Material Jacobian

$$\delta W = \int_V \underline{\sigma} : \delta \underline{D} \, dV$$

$$= \int_{V_0} \underline{\tau} : \delta \underline{D} \, dV^0 .$$

$$d\delta W = d \int_{V_0} \underline{\tau} : \delta \underline{D} \, dV^0$$

$$= \int_{V_0} d\underline{\tau} : \delta \underline{D} \, dV_0 + \int_V \underline{\sigma} : d\delta \underline{D} \, dV .$$

Now $\quad d\underline{\tau} = \underbrace{\sum_A d(\tau^A) \, \underline{n}_e^A \otimes \underline{n}_e^A}_{C_1} + \underbrace{\sum \tau^A d(\underline{n}_e^A \otimes \underline{n}_e^A)}_{C_2} .$

Now $\quad \underset{\sim}{n}_e^A = \dfrac{F^e \cdot \underset{\sim}{N}_*^A}{(\lambda_e^A)}$

$$\underset{\sim}{n}_e^A = \overset{*}{\underline{F}} \cdot \exp\left[\sum_B -\frac{\Delta\gamma S^B}{\sqrt{2}\,\sigma'} \underset{\sim}{N}_*^B \otimes \underset{\sim}{N}_*^B\right] \cdot \frac{\underset{\sim}{N}_*^A}{\lambda_e^A}$$

$$= \overset{*}{\underline{F}} \cdot \exp\left(\frac{-\Delta\gamma S^A}{\sqrt{2}\,\sigma'}\right) \frac{1}{\lambda_e^A} \underset{\sim}{N}_*^A$$

$$= \overset{*}{\underline{F}} \cdot \frac{\underset{\sim}{N}_*^A}{\lambda_*^A}$$

$$\therefore \quad \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A = \frac{1}{(\lambda_*^A)^2} \overset{*}{\underline{F}} \cdot \left(\underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \overset{*}{\underline{F}}^T .$$

$$\therefore \quad = \underline{F} \cdot \frac{1}{(\lambda_*^A)^2} \underline{F}_n^{p^{-1}} \cdot \left(\underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \underline{F}_n^{p^{-T}} \cdot \underline{F}^T ,$$

$$\therefore \quad d\left(\underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A\right) = d\underline{L} \cdot \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A + \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A \cdot d\underline{L}^T$$

$$+ \underline{F} \cdot d\left[\frac{1}{(\lambda_*^A)^2} \underline{F}_n^{p^{-1}} \cdot \left(\underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \underline{F}_n^{p^{-T}}\right] \cdot \underline{F}^T .$$

$$\therefore \quad \sum_A \tau^A d\left(\underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A\right)$$

$$= \cancel{d\underline{L}} \left(\sum_A \left(d\underline{L} \cdot \tau^A \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A + \tau^A \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A \cdot d\underline{L}^T\right)\right) \cancel{d\underline{L}^T}$$

$$+ \underline{F} \cdot \sum_A \tau^A d\left[\frac{1}{(\lambda_*^A)^2} \underline{F}_n^{p^{-1}} \cdot \left(\underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right) \cdot \underline{F}_n^{p^{-T}}\right] \cdot \underline{F}^T .$$

$$= C_{21} + C_{22}$$

$$\mathbb{C}_{21} = (d\underline{D} + d\underline{\Omega})\cdot\underline{\tau} + \underline{\tau}\cdot(d\underline{D} - d\underline{\Omega})$$

$$= d\underline{D}\cdot\underline{\tau} + \underline{\tau}\cdot d\underline{D} + \underbrace{J d\underline{\Omega}\cdot\underline{\sigma} - J\underline{\sigma}\cdot d\underline{\Omega}}_{\text{goes to Initial Stress Stiffness}}$$

So, Material Stiffness Contribution from $\mathbb{C}_2$

$$= \mathbb{C}_{22} + \boxed{J d\underline{D}\cdot\underline{\tau} + J\underline{\tau}\cdot d\underline{D}}$$

Now $\mathbb{C}_{22} = \underline{F}\cdot\sum_A J\tau^A d\left[\frac{1}{\mu_*^A}\underline{F}_n^{p^{-1}}\cdot(\underset{\sim}{N}_*^A\otimes\underset{\sim}{N}_*^A)\cdot\underline{F}_n^{p^{-T}}\right]\cdot\underline{F}^T$

$$= \overset{*}{\underline{F}}\cdot\sum_A \tau^A d\left[\frac{1}{\mu_*^A}\underset{\sim}{N}_*^A\otimes\underset{\sim}{N}_*^A\right]\cdot\overset{*}{\underline{F}}^T$$

