On the Compatibility Equations of Nonlinear and Linear Elasticity in the Presence of Boundary Conditions*

Arzhang Angoshtari† Arash Yavari‡

August 10, 2015

Abstract

We use Hodge-type orthogonal decompositions for studying the compatibility equations of the displacement gradient and the linear strain with prescribed boundary displacements. We show that the displacement gradient is compatible if and only if for any equilibrated virtual first-Piola Kirchhoff stress tensor field, the virtual work done by the displacement gradient is equal to the virtual work done by the prescribed boundary displacements. This condition is very similar to the classical compatibility equations for the linear strain. Since these compatibility equations for linear and nonlinear strains involve infinite-dimensional spaces and consequently are not easy to use in practice, we derive alternative compatibility equations, which are written in terms of some finite-dimensional spaces and are more useful in practice. Using these new compatibility equations, we present some non-trivial examples that show that compatible strains may become incompatible in the presence of prescribed boundary displacements.

Contents

1 Introduction

The classical compatibility problems of nonlinear and linear elasticity seek to determine the necessary and sufficient conditions that guarantee the existence of a displacement field that generates a given strain field. These compatibility problems have been the subject of various studies during the past two centuries, for example see [1] and references therein. The above statement of the compatibility problem ignores the fact that the values of the displacement is usually prescribed on the whole or parts of the boundary. Therefore, it is more realistic to consider the following compatibility problem:

> *Given a strain field and prescribed displacement values on (a part of) the boundary, determine the necessary and sufficient conditions for the existence of a displacement field that generates the given strain and satisfies the given boundary conditions.*

---

*To appear in *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*.

†School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332. E-mail: arzhang@gatech.edu

‡School of Civil and Environmental Engineering & The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332. E-mail: arash.yavari@ce.gatech.edu.

Dorn and Schild [2] and Rostamian [3] showed that a linear strain with a prescribed boundary displacement is compatible if and only if for any equilibrated virtual stress, the virtual work of the linear strain is the same as the virtual work of the prescribed boundary displacement.[1] Note that regardless of the topological properties of bodies, the space of equilibrated Cauchy stresses is infinite-dimensional.[2] Therefore, it is not clear how to use the above condition in practice, as one would need to check this condition for an infinite number of virtual stresses.

Main Results. In this paper, we show that a compatibility condition similar to the above condition for the linear strain also determines the compatibility of displacement gradients with prescribed boundary displacements (Remark 9).[3] We show that for both displacement gradients and linear strains, it is possible to write the compatibility equations in terms of only a finite-dimensional subspace of equilibrated stresses (Theorems 7 and 15). These new compatibility conditions are more practical as one only needs to verify them for a finite number of virtual stresses. This finite number is determined by the topological properties of domains and the regions on which boundary displacements are imposed. As an application of these compatibility equations, we present some non-trivial compatible strains that are incompatible in the presence of boundary conditions (Examples 10, 12, and 18).

Contents of the paper. The compatibility problem for the displacement gradient can be stated in terms of a boundary-value problem for differential forms. In §2, we study this boundary-value problem by using orthogonal decompositions. Since we may want to impose boundary conditions only on a part of the boundary and not necessarily on the whole boundary, we need to appropriately extend some standard results. In particular, we extend the classical Friedrichs decompositions in Theorem 1. This allows us to extend the Hodge-Morrey-Friedrichs decomposition of differential forms as well. Using this decomposition, we derive two sets of equivalent integrability conditions for the boundary-value problem (Theorems 3 and 4). In §3, we derive the compatibility equations for nonlinear and linear strains. First, we directly use the results of §2 to write the compatibility equations for the displacement gradient. Then, we exploit some Hodge-type decompositions for symmetric tensors and arguments similar to those of §2 to obtain the compatibility equations for the linear strain. Finally, in §4, we show that the results of §2 can be used to derive the necessary and sufficient conditions for the existence of a (first Piola-Kirchhoff or Cauchy) stress tensor that is in equilibrium with a given body force field and takes prescribed values of boundary tractions.

