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CRACK PROPAGATION LAWS CORRESPONDING TO A GENERALIZED EL HADDAD EQUATION

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The El Haddad equation permits to deal simply with both short and long cracks, and we have recently suggested a generalization for finite life, defining a "finite life intrinsic crack size", as a power law of number of cycles to failure. Here, we derive the corresponding crack propagation law, finding that it shows features similar to Paris' law in the limit of long cracks, but shows some dependence of the "equivalent" C, m Paris' material's "constants" with applied stress range. The increase of crack propagation speed is obtained for short cracks, but additional size effects are derived, which may require quantitative validation, and correspond to the intrinsic difference with respect to the standard Paris' law.

Keywords: Fatigue design; Crack propagation; Critical distance approach.

Nomenclature

a_0= El Haddad intrinsic crack size
$a_0\,(N)$= "finite life" El Haddad intrinsic crack size
C, m= Paris' "material constants"
K_f = fatigue strength reduction factor
$K_f\,(N)$= "finite life" fatigue strength reduction factor
ΔK_{th} = fatigue threshold
a = notch or crack size
$\Delta\sigma$ = range of the gross nominal stress
$\Delta\sigma_\infty$ = threshold value of the gross nominal stress range according to El Haddad equation
$\Delta\sigma_L$ = plain specimen fatigue limit (in terms of stress range)
$\Delta\sigma_{EH}\,(N)$= the new "El Haddad" stress range finite life equation
$R = \sigma_{\min}/\sigma_{\max}$= stress ratio
α = geometric shape factor

1. Introduction

Fatigue is due to different processes occurring at different scales. When dealing with plain specimen, fatigue occurs at mesoscopic level at the borders between grains, and in many cases we observe simple power-law equations as in the Basquin law,

$$N_S\left(\Delta\sigma_S\right)^k = N_\infty\left[\Delta\sigma_L\right]^k = N\left[\Delta\sigma\left(N\right)\right]^k = C_W \tag{1}$$

which defines the SN curve of the *uncracked* material under stress control. Here, both N_∞ and N_S are in general "conventional" values, particularly in case there is no real "infinite life" in the SN curve. N_∞ is generally about 10^7, whereas N_S is generally taken around 10^3 cycles, in which case $\Delta\sigma_S$ should be typically taken as about $0.9\sigma_R\left(1-R\right)$.

Another limit case is that of a distinctly cracked specimen, where geometrical self-similarity induces more naturally power-law behaviour of Paris' law [Paris and Erdogan 1963], which relates the crack advancement per cycle to the range of Irwin's stress intensity factors as

$$\frac{da}{dN} = C\Delta K^m \tag{2}$$

where C, m are Paris' "material constants". Deviations near threshold or near the critical condition of static failure have been observed quite early. Then, with improvements of measuring techniques, and interest in long lives, a different behaviour for short cracks emerged, showing not only they can propagate, but actually quite fast, *below* the threshold. This behaviour received significant attention but a simple type of modelling was only possible for infinite-life. Indeed, some authors [Kitagawa and Takahashi, 1976] plotted data for non-propagating short cracks in a $\Delta\sigma - a$ diagram which showed the transition from the two limits in a simple way. The transition from short crack (fatigue-limit dominated) to long-cracks (fatigue-threshold dominated), can be shown to occur at crack sizes of the order of the El Haddad intrinsic crack size [El Haddad *et al.* 1979]

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th}}{\Delta\sigma_L}\right)^2 \tag{3}$$

where ΔK_{th} is fatigue threshold and $\Delta\sigma_L$ fatigue limit (at a given load $R-$ratio), which is of the order of $100\mu m$ for many metals (for typical values of R between 0, -1). The reason for "intrinsic crack" denomination for a_0 comes from the simple interpolating formulae for the infinite life strenght $\Delta\sigma_\infty$which adds to the real crack size

$$\Delta\sigma_\infty = \Delta K_{th}/\sqrt{\pi\left(\alpha a + a_0\right)} \tag{4}$$

where α is a geometrical factor. Perhaps no "direct approach" for modelling short cracks has become equally successful in engineering terms as this, which essentially belongs to the class of "critical distance" heuristic methods starting from the early suggestions by Neuber and Peterson for the fatigue knock-down factor K_f [Ciavarella and Meneghetti 2004].