So, we need

$$d\left[\frac{1}{\mu_*^A}\underset{\sim}{N}_*^A\otimes\underset{\sim}{N}_*^A\right].$$

Important Formulae:

① $d\lambda$:-

$$\underline{C} \cdot \underset{\sim}{N}^A = (\lambda^A)^2 \underset{\sim}{N}^A$$

$$\Rightarrow d\underline{C} \cdot \underset{\sim}{N}^A + \underline{C} \cdot d\underset{\sim}{N}^A = 2\lambda^A d\lambda^A \underset{\sim}{N}^A + (\lambda^A)^2 d\underset{\sim}{N}^A.$$

Use $\underset{\sim}{N}^A \cdot d\underset{\sim}{N}^A = 0$

$$\Rightarrow \underset{\sim}{N}^A \cdot d\underline{C} \cdot \underset{\sim}{N}^A = 2\lambda^A d\lambda^A \Rightarrow d\lambda^A = \frac{1}{2(\lambda^A)}(\underset{\sim}{N}^A \otimes \underset{\sim}{N}^A) : d\underline{C}$$

Now $d\mathcal{E}^A = \frac{1}{\lambda^A} d\lambda^A$

$$\Rightarrow \boxed{d\mathcal{E}^A = \frac{1}{2(\lambda^A)^2}\left(\underset{\sim}{N}^A \otimes \underset{\sim}{N}^A\right) : d\underline{C}}$$

② $d(\underset{\sim}{N}^A \otimes \underset{\sim}{N}^A)$:-

$$\underline{C} - (\lambda^{A+1})^2 \mathbb{1} = \left[(\lambda^A)^2 - (\lambda^{A+1})^2\right] \underset{\sim}{N}^A \otimes \underset{\sim}{N}^A + \left[(\lambda^{A+2})^2 - (\lambda^{A+1})^2\right] \underset{\sim}{N}^{A+2} \otimes \underset{\sim}{N}^{A+2}$$

$$\underline{C} - (\lambda^{A+2})^2 \mathbb{1} = \left[(\lambda^A)^2 - (\lambda^{A+2})^2\right] \underset{\sim}{N}^A \otimes \underset{\sim}{N}^A + \left[(\lambda^{A+1})^2 - (\lambda^{A+2})^2\right] \underset{\sim}{N}^{A+1} \otimes \underset{\sim}{N}^{A+1}.$$

$$\therefore \left[\underline{C} - \mu^{A+1}\mathbb{1}\right] \cdot \left[\underline{C} - \mu^{A+2}\mathbb{1}\right]$$

$$= \left[\mu^{A} - \mu^{A+1}\right]\left[\mu^{A} - \mu^{A+2}\right] \underset{\sim}{N}^{A} \otimes \underset{\sim}{N}^{A}$$

$$\therefore \boxed{\underset{\sim}{N}^{A} \otimes \underset{\sim}{N}^{A} = \frac{\left[\underline{C} - \mu^{A+1}\mathbb{1}\right] \cdot \left[\underline{C} - \mu^{A+2}\mathbb{1}\right]}{\left[\mu^{A} - \mu^{A+1}\right]\left[\mu^{A} - \mu^{A+2}\right]}}$$

Now $\quad \overset{*}{\underline{F}} = \underline{F} \cdot \underline{F}_n^{p^{-1}}$

$$\overset{*}{\underline{C}} = \underline{F}_n^{p^{-T}} \cdot \underline{C} \cdot \underline{F}_n^{p^{-1}}$$

$$\therefore \quad d\overset{*}{\underline{C}} = \underline{F}_n^{p^{-T}} \cdot d\underline{C} \cdot \underline{F}_n^{p^{-1}}$$

Now $\quad \delta \underline{D} = \frac{1}{2}\left[\delta \underline{F} \cdot \underline{F}^{-1} + \underline{F}^{-T} \cdot \delta \underline{F}^{T} \right]$

$$= \frac{1}{2}\left[\underline{F}^{-T} \cdot \underline{F}^{T} \cdot \delta \underline{F} \cdot \underline{F}^{-1} + \underline{F}^{-T} \cdot \delta \underline{F}^{T} \cdot \underline{F} \cdot \underline{F}^{-1} \right]$$

$$= \frac{1}{2} \underline{F}^{-T} \cdot \left[\delta \underline{C} \right] \cdot \underline{F}^{-1}$$

$$\Rightarrow \quad \boxed{2\, \underline{F}^{T} \cdot d\underline{D} \cdot \underline{F} = d\underline{C}}$$

$$\therefore \quad d\overset{*}{\underline{C}} = 2\, \underline{F}_n^{p^{-T}} \cdot \underline{F}^{T} \cdot d\underline{D} \cdot \underline{F} \cdot \underline{F}_n^{p^{-1}}$$

$$= 2\, \overset{*}{\underline{F}}{}^{T} \cdot d\underline{D} \cdot \overset{*}{\underline{F}}.$$

