Bending of Bilayered Plates

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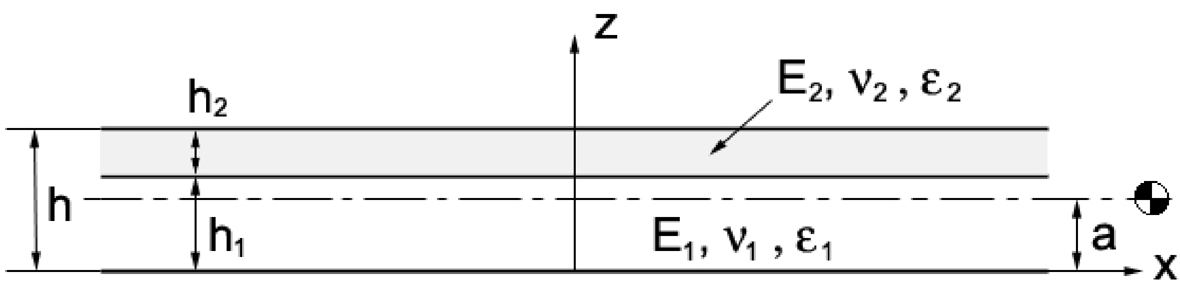

Figure 1. Illustration of a bilayered plate

As illustrated in the Fig. 1, a bilayered plate consists of lower layer and upper layer labeled by subscripts 1 and 2 respectively. ε_1 and ε_2 are equi-biaxial residual strains (e.g., due to lattice mismatch or thermal expansion mismatch). The location of the neutral plane is denoted by $z = a$, to be determined.

Under a combined in-plane deformation and bending, the in-plane strain components are

$$\varepsilon_{xx} = \begin{cases} \varepsilon_1 + \varepsilon_{nx} - w_{,xx}(z-a) & 0 \le z \le h_1 \\ \varepsilon_2 + \varepsilon_{nx} - w_{,xx}(z-a) & h_1 \le z \le h \end{cases} \tag{1}$$

$$\varepsilon_{yy} = \begin{cases} \varepsilon_1 + \varepsilon_{ny} - w_{,yy}(z-a) & 0 \le z \le h_1 \\ \varepsilon_2 + \varepsilon_{ny} - w_{,yy}(z-a) & h_1 \le z \le h \end{cases} \tag{2}$$

$$\varepsilon_{xy} = \varepsilon_{nxy} - w_{xy}(z-a) \tag{3}$$

where ε_{nx} ε_{ny} and ε_{nxy} are the strains at the neutral plane ($z = a$) and related to the displacements as

$$\varepsilon_{nx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \ \varepsilon_{nx} = \frac{\partial u_y}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \ \varepsilon_{nxy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right). \tag{4}$$

In the linear analysis, the nonlinear terms are ignored. As a result, the strain components at the neutral plane are independent of bending deflection *w*. In other words, the strain due to bending is zero at the neutral plane.

By Hooke's law the in-plane stresses are:

$$
\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \\
&= \begin{cases} \dfrac{E_1\varepsilon_1}{1-\nu_1} + \dfrac{E_1}{1-\nu_1^2}\left(\dfrac{\partial u_x}{\partial x} + \nu_1\dfrac{\partial u_y}{\partial y}\right) - \dfrac{E_1}{1-\nu_1^2}\left(w_{,xx} + \nu_1 w_{,yy}\right)(z-a) & 0 \le z \le h_1 \\ \dfrac{E_2\varepsilon_2}{1-\nu_2} + \dfrac{E_2}{1-\nu_2^2}\left(\dfrac{\partial u_x}{\partial x} + \nu_2\dfrac{\partial u_y}{\partial y}\right) - \dfrac{E_2}{1-\nu_2^2}\left(w_{,xx} + \nu_2 w_{,yy}\right)(z-a) & h_1 \le z \le h \end{cases}
\end{aligned} \tag{5}
$$