Integration of a crack propagation law is the approach attempted in the *damage tolerance* programs, and in principle could lead to generalized "finite life" Kitagawa diagram. These approaches practically do not employ SN curves and Basquin type laws, but base their calculations upon various modifications of Paris law. This approach becomes more complicated and less reliable in the case the initial crack size is short, and is anyway quite sensitive to the Paris' constants — inevitably for high power coefficients m. A number of specialized models and software programs exist (well known are the NASA and AIRFORCE ones, NASGROW and AFGROW, respectively), which perhaps not surprisingly may give significantly differrent predictions. Alternatively, one could use the idea of splitting life into initiation and propagation, in which case there is need to specify the length of "initiated" cracks.

In recent attempts, we looked at the possibility to deal with short cracks in simplified ways, looking both at the SN plane, and in the $\Delta\sigma - a$ Kitagawa plane. In other words, we suggested that the apparent bizzarre behaviour of "short cracks" is nothing but a behaviour *intermediate* between the Wohler curve of the nominally uncracked material and the Paris' "integrated" curve or a cracked material, as an extension of the idea that, for infinite life, there is a behaviour intermediate between fatigue limit and fatigue threshold. The first two attempts interpolate between Basquin's law and Paris' law [Pugno *et al.* 2006, Ciavarella and Monno, 2006]. Considering that Paris' law constants are not true material constants [Ciavarella *et al.* 2008][a], in a third attempt [Ciavarella 2011] we attempted to avoid Paris' constants, and directly moved to extending El Haddad using only Basquin's law for the uncracked material, and a free parameter r. The basic idea was to define a finite life intrinsic crack, $a_0(N)$ as a power law increasing from the known value at infinite life (given by N_∞)

$$a_0(N) = \frac{1}{\pi}\left(\frac{\Delta K_{th}(N)}{\Delta\sigma(N)}\right)^2 = a_0\left(\frac{(N_\infty/N)^{1/r}}{(N_\infty/N)^{1/k}}\right)^2 \simeq a_0\left(\frac{N_\infty}{N}\right)^{2(1/r-1/k)} \tag{5}$$

where we have implicitely defined a "finite-life threshold" $\Delta K_{th}(N)$, also as a power law. As discussed in our previous paper [8], r turns out of the same order of Paris' plot slope, m, for which indeed [Fleck *et al.* 1994] estimate (see their Fig. 16),

$$\log\left(\frac{\Delta K_{th}}{K_{Ic}}\right) \simeq -\frac{4}{m} \tag{6}$$

[a]In particular, with larger sizes, we expect for metallic materials an increase of m and decrease of C to move towards more "static" modes of failure.

The generalized El Haddad formula was obtained therefore in the form (EHG, in the following and in the caption)

$$\frac{\Delta\sigma_{EH}(N)}{\Delta\sigma_L} = \frac{\left(\frac{N_\infty}{N}\right)^{1/r}}{\sqrt{\frac{a}{a_0} + \left(\frac{N}{N_\infty}\right)^{2/k-2/r}}} \tag{7}$$

which can be considered an implicit SN(a) curve, ie. a SN curve which depends on the initial crack size a. In the previous paper, plots are shown in the Kitagawa diagram or as a SN curve for representative cases: a "metal" $r = 3, k = 10$ or "ceramic material" $r = 10, k = 10$. In the former case the transition from the Basquin power-law, to the "Paris-like" power law is clear in the SN curves, whereas this is virtually not distinguishable in the ceramic material case.