Also denote $\boxed{\dfrac{\overset{*}{\underline{F}} \cdot \underset{\sim}{N}{}_{*}^{A}}{\lambda_{*}^{A}} =: \underset{\sim}{n}{}_{*}^{A}}$

$$\& \quad d_1^{A} := \left(\lambda_{*}^{A}\right)^2 - \left(\lambda_{*}^{A+1}\right)^2$$

$$d_2^{A} := \left(\lambda_{*}^{A}\right)^2 - \left(\lambda_{*}^{A+2}\right)^2$$

Now, $$\underset{\sim}{N}^A \otimes \underset{\sim}{N}^A = \frac{1}{d_1^A d_2^A}\left[\underline{C}^2 - \left(\mu^{A+2} + \mu^{A+1}\right)\underline{C} + \mu^{A+1}\mu^{A+2}\,\mathbb{1}\right]$$

Characteristic Equ:

$$-\mu_A^3 + I_1\mu_A^2 - I_2\mu_A + I_3 = 0.$$

Cayley-Hamilton:

$$-\underline{C}^3 + I_1\underline{C}^2 - I_2\underline{C} + I_3\mathbb{1} = 0$$

$$\Rightarrow -\underline{C}^2 + I_1\underline{C} - I_2\mathbb{1} + I_3\underline{C}^{-1} = 0.$$

$$\Rightarrow \boxed{\underline{C}^2 = I_1\underline{C} - I_2\mathbb{1} + I_3\underline{C}^{-1}.}$$

$$\therefore \underset{\sim}{N}^A \otimes \underset{\sim}{N}^A = \frac{1}{d_1^A d_2^A}\left[\cancel{I_1}\underline{C} - I_2\mathbb{1} + I_3\underline{C}^{-1} - (\cancel{I_1} - \mu^A)\underline{C} + I_3\mu_A^{-1}\mathbb{1}\right]$$

$$\therefore \frac{1}{\mu^A}\underset{\sim}{N}^A \otimes \underset{\sim}{N}^A = \frac{1}{d_1^A d_2^A}\left[\underline{C} + I_3\mu_A^{-1}\underline{C}^{-1} + \left(-I_2 + I_3\mu_A^{-1}\right)\mu_A^{-1}\mathbb{1}\right]$$

But again ~~Cayley-Hamilton~~ the Characteristic Equ. gives

$$\left(-I_2 + I_3\mu_A^{-1}\right) = \mu_A^2 - I_1\mu_A$$

$$\therefore \boxed{\frac{1}{\mu_A} \underset{\sim}{N}^A \otimes \underset{\sim}{N}^A = \frac{1}{d_1^A d_2^A}\left[\underset{\sim}{C} - (I_1 - \mu_A)\mathbb{1} + I_3 \mu_A^{-1} \underset{\sim}{C}^{-1}\right]}$$

Hence $d\left(\frac{1}{\mu_*^A} \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right)$

Consider $\left[\mu^A - \mu^{A+1}\right]\left[\mu^A - \mu^{A+2}\right]$

$$= \mu_A^2 - \mu_A\left(\mu^{A+1} + \mu^{A+2}\right) + \mu^{A+1}\mu^{A+2}$$

$$= \mu_A^2 - \mu_A(I_1 - \mu_A) + I_3 \mu_A^{-1}.$$

$\therefore$ define $\boxed{d_1^A d_2^A = D^A := \mu_A^2 - \mu_A(I_1 - \mu_A) + I_3 \mu_A^{-1}}$

$$\therefore \boxed{\frac{1}{\mu_*^A} \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A = \frac{1}{D_*^A}\left[\underset{\sim}{C}^* - (\overset{*}{I}_1 - \mu_*^A)\mathbb{1} + \overset{*}{I}_3 \overset{*}{\mu}_A{}^{-1} \overset{*}{\underset{\sim}{C}}{}^{-1}\right]}$$

& we need

$$d\left(\frac{1}{\mu_*^A} \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right).$$

Notice that the whole expression is only in "terms of Index A".