Notation. We assume a body $\mathcal{B} \subset \mathbb{R}^n$, $n = 2, 3$, is an open subset, which is bounded and connected and has a smooth boundary $\partial\mathcal{B}$. We also assume that $\partial\mathcal{B} = \mathcal{S}_1 \cup \mathcal{S}_2$, where the disjoint subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are either empty or $(n-1)$-dimensional surfaces without boundary.[4] The closure of $\mathcal{B}$ is denoted by $\bar{\mathcal{B}}$. The Cartesian coordinates, the standard orthonormal basis, and the standard inner product of $\mathbb{R}^n$ are denoted by $\{X^I\}$, $\{\mathbf{E}_I\}$, and $\langle\!\langle , \rangle\!\rangle$, respectively. For any non-negative integer s, the spaces of vector fields, $\binom{2}{0}$-tensors, and symmetric $\binom{2}{0}$-tensors of Sobolev class H^s ($L^2 := H^0$) are denoted by $H^s(T\mathcal{B})$, $H^s(\otimes^2 T\mathcal{B})$, and $H^s(S^2 T\mathcal{B})$ (the Cartesian components of H^s-fields belong to the standard Sobolev space $H^s(\mathcal{B})$). The partial derivative $\partial f/\partial X^I$ is denoted by $f_{,I}$. The subspace H^s_0 is the space of H^s-tensors with all partial derivatives of their components of order $< s$ vanishing on $\partial\mathcal{B}$. For any $\boldsymbol{U}, \boldsymbol{Z} \in L^2(T\mathcal{B})$ and $\boldsymbol{R}, \boldsymbol{T} \in L^2(\otimes^2 T\mathcal{B})$, we have the inner products $\langle\!\langle \boldsymbol{U}, \boldsymbol{Z} \rangle\!\rangle_{L^2} := \sum_I \langle\!\langle U^I, Z^I \rangle\!\rangle_{L^2}$, and $\langle\!\langle \boldsymbol{R}, \boldsymbol{T} \rangle\!\rangle_{L^2} := \sum_{I,J} \langle\!\langle R^{IJ}, T^{IJ} \rangle\!\rangle_{L^2}$. The summation convention is assumed on repeated indices.

[1] Dorn and Schild [2] derived this condition for simply-connected bodies. Later, Rostamian [3] showed that the same condition works for non-simply-connected bodies as well.

[2] The space of harmonic fields is a subspace of the space of divergence-free fields. It is well-known that on compact domains with boundary, the former space is infinite-dimensional, e.g. see [4, Theorem 3.4.2].

[3] We have $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{K}$, where $\boldsymbol{F}$ and $\boldsymbol{K}$ are the deformation gradient and the displacement gradient of a motion, respectively. Therefore, by replacing $\boldsymbol{K}$ with $\boldsymbol{F} - \boldsymbol{I}$ in the compatibility equations of $\boldsymbol{K}$, one can obtain the compatibility equations of $\boldsymbol{F}$.

[4] These assumptions are used to avoid some technical difficulties. We believe that by considering some modifications regarding the smoothness of solutions, the results of this paper are still valid under less restrictive assumptions that $\mathcal{B}$ has a locally Lipschitz boundary and that $\mathcal{S}_1$ and $\mathcal{S}_2$ are admissible patches in the sense of [5, page 377], which allows $\mathcal{S}_1$ and $\mathcal{S}_2$ to have non-empty boundaries.