Notice that (7) obviously returns to the well known original El Haddad equation (4) when $\frac{N}{N_\infty} = 1$. Also, for $a \to 0$ we reobtain the expected SN Basquin law, and this is also the asymptotic limit for $\frac{N}{N_\infty} \to 0$, reflecting the fact that the effect of a crack is less important at static failure than in fatigue. Finally, the asymptotic limit for $a \to \infty$, is instead

$$\frac{\Delta\sigma_{EH}(N)}{\Delta\sigma_L} = \left(\frac{N_\infty}{N}\right)^{1/r}\left(\frac{a}{a_0}\right)^{-1/2} \tag{8}$$

which gives a different size effect with respect to the more esthablished Paris' regime.

We shall obtain from this equation an equivalent Paris' law in the present note. To do so, we have to generate the general derivation procedure, starting from the classical Paris' law.

2. Crack Propagation from SN(a) Curves

Integration of the standard Paris' law (2), neglecting the change of geometric factors α with crack size, gives for a constant stress range, the number of cycles to failure as

$$N_f = \frac{a_i^{1-m/2} - a_f^{1-m/2}}{\pi^{m/2}(m/2-1)\,C\,(\alpha\Delta\sigma)^m} \simeq \frac{a_i^{1-m/2}}{\beta\,(\alpha\Delta\sigma)^m} \tag{9}$$

where a_i, a_f, are initial and final length of the crack. In the second step, we have neglected the final size of crack a_f, which is convenient for conditions not too close to static failure (i.e. for a given sufficiently large number of cycles). Also, we have used the notation $\beta = \pi^{m/2}(m/2-1)\,C$.

We can consider the obtained equation as a general function of three variables

$$F(a_i, N_f, \Delta\sigma) = a_i^{1-m/2} - \beta\,(\alpha\Delta\sigma)^m N_f = 0 \tag{10}$$

To re-obtain Paris' law, as a general method, we differentiate F, keeping constant $\Delta\sigma(N_f)$, obtaining for an increase of a_i, a decrease of the number of cycles to failure

N_f. Hence, we could write in general

$$\frac{da}{dN} = \frac{\frac{\partial}{\partial N}F}{\frac{\partial}{\partial a}F} \tag{11}$$

In fact, for the standard Paris' law, we have $\frac{\partial}{\partial a}F = (1 - m/2)\, a^{-m/2}$, and $\frac{\partial}{\partial N}$ $F = -\beta\,(\alpha\Delta\sigma)^m$. Hence, (2) follows easily.

In the EHG case, we can square (7) to obtain

$$F\,(a_i, N_f, \Delta\sigma) = \frac{a_i}{a_0} - \left(\frac{N_\infty}{N_f}\right)^{2/r}\left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2 + \left(\frac{N_f}{N_\infty}\right)^{2/k-2/r} = 0 \tag{12}$$

The differentiation of F according to (11) leads to

$$\begin{aligned}\frac{da}{dN} &= \frac{a_0}{N_\infty} f\left(\frac{\Delta\sigma_L}{\Delta\sigma}, \frac{N_\infty}{N}, r, k\right) \\ &= \frac{a_0}{N_\infty}\left[\left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2 \frac{2}{r}\left(\frac{N_\infty}{N_f}\right)^{\frac{2}{r}+1} + 2\left(\frac{1}{k} - \frac{1}{r}\right)\left(\frac{N_f}{N_\infty}\right)^{\frac{2}{k}-\frac{2}{r}-1}\right]\end{aligned} \tag{13}$$

which unfortunately we are not able to put in explicit standard form in terms of $\frac{a_i}{a_0}$ from inverting (7) or (12), except approximately or in special cases.