$$
\sigma_y = \begin{cases} \dfrac{E_1\varepsilon_1}{1-\nu_1} + \dfrac{E_1}{1-\nu_1^2}\left(\dfrac{\partial u_y}{\partial y} + \nu_1\dfrac{\partial u_x}{\partial x}\right) - \dfrac{E_1}{1-\nu_1^2}\left(w_{,yy} + \nu_1 w_{,xx}\right)(z-a) & 0 \le z \le h_1 \\ \dfrac{E_2\varepsilon_2}{1-\nu_2} + \dfrac{E_2}{1-\nu_2^2}\left(\dfrac{\partial u_y}{\partial y} + \nu_2\dfrac{\partial u_x}{\partial x}\right) - \dfrac{E_2}{1-\nu_2^2}\left(w_{,yy} + \nu_2 w_{,xx}\right)(z-a) & h_1 \le z \le h \end{cases} \tag{6}
$$

$$
\sigma_{xy} = \begin{cases} \dfrac{E_1\varepsilon_{nxy}}{1+\nu_1} - \dfrac{E_1}{1+\nu_1} w_{,xy}(z-a) & 0 \le z \le h_1 \\ \dfrac{E_2\varepsilon_{nxy}}{1+\nu_2} - \dfrac{E_2}{1+\nu_2} w_{,xy}(z-a) & h_1 \le z \le h \end{cases} \tag{7}
$$

The in-plane membrane forces are,

$$
\begin{aligned}
N_x &= \int_0^h \sigma_x dz \\
&= \frac{E_1h_1\varepsilon_1}{1-\nu_1} + \frac{E_2h_2\varepsilon_2}{1-\nu_2} + \frac{E_1h_1}{1-\nu_1^2}\left(\frac{\partial u_x}{\partial x} + \nu_1\frac{\partial u_y}{\partial y}\right) + \frac{E_2h_2}{1-\nu_2^2}\left(\frac{\partial u_x}{\partial x} + \nu_2\frac{\partial u_y}{\partial y}\right) \\
&\quad - \frac{E_1}{1-\nu_1^2}\left(w_{,xx} + \nu_1 w_{,yy}\right)\left(\frac{h_1^2}{2} - ah_1\right) - \frac{E_2}{1-\nu_2^2}\left(w_{,xx} + \nu_2 w_{,yy}\right)\left(\frac{h^2 - h_1^2}{2} - a(h-h_1)\right)
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
N_y &= \int_0^h \sigma_y dz \\
&= \frac{E_1h_1\varepsilon_1}{1-\nu_1} + \frac{E_2h_2\varepsilon_2}{1-\nu_2} + \frac{E_1h_1}{1-\nu_1^2}\left(\frac{\partial u_y}{\partial y} + \nu_1\frac{\partial u_x}{\partial x}\right) + \frac{E_2h_2}{1-\nu_2^2}\left(\frac{\partial u_y}{\partial y} + \nu_2\frac{\partial u_x}{\partial x}\right) \\
&\quad - \frac{E_1}{1-\nu_1^2}\left(w_{,yy} + \nu_1 w_{,zz}\right)\left(\frac{h_1^2}{2} - ah_1\right) - \frac{E_2}{1-\nu_2^2}\left(w_{,yy} + \nu_2 w_{,zz}\right)\left(\frac{h^2 - h_1^2}{2} - a(h-h_1)\right)
\end{aligned} \tag{9}
$$

$$N_{xy} = \int_0^h \sigma_{xy} dz$$
$$= \left(\frac{E_1 h_1}{1+\nu_1} + \frac{E_2 h_2}{1+\nu_2} \right) \varepsilon_{nxy} - w_{,xy} \left[\frac{E_1}{1+\nu_1} \left(\frac{h_1^2}{2} - a h_1 \right) + \frac{E_1}{1+\nu_2} \left(\frac{h^2 - h_1^2}{2} - a(h - h_1) \right) \right] \quad (10)$$