2 A Boundary-Value Problem for Differential Forms

The compatibility problem for the displacement gradient can be stated in terms of a boundary-value problem for differential forms. In this section, we derive some necessary and sufficient conditions for the existence of a solution to this problem. Let $L^2(\Lambda^k T^*\mathcal{B})$ and $H^s(\Lambda^k T^*\mathcal{B})$ denote the standard Sobolev spaces of k-forms of class L^2 and H^s, which are equipped with the inner products $\langle\!\langle , \rangle\!\rangle_{L^2}$ and $\langle\!\langle , \rangle\!\rangle_{H^s}$.[5] At the boundary $\partial\mathcal{B}$, any $\boldsymbol{\alpha} \in H^{s+1}(\Lambda^k T^*\mathcal{B})$ can be uniquely decomposed as $\boldsymbol{\alpha}|_{\partial\mathcal{B}} = \mathsf{t}\boldsymbol{\alpha} + \mathsf{n}\boldsymbol{\alpha}$, where $\mathsf{t}\boldsymbol{\alpha}$ ($\mathsf{n}\boldsymbol{\alpha}$) is called the tangent (normal) component of $\boldsymbol{\alpha}$ at $\partial\mathcal{B}$. One can show that $\mathsf{t}\boldsymbol{\alpha}$ and $\mathsf{n}\boldsymbol{\alpha}$ belong to the fractional Sobolev space $H^{s+\frac{1}{2}}(\Lambda^k T^*\mathcal{B}|_{\partial\mathcal{B}})$ [4, Theorem 1.3.7]. The exterior derivative d and the codifferential operator δ can be extended to the continuous mappings $d : H^{s+1}(\Lambda^k T^*\mathcal{B}) \to H^s(\Lambda^{k+1} T^*\mathcal{B})$ and $\delta : H^{s+1}(\Lambda^k T^*\mathcal{B}) \to H^s(\Lambda^{k-1} T^*\mathcal{B})$. We will study the following boundary-value problem:

Given $\boldsymbol{\gamma} \in H^1(\Lambda^{k+1} T^*\mathcal{B})$ *and* $\boldsymbol{\eta} \in H^{\frac{3}{2}}(\Lambda^k T^*\mathcal{B}|_{\mathcal{S}_1})$*, find* $\boldsymbol{\alpha} \in H^2(\Lambda^k T^*\mathcal{B})$ *such that*

$$\begin{aligned} d\boldsymbol{\alpha} &= \boldsymbol{\gamma}, \quad \text{on } \mathcal{B}, \\ \mathsf{t}\boldsymbol{\alpha} &= \mathsf{t}\boldsymbol{\eta}, \quad \text{on } \mathcal{S}_1. \end{aligned} \tag{2.1}$$

Our main tool is the Hodge-Morrey decomposition, which states that any $\boldsymbol{\alpha} \in H^1(\Lambda^k T^*\mathcal{B})$ admits the following L^2-orthogonal decomposition:

$$\boldsymbol{\alpha} = d\boldsymbol{\phi}_{\boldsymbol{\alpha}} + \delta\boldsymbol{\psi}_{\boldsymbol{\alpha}} + \boldsymbol{\lambda}_{\boldsymbol{\alpha}}, \tag{2.2}$$

where $d\boldsymbol{\phi}_{\boldsymbol{\alpha}}, \delta\boldsymbol{\psi}_{\boldsymbol{\alpha}}, \boldsymbol{\lambda}_{\boldsymbol{\alpha}} \in H^1(\Lambda^k T^*\mathcal{B})$, with

$$\begin{aligned} \boldsymbol{\phi}_{\boldsymbol{\alpha}} &\in H^1_{\mathrm{n}}(\Lambda^{k-1} T^*\mathcal{B}, \partial\mathcal{B}) := \left\{\boldsymbol{\beta} \in H^1(\Lambda^{k-1} T^*\mathcal{B}) : \ \mathsf{t}\boldsymbol{\beta}|_{\partial\mathcal{B}} = 0\right\}, \\ \boldsymbol{\psi}_{\boldsymbol{\alpha}} &\in H^1_{\mathrm{t}}(\Lambda^{k+1} T^*\mathcal{B}, \partial\mathcal{B}) := \left\{\boldsymbol{\beta} \in H^1(\Lambda^{k+1} T^*\mathcal{B}) : \ \mathsf{n}\boldsymbol{\beta}|_{\partial\mathcal{B}} = 0\right\}, \\ \boldsymbol{\lambda}_{\boldsymbol{\alpha}} &\in \mathcal{H}^k(\bar{\mathcal{B}}) := \left\{\boldsymbol{\lambda} \in H^1(\Lambda^k T^*\mathcal{B}) : d\boldsymbol{\lambda} = 0 \text{ and } \delta\boldsymbol{\lambda} = 0\right\}. \end{aligned}$$

