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In a recent letter [1], Fontana and Palffy-Muhoray proposed a connection between materials failure statistics and the St. Petersburg paradox by linking the average load carrying capacity of a wire to its length. The result, however, was derived assuming that "the force required to fracture the fiber is a linear function of the defect size" [1], which is in glaring contrast with fracture mechanics. Here we address the problem combining extreme value theory (EVT) [2] with the Griffith's stability crack criterion [3]. According to the Griffith's assumption, the failure stress should be inversely proportional to the square root of the largest defect size. We also show that in the asymptotic limit, the wire strength follows the Gumbel's distribution, in full agreement with the data reported in [1], as we demonstrate using the maximum likelihood method. We thus conclude that the load carrying capacity of the wires studied in [1] follows EVT, in agreement with previous observations for different materials [2].

We consider a wire of length L which we divide in $N = L/L_0$ independent elements of size L_0. Following Ref. [1], we want to relate the statistics of the micro-cracks present in the wire with its failure strength. Defining $P(n)$ as the probability density function (pdf) of micro-cracks of length $w \equiv nL_0$, $F(z) = \int_0^z dnP(n)$ is the probability that no micro-crack larger than z will be found in the wire. We then define n_{max} as the largest micro-crack in the wire, with the only constraint that $w_{max} = n_{max}L_0 \ll L$ [4]. If $\rho_N(n_{max})$ is the pdf for the largest micro-crack, then $F_N(z) = \int_0^z dn_{max}\, \rho_N(n_{max}) = [F(z)]^N$. The Fisher-Gnedenko-Tippet theorem ensures that $F_N(z) \to G(z)$ for large N, where G belongs to one of three families only: Weibull, Fréchet or Gumbel [5, 6]. The convergence to either one of these universal distributions depends on the asymptotic properties of $P(n)$ [7, 8]. If the distribution of micro-cracks has an exponential tail [4, 8, 9], $G(z)$ converges asymptotically to the Gumbel distribution [10]. To derive the fracture strength, the authors of Ref. [1] assume that it is linearly dependent on the size of a defect, obtained through the St. Petersburg model, but this assumption is not justified by fracture mechanics. A relation between crack length and fracture strength in an elastic medium is provided by the Griffith's stability criterion, for which a crack of length w subject to a normal stress σ is stable as long as $\sigma < K_{1C}/Y\, w^{-1/2}$[3], where K_{1C} is the critical stress intensity factor and Y is a geometric factor. In our context, the wire should break when the largest micro-crack becomes unstable, hence the probability that a wire of length L does not fail under a stress σ is given by $\Sigma_L(\sigma) \sim \exp\left[-L/L_0\, e^{-(\sigma_0/\sigma)^2}\right]$. This is the Duxbury-Leath-Beale distribution [4], which was shown to converge to the Gumbel distribution as $L \gg L_0$, i.e. $\Sigma_L(\sigma) \to \exp\left[-L/L_0\, e^{(\sigma-\mu)/\beta}\right]$ [8]. The average breaking stress is then given by $\langle\sigma\rangle = \beta\left[\gamma - \ln\left(L/L_0\right)\right] + \mu$ which recovers Eq.(1) of [1]: γ is the Euler-Mascheroni constant and β and μ are Gumbel's parameters.

The tensile experiments performed in [1] on polyester and polyamide wires, corroborate the extreme value statistics over 6 order of magnitude. We fitted the data using the Gumbel form of the generalized EVT, which is purposely designed to account for strain and thermal effects (Fig.1 main panels) [2, 11]. Since the experiments were performed with different strain rate for each sample size, the statistical analysis can only be performed using the maximum likelihood method for parameters estimation [2]. Our fit would indicate a very small strain rate dependence in the parameters in agreement with the experiments on polyester (inset Fig.1a). The experiments on polyamide indicate a strain-rate dependence that is not captured by the fit. A possible reason for this discrepancy is that the precise value of the strain-rate is not known and can only be estimated indirectly by $1/t_{rup}$, as also acknowledged by the authors [12].

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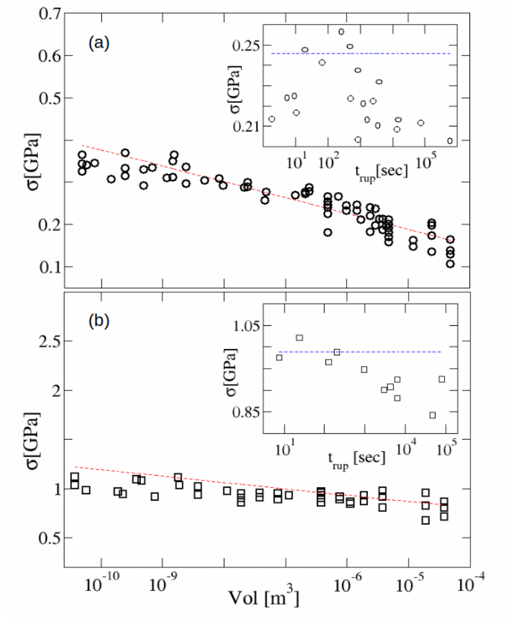

FIG. 1. Main panels: fracture stresses of polyester (a) and polyamide (b) fibers from the experiments in [1] (Fig.2). The maximum likelihood estimates were used to calculate $\langle\sigma\rangle$ (red dashed line). Insets: breaking stresses as a function of the rupture times (Fig.3 in [1]). Dashed blue lines are $\langle\sigma\rangle$, evaluated using the same parameters values fitted in the main panels, and using $1/t_{rup}$ as strain rate.