ \hline \delta\Delta\gamma \end{array}\right]$$

$$\left(\varepsilon_A^B\right)_{K+1} = \left(\varepsilon_A^B\right)_K + \delta\varepsilon_A^B$$

$$\left(\Delta\gamma\right)_{K+1} = \left(\Delta\gamma\right)_K + \delta\Delta\gamma.$$

Need :- I $\quad \dfrac{\partial f^A}{\partial \varepsilon_e^B} = -\dfrac{\Delta\gamma}{\sqrt{2}} \dfrac{\partial}{\partial \varepsilon_e^B}\left\{\dfrac{S^A}{\sigma'}\right\} - \delta^A_B$

II $\quad \dfrac{\partial f^A}{\partial \Delta\gamma} = -\dfrac{S^A}{\sqrt{2}\,\sigma'}$

III $\quad \dfrac{\partial g}{\partial \varepsilon_e^B} = -\hat{C}_1 \dfrac{\partial}{\partial \varepsilon_e^B}\left\{\left[\sqrt{\dfrac{I_1^p}{3}} - 1\right]^{C_2}\right\}\sigma'^m \Delta t$

$$- \hat{C}_1\left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{C_2} \frac{\partial}{\partial \varepsilon_e^B}\left(\sigma'^m\right)\Delta t$$

IV $\quad \dfrac{\partial g}{\partial \Delta\gamma} = 1 - \hat{C}_1 \dfrac{\partial}{\partial \Delta\gamma}\left\{\left[\sqrt{\dfrac{I_1^p}{3}} - 1\right]^{C_2}\right\}\sigma'^m \Delta t$

I :- $\quad \dfrac{\partial f^A}{\partial \varepsilon_e^B} = -\dfrac{\Delta\gamma}{\sqrt{2}}\left\{\dfrac{1}{\sigma'}\dfrac{\partial S^A}{\partial \varepsilon_e^B} - \dfrac{S^A}{\sigma'^2}\cdot\dfrac{1}{2}\cdot\dfrac{1}{\sigma'}\sum_C S^C 2\dfrac{\partial S^C}{\partial \varepsilon_e^B}\right\} - \delta^A_B$

$$\Rightarrow \boxed{\frac{\partial f^A}{\partial \varepsilon_e^B} = -\frac{\Delta\gamma}{\sqrt{2}}\left\{\frac{1}{\sigma'}\frac{\partial S^A}{\partial \varepsilon_e^B} - \frac{S^A}{2\sigma'^3}\sum_C S^C\frac{\partial S^C}{\partial \varepsilon_e^B}\right\} - \delta^A_B}$$

$\boxed{\text{So we need } \dfrac{\partial S^A}{\partial \varepsilon_e^B}.}$

From pg. 36 :-

$$S^A = \frac{2}{J}\left[\left(\frac{\partial U}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial U}{\partial \bar{I}_2}\right)(\bar{\lambda}_e^A)^2 - \frac{\partial U}{\partial \bar{I}_2}(\bar{\lambda}_e^A)^4 - \frac{1}{3}\frac{\partial U}{\partial \bar{I}_1}\bar{I}_1 - \frac{2}{3}\frac{\partial U}{\partial \bar{I}_2}\bar{I}_2\right]$$

$$\Rightarrow \boxed{S^A = \frac{2}{J}\left[\frac{\partial U}{\partial \bar{I}_{1e}}\left(\{\bar{\lambda}_e^A\}^2 - \frac{1}{3}\bar{I}_{1e}\right) + \frac{\partial U}{\partial \bar{I}_{2e}}\left(\bar{I}_{1e}\{\bar{\lambda}_e^A\}^2 - \{\bar{\lambda}_e^A\}^4 - \frac{2}{3}\bar{I}_{2e}\right)\right]}$$

We use this expression regardless of the fact whether the material is compressible or incompressible. In the FEM context, incompressibility is imposed in the weak sense.

In any case, if $J = 1$ at a material pt., the expression for S^A coincides for both cases.