We also use decompositions of harmonic fields $\mathcal{H}^k(\bar{\mathcal{B}})$, which can be derived using Dirichlet-Neumann (DN) potentials. These potentials are certain solutions of the following boundary-value problem for the Laplacian $\Delta := d \circ \delta + \delta \circ d$:

Given $\boldsymbol{\gamma} \in L^2(\Lambda^k T^*\mathcal{B})$*, find a k-form* $\boldsymbol{\mu}$ *such that*

$$\begin{aligned} \Delta\boldsymbol{\mu} &= \boldsymbol{\gamma}, && \text{on } \mathcal{B}, \\ \mathsf{t}\boldsymbol{\mu} &= 0, \ \mathsf{t}(\delta\boldsymbol{\mu}) = 0, && \text{on } \mathcal{S}_1, \\ \mathsf{n}\boldsymbol{\mu} &= 0, \ \mathsf{n}(d\boldsymbol{\mu}) = 0, && \text{on } \mathcal{S}_2. \end{aligned} \tag{2.3}$$

The problem (2.3) is an elliptic boundary-value problem [7, page 330]. Let $\mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ be the space of solutions of (2.3) for $\boldsymbol{\gamma} = 0$. Standard results for elliptic boundary-value problems suggest that $\mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional and contains only C^∞ forms. In particular, one can show that

$$\dim \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) = b_k(\bar{\mathcal{B}}, \mathcal{S}_1),$$

where $b_k(\bar{\mathcal{B}}, \mathcal{S}_1)$ is the k-th relative Betti number of the pair $(\bar{\mathcal{B}}, \mathcal{S}_1)$ [5, Theorems 5.3 and 6.1]. Green's formula for differential forms states that for any $\boldsymbol{\alpha} \in H^1(\Lambda^k T^*\mathcal{B})$ and $\boldsymbol{\beta} \in H^1(\Lambda^{k+1} T^*\mathcal{B})$, we have (e.g. see [4, page 60])

$$\langle\!\langle d\boldsymbol{\alpha}, \boldsymbol{\beta} \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{\alpha}, \delta\boldsymbol{\beta} \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \mathsf{t}\boldsymbol{\alpha} \wedge (*\mathsf{n}\boldsymbol{\beta}).$$

[5] We refer the reader to Morrey [6] and Schwarz [4] for the definition of these spaces and also for more discussions on notions that we use throughout this section.

Using Green's formula, it is straightforward to show that

$$\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)=\mathcal{H}^k(\bar{\mathcal{B}})\cap H^1_{\mathrm{n}}(\Lambda^kT^*\mathcal{B},\mathcal{S}_1)\cap H^1_{\mathrm{t}}(\Lambda^kT^*\mathcal{B},\mathcal{S}_2). \tag{2.4}$$

Since $\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$ is finite-dimensional, it is a closed subspace of L^2 forms and hence one can write the following orthogonal decomposition

$$L^2(\Lambda^kT^*\mathcal{B})=\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)\oplus\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp,$$

where $\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp$ is the orthogonal complement of $\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$ in $L^2(\Lambda^kT^*\mathcal{B})$. The necessary and sufficient condition for the existence of a solution to (2.3) is $\boldsymbol{\gamma}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp$ [5, Theorem 6.1]. The DN-potential of any $\boldsymbol{\gamma}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp$ is defined to be the unique solution $\boldsymbol{\mu}_{\boldsymbol{\gamma}}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp$ of (2.3) associated to $\boldsymbol{\gamma}$. Next, we derive some orthogonal decompositions for $\mathcal{H}^k(\bar{\mathcal{B}})$.