The location of the neutral plane should be determined such that the resultant membrane forces (N_x, N_y, N_{xy}) vanish under an arbitrary pure-bending deformation (i.e, $\varepsilon_{nx} = \varepsilon_{ny} = \varepsilon_{nxy} = 0$). This condition leads to three equations:

$$\frac{E_1}{1-\nu_1^2} \left(\frac{h_1^2}{2} - a h_1 \right) + \frac{E_2}{1-\nu_2^2} \left(\frac{h^2 - h_1^2}{2} - a(h - h_1) \right) = 0 \quad (11)$$

$$\frac{E_1 \nu_1}{1-\nu_1^2} \left(\frac{h_1^2}{2} - a h_1 \right) + \frac{E_2 \nu_2}{1-\nu_2^2} \left(\frac{h^2 - h_1^2}{2} - a(h - h_1) \right) = 0 \quad (12)$$

$$\frac{E_1}{1+\nu_1} \left(\frac{h_1^2}{2} - a h_1 \right) + \frac{E_2}{1+\nu_2} \left(\frac{h^2 - h_1^2}{2} - a(h - h_1) \right) = 0 \quad (13)$$

In general, however, a single-valued location *z* = *a* cannot satisfy all the three equations and thus an ideal neutral plane does not exist. Two special cases are noted here. First, under a cylindrical bending of the bilayered plate (e.g., $w_{,yy} = w_{,xy} = 0$), only Eq. (11) needs to be satisfied for the neutral plane, and thus the location of the neutral plane is determined:

$$a = \frac{\bar{E}_1 h_1^2 + \bar{E}_2 h^2 - \bar{E}_2 h_1^2}{2\left(\bar{E}_1 h_1 + \bar{E}_2 h - \bar{E}_2 h_1 \right)} \quad (14)$$

where $\bar{E} = E/(1-\nu^2)$ is the plane strain modulus for each layer. For a narrow beam, the location of the neutral plane is determined by replacing the plane strain modulus with Young's modulus in Eq. (14). Second, under a general two-dimensional bending, Eqs. (11-13) collapse into one when the Poisson's ratios for the two layers are identical (i.e., $\nu_1 = \nu_2$). In this case, the location of the neutral plane is uniquely determined by Eq. (14), irrespective to plane-strain or plane-stress modulus.

For the case with identical Poisson's ration, the bending moments are:

$$M_x = -\int_0^h \sigma_x (z-a)dz = -\frac{E_1\varepsilon_1}{1-\nu_1}\left(\frac{h_1^2}{2} - ah_1\right) - \frac{E_2\varepsilon_2}{1-\nu_2}\left(\frac{h^2 - h_1^2}{2} - ah_2\right)$$
$$+ w_{,xx}\left[\frac{E_1}{1-\nu_1^2}\left(\frac{(h_1-a)^3 + a^3}{3}\right) + \frac{E_2}{1-\nu_2^2}\left(\frac{(h-a)^3 - (h_1-a)^3}{3}\right)\right]$$
$$+ w_{,yy}\left[\frac{\nu_1 E_1}{1-\nu_1^2}\left(\frac{(h_1-a)^3 + a^3}{3}\right) + \frac{\nu_2 E_2}{1-\nu_2^2}\left(\frac{(h-a)^3 - (h_1-a)^3}{3}\right)\right]$$

$$M_y = -\int_0^h \sigma_y (z-a)dz = -\frac{E_1\varepsilon_1}{1-\nu_1}\left(\frac{h_1^2}{2} - ah_1\right) - \frac{E_2\varepsilon_2}{1-\nu_2}\left(\frac{h^2 - h_1^2}{2} - ah_2\right)$$
$$+ w_{,yy}\left[\frac{E_1}{1-\nu_1^2}\left(\frac{(h_1-a)^3 + a^3}{3}\right) + \frac{E_2}{1-\nu_2^2}\left(\frac{(h-a)^3 - (h_1-a)^3}{3}\right)\right]$$
$$+ w_{,xx}\left[\frac{\nu_1 E_1}{1-\nu_1^2}\left(\frac{(h_1-a)^3 + a^3}{3}\right) + \frac{\nu_2 E_2}{1-\nu_2^2}\left(\frac{(h-a)^3 - (h_1-a)^3}{3}\right)\right]$$

$$M_{xy} = \int_0^h \sigma_{xy} (z-a)dz$$
$$= w_{,xy}\left[\frac{E_1}{1+\nu_1}\left(\frac{(h_1-a)^3 + a^3}{3}\right) + \frac{E_2}{1+\nu_2}\left(\frac{(h-a)^3 - (h_1-a)^3}{3}\right)\right]$$