2.1. *Special Case $r = k$*

When $r = k$ (what we can call "ceramic" material since in this case the constants approach each other), the two equations (12) and (13) simplify and hence we can combine them into

$$\frac{da}{dN} = \frac{2a_0}{N_\infty r}\left(\frac{\Delta\sigma}{\Delta\sigma_L}\right)^r\left(\frac{a}{a_0} + 1\right)^{1+r/2} \tag{14}$$

In turn this, for small cracks, approaches a value independent on crack size

$$\left.\frac{da}{dN}\right|_{a\to 0} \to \frac{2a_0}{N_\infty k}\left(\frac{\Delta\sigma}{\Delta\sigma_L}\right)^k \tag{15}$$

which as obtained in similar approaches [5,6] gives an exponent related to Basquin's coefficient.

For large crack sizes (and large r as is indeed the case for ceramic materials), the general equation approaches approximately a Paris' like behaviour

$$\frac{da}{dN} \to \frac{2a_0}{N_\infty r}\left(\frac{\Delta K}{\Delta K_{th}}\right)^r = C\,(\Delta K)^r \tag{16}$$

but other than in this limit, there are some deviations.

2.2. *Low $\Delta\sigma$ Limit*

We start from writing the EHG law (7) or (12) in the form

$$\frac{a_i}{a_0} = \left(\frac{N_\infty}{N_f}\right)^{2/r} \left[\left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2 - \left(\frac{N_f}{N_\infty}\right)^{2/k}\right], \tag{17}$$

The second term in the parenthesis can be neglected for low applied stress ranges $\Delta\sigma << \left(\frac{N_\infty}{N_f}\right)^{1/k} \Delta\sigma_L$, where we expect no dependence on the original Basquin regime. In fact, upon rearranging (17), we reobtain the original "assumed" $\Delta K - N$ law

$$\left(\frac{\Delta K_i}{\Delta K_{th}}\right)^r \simeq \frac{N_\infty}{N_f} \tag{18}$$

In the original Kitagawa plane, we are simply looking at the lines inclined by slope -1/2 as the original threshold line. Moreover, when the first term dominate in (17), we obtain the largest $\frac{a_i}{a_0}$. Hence, we are looking, for a given $\Delta\sigma$, to large crack sizes and small N_f which in turn means moving to the right in the Kitagawa plane, towards static failure.

Substituting (18) into the crack propagation law (13), we get

$$\frac{da}{dN} = \frac{a_0}{N_\infty}\left(\frac{\Delta K_i}{\Delta K_{th}}\right)^{r+2} \left[\frac{2}{r}\left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2 + 2\left(\frac{1}{k} - \frac{1}{r}\right)\left(\frac{\Delta K_i}{\Delta K_{th}}\right)^{-\frac{2r}{k}}\right] \tag{19}$$

whose term outside the parenthesys looks similar to a Paris' law, but the dependence is not exactly on a ΔK_i because of the $\Delta\sigma$ terms. Assuming the first term dominates, the law can also be written as

$$\frac{da}{dN} = \frac{a_0}{N_\infty}\frac{2}{r}\frac{a_i}{a_0}\left(\frac{\Delta K_i}{\Delta K_{th}}\right)^r = \frac{a_0}{N_\infty}\frac{2}{r}\left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2\left(\frac{\Delta K_i}{\Delta K_{th}}\right)^{r+2} \tag{20}$$

which is a Paris law with an additional dependence on crack size, or on $\Delta\sigma$. In particular, it would seem the crack propagation rate decrease with short cracks, but it should be borne in mind this is a limit case for low $\Delta\sigma$ and hence cannot be estrapolated for low $\frac{a_i}{a_0}$. Instead, considering large sizes, the law is the correct limit of our new equation, and it does predict an increase of propagation speed with size (or equivalently, a decrease with $\Delta\sigma$). Compared to the classical results of Barenblatt and Botvina (recently reviewed in [7]), this is not entirely unreasonable, although more often it is expected that both the exponent m of the law and the coefficient C should change to move towards more "static" modes of failure. Practical ranges of this effect will be seen in the example plots.