Let $\bar{\mu}^A_e := (\bar{\lambda}^A_e)^2$ & $\boxed{\mu^A_e := (\lambda^A_e)^2}$

We also have $\bar{\lambda}^A_e = \dfrac{\lambda^A_e}{J^{1/3}}$ since $J_e = J$.

$$\boxed{\begin{aligned} &\bar{I}_{1e} = \frac{I_{1e}}{J^{2/3}} \; ; \quad \bar{I}_{2e} = \frac{I_{2e}}{J^{4/3}} \\ \text{where} \quad &I_{1e} = \sum_B \mu^B_e \; ; \quad I_{2e} = \sum_B \mu^{B+1} \mu^{B+2} \end{aligned}}$$

Note that for the purpose of the material update we do not consider the λ^A_e to be constrained to satisfy $\lambda^1_e \lambda^2_e \lambda^3_e = J$. This property is satisfied by the soln to the material update. So, for the material update, we look for a solution for $\{\varepsilon^1_e, \varepsilon^2_e, \varepsilon^3_e\}$ in $\mathbb{R}^3$. Also, for the purpose of the material update J is not a fn of the solution variables $\{\varepsilon^1_e, \varepsilon^2_e, \varepsilon^3_e, \Delta\gamma\}$.

$$\frac{\partial S^A}{\partial \varepsilon_e^B} = \frac{2}{J}\left[\frac{\partial U}{\partial \bar{I}_{1e}}\left(2\bar{\lambda}_e^{A^2}\frac{\partial \varepsilon_e^A}{\partial \varepsilon_e^B} - \frac{1}{3}\frac{\partial \bar{I}_{1e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2}\right)\right.$$

$$+ \frac{\partial U}{\partial \bar{I}_{2e}}\left(\frac{\partial \bar{I}_{1e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2}\bar{\lambda}_e^{A^2} + \bar{I}_{1e}\, 2(\bar{\lambda}_e^A)^2 \frac{\partial \varepsilon_e^A}{\partial \varepsilon_e^B}\right.$$

$$\left. - 4\bar{\lambda}_e^{A^4}\frac{\partial \varepsilon_e^A}{\partial \varepsilon_e^B} - \frac{2}{3}\frac{\partial \bar{I}_{2e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2}\right)$$

$$+ \left(\bar{\lambda}_e^{A^2} - \frac{1}{3}\bar{I}_{1e}\right)\left(\frac{\partial^2 U}{\partial \bar{I}_{1e}\partial \bar{I}_{1e}}\frac{\partial \bar{I}_{1e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2} + \frac{\partial^2 U}{\partial \bar{I}_{1e}\partial \bar{I}_{2e}}\frac{\partial \bar{I}_{2e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2}\right)$$

$$\left. + \left(\bar{I}_{1e}\bar{\lambda}_e^{A^2} - \bar{\lambda}_e^{A^4} - \frac{2}{3}\bar{I}_{2e}\right)\left(\frac{\partial^2 U}{\partial \bar{I}_{2e}\partial \bar{I}_{1e}}\frac{\partial \bar{I}_{1e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2} + \frac{\partial^2 U}{\partial \bar{I}_{2e}\partial \bar{I}_{2e}}\frac{\partial \bar{I}_{2e}}{\partial \bar{\mu}_e^B} 2\bar{\lambda}_e^{B^2}\right)\right]$$

Now $\bar{I}_{1e} = \sum_C \bar{\mu}_e^C$

$$\boxed{\frac{\partial \bar{I}_{1e}}{\partial \bar{\mu}_e^B} = \sum_C \frac{\partial \bar{\mu}_e^C}{\partial \bar{\mu}_e^B} = 1}$$

also $\bar{I}_{2e} = \sum_C \bar{\mu}^{C+1}\bar{\mu}^{C+2}$

$$\frac{\partial \bar{I}_{2e}}{\partial \bar{\mu}_e^B} = \sum_C \left(\frac{\partial \bar{\mu}^{C+1}}{\partial \bar{\mu}_e^B}\mu^{C+2} + \bar{\mu}^{C+1}\frac{\partial \bar{\mu}^{C+2}}{\partial \bar{\mu}_e^B}\right)$$