$$\mu_*^A = \lambda_*^{A\,2}$$

Also denote (20)

$$\underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A = \underline{M}_*^A$$

$$d\mu_A^* = \underline{M}^{A*} : d\overset{*}{C}$$

$$\therefore\ d\left(\frac{1}{\mu_*^A}\, \underset{\sim}{N}_*^A \otimes \underset{\sim}{N}_*^A\right)$$

$$= \frac{1}{D_*^A}\Big[\mathbb{I} - \mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes \underline{M}_*^A + \overset{*}{I}_3\, \mu_A^{*-1}\, \overset{*}{\underline{C}}^{-1} \otimes \overset{*}{\underline{C}}^{-1} + \overset{*}{I}_3 (-1)\cdot \mu_A^{*-2}\, \overset{*}{\underline{C}}^{-1} \otimes \underline{M}_*^A + \overset{*}{I}_3\, \mu_A^{*-1}\, \frac{\partial \overset{*}{\underline{C}}^{-1}}{\partial \overset{*}{\underline{C}}}\Big] : d\overset{*}{\underline{C}}$$

$$+ \underbrace{D_*^A\, \mu_A^{*-1}}_{(-1)\, D_*^{A\,-2}}\, \underline{M}_*^A \otimes \Big[4\mu_A^*\, \underline{M}_*^A - I_1\, \underline{M}_*^A - \mu_A^*\, \mathbb{1} + \mu_A^{*-1}\, \overset{*}{I}_3\, \overset{*}{\underline{C}}^{-1} + I_3 (-1)\, \mu_A^{*-2}\, \underline{M}_*^A\Big] : d\overset{*}{\underline{C}}$$

Now

$$C_{im} C_{mj}^{-1} = \delta_{ij}$$

$$\delta_{ik}\delta_{ml}\, C_{mj}^{-1} + C_{im}\, \frac{\partial C_{mj}^{-1}}{\partial C_{kl}} = 0$$

$$\Rightarrow \frac{\partial C_{mj}^{-1}}{\partial C_{kl}} = -C_{mi}^{-1}\, \delta_{ik}\, \delta_{rl}\, C_{rj}^{-1}$$

$$= -C_{mk}^{-1}\, C_{lj}^{-1}.$$

Denote $\boxed{\partial_{\overset{*}{C}} \overset{*}{C}^{-1} = -C_{ik}^{-1}\, C_{jl}^{-1}\, \underset{\sim}{E}_i \otimes \underset{\sim}{E}_j \otimes \underset{\sim}{E}_k \otimes \underset{\sim}{E}_l}$.

Has major symmetries

$$\therefore\ d\left(\frac{1}{\mu_*^A}\, \tilde{N}_*^A \otimes \tilde{N}_*^A\right)$$

$$= \frac{1}{D_*^A}\left[\mathbb{I} - \mathbb{1}\otimes\mathbb{1} + \overset{*}{I}_3\, \mu_A^{*-1}\left(\overset{*}{\underline{C}}{}^{-1}\otimes \overset{*}{\underline{C}}{}^{-1} + \partial_{\overset{*}{\underline{C}}}\, \overset{*}{C}{}^{-1}\right)\right] : d\overset{*}{\underline{C}}$$

$$+ \frac{1}{D_*^A}\left[\left(\mathbb{1}\otimes \underline{M}_*^A + \underline{M}_*^A\otimes\mathbb{1}\right) + \mu_A^{*-1}\left(\overset{*}{I}_1 - 4\mu_*^A + \overset{*}{I}_3\, \mu_*^{A\,-2}\right) \underline{M}_*^A \otimes \underline{M}_*^A\right] : d\overset{*}{\underline{C}}$$

$$- \frac{1}{D_*^A}\left[\overset{*}{I}_3\, \mu_A^{*-2}\left(\overset{*}{\underline{C}}{}^{-1}\otimes \underline{M}_*^A + \underline{M}_*^A \otimes \overset{*}{\underline{C}}{}^{-1}\right)\right] : d\overset{*}{\underline{C}}.$$

$$\therefore\ \mathbb{C}_{22} = \sum_A S^A\, \overset{*}{\underline{F}}\cdot d\left[\frac{1}{\mu_*^A}\, \tilde{N}_*^A\otimes \tilde{N}_*^A\right]\cdot \overset{*}{\underline{F}}{}^T.$$

$$\mathbb{I}_{ijkl} = \delta_{ik}\delta_{jl}.$$

So $F_{ir}\,\delta_{rm}\delta_{sn}\,F^T_{sj} : dC_{mn}$

$$= 2\, F_{ir}\,\delta_{rm}\delta_{sn}\,F^T_{sj}\; F^T_{mk}\, dD_{kl}\, F_{ln}$$

$$= 2\, F_{im}\, F^T_{nj}\, F^T_{mk}\, F_{ln}\, dD_{kl}$$

$$= 2\, B_{ik}\, B_{lj}\, dD_{kl}$$

Denote $\boxed{\mathbb{I}_B = B_{ik}B_{jl}\; \tilde{e}_i\otimes \tilde{e}_j\otimes \tilde{e}_k\otimes \tilde{e}_l.}$

$$\text{Denote} \quad \underset{\sim}{n}^A \otimes \underset{\sim}{n}^A = \underset{\sim}{m}^A$$