3. A Remark on Crack Lenght

Finally, we give a small remark on integrating Paris' law to get a current evaluation of the crack lenght, from equations like (10,12). These effectively return the number of cycles to failure N_f for a given initial crack size a_i. Since instead we are looking for the actual crack size as a number of cycles N, we effectively need to consider that for every given remaining cycles to failure, we have consumed the number N of cycles. Hence, in the case of the new EHG

$$F\left(a, N_f - N, \Delta\sigma\right) = \frac{a}{a_0} - \left(\frac{N_\infty}{N_f - N}\right)^{2/r} \left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2 + \left(\frac{N_f - N}{N_\infty}\right)^{2/k-2/r} = 0 \quad (21)$$

or else

$$\frac{a}{a_0} = \left(\frac{N_\infty}{N_f - N}\right)^{2/r} \left(\frac{\Delta\sigma_L}{\Delta\sigma}\right)^2 - \left(\frac{N_f - N}{N_\infty}\right)^{2/k-2/r} \quad (22)$$

where the N_f has to be computed implicitely from the same function. Alternatively, one can fix N_f and hence compute $a_{i.}$

4. Example Plots

In order to plot the obtained equation da/dN, we suggest the following procedure which avoids solving numerically non-linear equations

1) Fix $\left(\frac{\Delta\sigma_L}{\Delta\sigma}, \frac{N_\infty}{N}, r, k\right)$;

2) Compute independently the dimensionless quantities $\frac{a}{a_0}$ from (12), and $\frac{da}{dN}/\frac{a_0}{N_\infty} = f$ from (13);

3) define pairs $(\Delta\sigma, a)$ in the Kitagawa diagram between the curves for infinite life, and static life or at least, above the El Haddad infinite life limit. The latter implies for a given $\Delta\sigma$, that $a > a_{EH}$ where a_{EH} is from the original El Haddad equation (4).

4) we can plot the crack propagation law as $\frac{da}{dN}/\frac{a_0}{N_\infty}$ as a function of

$$\frac{\Delta K}{\Delta K_{th}} = \frac{\alpha\Delta\sigma\sqrt{\pi a}}{\alpha\Delta\sigma_L\sqrt{\pi a_0}} = \frac{\Delta\sigma}{\Delta\sigma_L}\sqrt{\frac{a}{a_0}}$$

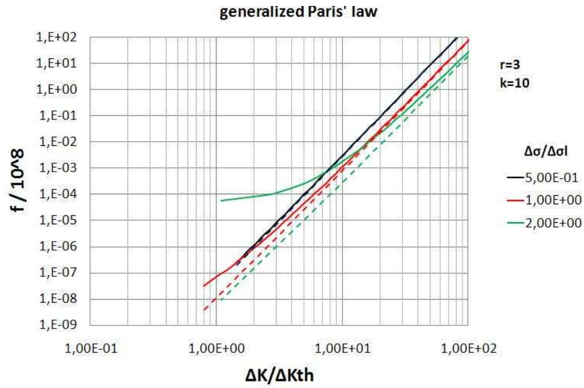
Figure 1 shows a couple of examples where we plot for convenience, $f/10^8 = \frac{da}{dN}/\left(10^8\frac{a_0}{N_\infty}\right)$ since we expect $\frac{a_0}{N_\infty} \sim 10^{-8} mm/cycle$ in many cases ($a_0 \sim 100\mu m = 0.1mm$ and $N_\infty = 10^7$). Hence,

$$\frac{da}{dN} = \left(\frac{a_0}{N_\infty}10^8\right)\frac{f}{10^8} \quad (23)$$