$$\boxed{\frac{\partial \bar{I}_{2e}}{\partial \bar{\mu}_e^B} = \left(\bar{\mu}^{B+1} + \bar{\mu}^{B-1}\right)}$$

$$\frac{\partial S^A}{\partial \varepsilon_e^B} = \frac{2}{J}\Bigg[2\frac{\partial U}{\partial \bar{I}_{1e}}\left(\bar{\mu}_e^A \delta_B^A - \frac{1}{3}\bar{\mu}_e^B\right)$$

$$+ \frac{\partial U}{\partial \bar{I}_{2e}}\left\{ 2\bar{\mu}_e^A\bar{\mu}_e^B + 2\bar{I}_{1e}\bar{\mu}_e^A\delta_B^A - 4\bar{\mu}_e^{A^2}\delta_B^A - \frac{4}{3}\left(\bar{\mu}_e^B\bar{\mu}_e^{B+1} + \bar{\mu}_e^B\bar{\mu}^{B-1}\right)\right\}$$

$$+ 2\left(\bar{\mu}_e^A - \frac{1}{3}\bar{I}_{1e}\right)\left\{\frac{\partial^2 U}{\partial \bar{I}_{1e}\partial \bar{I}_{1e}}\bar{\mu}_e^B + \frac{\partial^2 U}{\partial \bar{I}_{1e}\partial \bar{I}_{2e}}\left(\bar{\mu}_e^B\bar{\mu}^{B+1} + \bar{\mu}_e^B\bar{\mu}^{B-1}\right)\right\}$$

$$+ 2\left(\bar{I}_{1e}\bar{\mu}_e^A - \bar{\mu}_e^{A^2} - \frac{2}{3}\bar{I}_{2e}\right)\left\{\frac{\partial^2 U}{\partial \bar{I}_{2e}\partial \bar{I}_{1e}}\bar{\mu}_e^B + \frac{\partial^2 U}{\partial \bar{I}_{2e}\partial \bar{I}_{2e}}\left(\bar{\mu}_e^B\bar{\mu}^{B+1} + \bar{\mu}_e^B\bar{\mu}^{B-1}\right)\right\}\Bigg]$$

Substituting above in expression for (I) on page (5) $\frac{\partial f^A}{\partial \varepsilon_e^B}$ can be obtained.

III :
$$\frac{\partial g}{\partial \varepsilon_e^B} = -\hat{c}_1 \frac{\partial}{\partial \varepsilon_e^B}\left\{\left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{c_2}\right\}\sigma'^m$$
$$-\hat{c}_1\left[\sqrt{\frac{I^p}{3}} - 1\right]^{c_2}\frac{\partial}{\partial \varepsilon_e^B}\left(\sigma'^m\right)$$

Now
$$\frac{\partial \sigma'^m}{\partial \varepsilon_e^B} = m\,\sigma'^{(m-1)}\frac{\partial \sigma'}{\partial \varepsilon_e^B}$$
$$= m\,\sigma'^{(m-1)} \cdot \frac{1}{2}\,\frac{1}{\sigma'} \cdot \frac{1}{\cancel{2}}\sum_c \cancel{2}\,\frac{\partial S^c}{\partial \varepsilon_e^B}\,S^c$$

$$\boxed{\frac{\partial \sigma'^m}{\partial \varepsilon_e^B} = \frac{m\,\sigma'^{(m-2)}}{2}\sum_c S^c\frac{\partial S^c}{\partial \varepsilon_e^B}}$$

Now
$$\frac{\partial}{\partial \varepsilon_e^B}\left\{\left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{c_2}\right\}$$
$$= c_2\left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{(c_2-1)}\left\{\frac{1}{2}\left(\frac{I_1^p}{3}\right)^{-1/2}\frac{1}{3}\,\frac{\partial I_1^p}{\partial \varepsilon_e^B}\right\}$$
$$= \frac{c_2}{2\sqrt{3}}\left(I_1^p\right)^{-1/2}\left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{(c_2-1)}\frac{\partial I_1^p}{\partial \varepsilon_e^B}.$$