(22)

Also $\quad \partial_{\underset{\sim}{C}} \overset{*}{C}^{-1} : d\overset{*}{\underset{\sim}{C}} = (-1).2.\, C^{-1}_{rm} C^{-1}_{sn} F_{ir} F^{T}_{sj} F^{T}_{mk} dD_{kl} F_{ln}$

$$= -2 \quad F'^{-1}_{ra} F'^{-T}_{am} F'^{-1}_{sb} F'^{-T}_{bn} F'_{ir} F'^{T}_{sj} F'^{T}_{mk} F'_{ln}\, dD_{kl}$$

$$= -2 \quad \delta_{ia} \delta_{ak} F^{-1}_{sb} F^{T}_{sj} F^{-T}_{bn} F^{T}_{nl}\, dD_{kl}$$

$$= -2 \quad \delta_{ik} \delta_{jb} \delta_{bl}\, dD_{kl}$$

$$\boxed{\left(\partial_{\underset{\sim}{C}} \overset{*}{C}^{-1} : d\overset{*}{\underset{\sim}{C}}\right) = -2\, \delta_{ik} \delta_{jl}\, dD_{kl} = -2\, \mathbb{I} : d\underset{\sim}{D}}$$

$$\boxed{\begin{aligned} \mathcal{C}_{22} = \sum_A 2 \frac{\tau^A}{D^A_*} \Bigg[& \underset{\sim}{\mathbb{I}}_{\overset{*}{B}} - \overset{*}{\underset{\sim}{B}} \otimes \overset{*}{\underset{\sim}{B}} + \frac{\overset{*}{I}_3}{\overset{*}{\mu}_A} \left(\underset{\sim}{1} \otimes \underset{\sim}{1} - \underset{\sim}{\mathbb{I}}\right) \\ & + \overset{*}{\mu}_A \left(\overset{*}{\underset{\sim}{B}} \otimes \overset{*}{\underset{\sim}{m}}^A + \overset{*}{\underset{\sim}{m}}^A \otimes \overset{*}{\underset{\sim}{B}}\right) \\ & + \left(\overset{*}{I}_1 \overset{*}{\mu}_A - 4 \mu_*^{A^2} + \overset{*}{I}_3 \mu_*^{A^{-1}}\right) \overset{*}{\underset{\sim}{m}}^A \otimes \overset{*}{\underset{\sim}{m}}^A \\ & - \frac{\overset{*}{I}_3}{\overset{*}{\mu}_A} \left(\underset{\sim}{1} \otimes \overset{*}{\underset{\sim}{m}}^A + \overset{*}{\underset{\sim}{m}}^A \otimes \underset{\sim}{1}\right) \Bigg] : d\underset{\sim}{D} \end{aligned}}$$

Now $\mathbb{C}_{21(m)} = d\underline{D} \cdot \underline{\tau} + \underline{\tau} \cdot d\underline{D}$

$$\left[\mathbb{C}_{21(m)}\right]_{ij} = dD_{kl}\,\delta_{ik}\,\tau_{lj} + \tau_{ik}\,dD_{kl}\,\delta_{lj}$$
$$= \left(\delta_{ik}\tau_{lj} + \tau_{ik}\delta_{lj}\right) dD_{kl}$$

Denote $\mathbb{S} = \left(\delta_{ik}\tau_{lj} + \tau_{ik}\delta_{lj}\right) \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \otimes \underset{\sim}{e}_l$

Then $\mathbb{C}_{21(m)} = \mathbb{S} : d\underline{D}$

Need to calculate $\mathbb{C}_1$.

$$\mathbb{C}_1 = \sum_A d\tau^A\, \underset{\sim}{n}_e^A \otimes \underset{\sim}{n}_e^A$$

Now $\underset{\sim}{n}_e^A = \underline{n}_*^A$

$$\therefore\ \mathbb{C}_1 = \sum_A d\tau^A\, \underset{\sim}{n}_*^A \otimes \underset{\sim}{n}_*^A$$