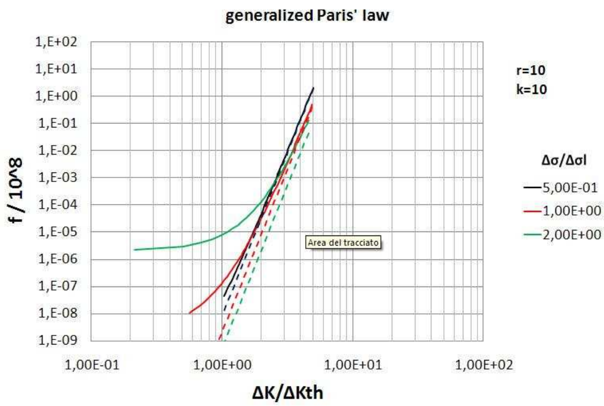
This way, apart from small deviations, the scale is directly in mm/cycles.

In Fig. 1a we have a case of metallic material with $r = 3$ and $k = 10$. Instead, in Fig. 1b we have a "ceramic"-like material, with $r = k = 10$. We have used three values of $\frac{\Delta\sigma}{\Delta\sigma_L} = 0.5, 1, 2$ to span reasonably the range of interesting values. We clearly see the following results:-

- for the low value of $\frac{\Delta\sigma}{\Delta\sigma_L} = 0.5$, the agreement between the approximate eqt.(19) in dashed lines and the full new crack propagation law is perfect, and the curve is anyway very close to a power law like Paris', all the way



(a)



(b)

Fig. 1. The generalized Paris' law $f/10^8 = \frac{da}{dN} / \left(10^8 \frac{a_0}{N_\infty}\right)$ for a case with $r = 3, k = 10$ (a), and for another with $r = k = 10$. Dashed lines correspond to approx.eqt.(19)

down to the truncation at $\Delta K/\Delta K_{th} \sim 1$ (or, more precisely, in correspondence to the El Haddad equation);

- for higher values of $\frac{\Delta\sigma}{\Delta\sigma_L}$, the curve tends to show a marked increase at low ΔK range, similarly to Pugno *et al.*[5] and qualitatively in agreement with experiments. However, there is also a smaller but sensible decrease in the high ΔK range, which is not equally expected. This decrease is of 1 order of magnitude in crack propagation speed moving from $\frac{\Delta\sigma}{\Delta\sigma_L} = 0.5$ to $\frac{\Delta\sigma}{\Delta\sigma_L} = 2$ and is the core effect of assuming the generalized El Haddad law, instead of a Paris curve.
- because of the normalization used, we see reasonable agreement with the usual finding, reported for example in Fleck *et al.*[9], i.e. that the fatigue thereshold corresponds to a propagation below 10^{-7} mm/cycles, and notice

that our equation provides this approximately for both ranges of r, without the need of changing any adjusting factor.

It is not expected that this form of crack propagation law could be sufficiently accurate in general, since we can assume that in its range of validity, the "proper" Paris' law would describe more accurately the data. However, the present law is obtained differently, and could be used in the range where the "proper" Paris law would *also* show some uncertainties. This note is aimed at showing the type of crack propagation law implicit in assuming the El Haddad generalized equation.

Notice in particular that the figures span a range of both x and y-axes too large in practice, where typically we measure up to 5 orders of magnitude in crack propagation speeds, and perhaps 1 or 2 orders between threshold and critical stress intensity factor range. Hence the figures are only illustrative.

5. Conclusions

We have recently generalized the El Haddad equation to finite life, avoiding Paris' law constants. The implication in terms of crack propagation law are examined here, finding that the resulting crack propagation law is a power law of the ΔK, only in the special case $r = k$ and in the limit of large crack. For low $\Delta\sigma$ we also derived a power law equation of ΔK, which however additionaly depends on the crack size a, predicting an increase of propagation rate with scale, which is qualitatively possible. At the other extreme, of high $\Delta\sigma$ load range, there is a large increase of crack propagation rate for short cracks. These effects require further investigations, and correspond to the ranges where deviation from Paris law is intrinsic in the assumed law.

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