Now $\quad I_1^p = C^p : \mathbb{1}$

$$\underset{\sim}{C}^p = \underline{F}_n^{p^T} \cdot \exp^2\left[\sum_A \frac{\Delta\gamma}{\sqrt{2}\,\sigma'} S^A \, \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right] \cdot \underline{F}_n^p$$

$$= \underline{F}_n^{p^T} \cdot \left(\sum_A \exp^2\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \underline{F}_n^p .$$

$$\therefore \frac{\partial I_1^p}{\partial \varepsilon_e^B} = \mathbb{1} : \left[\underline{F}_n^{p^T} \cdot \left(\sum_A 2\exp^2\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \frac{\partial}{\partial \varepsilon_e^B}\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \underline{F}_n^p\right]$$

Now $\quad f^A = \varepsilon_*^A - \dfrac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'} - \varepsilon_e^A$

$$\Rightarrow \frac{\partial f^A}{\partial \varepsilon_e^B} + \delta_B^A = -\frac{\partial}{\partial \varepsilon_e^B}\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right)$$

Consequently

$$\frac{\partial I_1^p}{\partial \varepsilon_e^B} = \mathbb{1} : \left[\underline{F}_n^{p^T} \cdot \left(\sum_A -2\exp^2\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right)\left(\frac{\partial f^A}{\partial \varepsilon_e^B} + \delta_B^A\right) \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \underline{F}_n^p\right]$$

$\therefore$ III

$$\frac{\partial g}{\partial \varepsilon_e^B} =$$

$$\left\{ -\frac{\hat{C}_1 C_2}{2\sqrt{3}} \left(I_1^p\right)^{-1/2} \left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{(C_2-1)} \sigma'^{\,m} \right\} \Delta t$$

$$\left\{ \mathbb{1} : \left[\underset{\sim}{F}_n^{p\,T} \cdot \left(\sum_A -2\, \exp^2\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \left(\frac{\partial f^A}{\partial \varepsilon_e^B} + \delta_B^A\right) \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A \right) \cdot \underset{\sim}{F}_n^p \right] \right\}$$

$$- \Delta t\, \hat{C}_1 \left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{C_2} \frac{m\, \sigma'^{(m-2)}}{2} \sum_C \frac{\partial S^C}{\partial \varepsilon_e^B} S^C$$

IV : $$\frac{\partial g}{\partial \Delta\gamma} = 1 - \hat{C}_1 \sigma'^{\,m} \frac{\partial}{\partial \Delta\gamma} \left\{ \left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{C_2} \right\}$$

Now $$\frac{\partial}{\partial \Delta\gamma} \left\{ \left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{C_2} \right\} = \frac{C_2}{2\sqrt{3}} \left(I_1^p\right)^{-1/2} \left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{(C_2-1)} \frac{\partial I_1^p}{\partial \Delta\gamma}$$

$$\frac{\partial I_1^p}{\partial \Delta\gamma} = \mathbb{1} : \left[\underset{\sim}{F}_n^{p\,T} \cdot \left(\sum_A 2 \exp^2\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \frac{S^A}{\sqrt{2}\,\sigma'} \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A \right) \cdot \underset{\sim}{F}_n^p \right]$$

$\therefore$ IV

$$\frac{\partial g}{\partial \Delta\gamma} = 1 - \left\{ \frac{\hat{C}_1 C_2}{2\sqrt{3}} \sigma'^{\,m} \left(I_1^p\right)^{-1/2} \left[\sqrt{\frac{I_1^p}{3}} - 1\right]^{(C_2-1)} \right\} \Delta t$$

$$\left\{ \mathbb{1} : \left[\underset{\sim}{F}_n^{p\,T} \cdot \left(\sum_A 2 \exp^2\left(\frac{\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \frac{S^A}{\sqrt{2}\,\sigma'} \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A \right) \cdot \underset{\sim}{F}_n^p \right] \right\}$$