Theorem 1. *The following L^2-orthogonal decompositions hold:*

$$\mathcal{H}^k(\bar{\mathcal{B}})=\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)\oplus\mathcal{H}^k_\delta(\bar{\mathcal{B}},\mathcal{S}_2)\oplus\mathcal{H}^k_d(\bar{\mathcal{B}},\mathcal{S}_1), \tag{2.5a}$$
$$\mathcal{H}^k(\bar{\mathcal{B}})=\mathcal{H}^k_{\mathrm{t}}(\bar{\mathcal{B}},\mathcal{S}_2)\oplus\mathcal{H}^k_d(\bar{\mathcal{B}},\mathcal{S}_1), \tag{2.5b}$$
$$\mathcal{H}^k(\bar{\mathcal{B}})=\mathcal{H}^k_{\mathrm{n}}(\bar{\mathcal{B}},\mathcal{S}_1)\oplus\mathcal{H}^k_\delta(\bar{\mathcal{B}},\mathcal{S}_2), \tag{2.5c}$$

where

$$\begin{aligned}
\mathcal{H}^k_\delta(\bar{\mathcal{B}},\mathcal{S}_2)&:=\left\{\boldsymbol{\lambda}\in\mathcal{H}^k(\bar{\mathcal{B}}):\boldsymbol{\lambda}=\delta\boldsymbol{\zeta},\ \mathsf{n}\boldsymbol{\zeta}|_{\mathcal{S}_2}=0\right\},\\
\mathcal{H}^k_d(\bar{\mathcal{B}},\mathcal{S}_1)&:=\left\{\boldsymbol{\lambda}\in\mathcal{H}^k(\bar{\mathcal{B}}):\boldsymbol{\lambda}=d\boldsymbol{\omega},\ \mathsf{t}\boldsymbol{\omega}|_{\mathcal{S}_1}=0\right\},\\
\mathcal{H}^k_{\mathrm{t}}(\bar{\mathcal{B}},\mathcal{S}_2)&:=\left\{\boldsymbol{\lambda}\in\mathcal{H}^k(\bar{\mathcal{B}}):\mathsf{n}\boldsymbol{\lambda}|_{\mathcal{S}_2}=0\right\},\\
\mathcal{H}^k_{\mathrm{n}}(\bar{\mathcal{B}},\mathcal{S}_1)&:=\left\{\boldsymbol{\lambda}\in\mathcal{H}^k(\bar{\mathcal{B}}):\mathsf{t}\boldsymbol{\lambda}|_{\mathcal{S}_1}=0\right\}.
\end{aligned}$$

Proof. We first prove the decomposition (2.5a). For any $\boldsymbol{\xi}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$, $\delta\boldsymbol{\zeta}\in\mathcal{H}^k_\delta(\bar{\mathcal{B}},\mathcal{S}_2)$, and $d\boldsymbol{\omega}\in\mathcal{H}^k_d(\bar{\mathcal{B}},\mathcal{S}_1)$, one can use Green's formula to write

$$\begin{aligned}
\langle\!\langle\boldsymbol{\xi},\delta\boldsymbol{\zeta}\rangle\!\rangle_{L^2}&=\langle\!\langle d\boldsymbol{\xi},\boldsymbol{\zeta}\rangle\!\rangle_{L^2}-\int_{\mathcal{S}_1}\mathsf{t}\boldsymbol{\xi}\wedge*\mathsf{n}\boldsymbol{\zeta}-\int_{\mathcal{S}_2}\mathsf{t}\boldsymbol{\xi}\wedge*\mathsf{n}\boldsymbol{\zeta}=0,\\
\langle\!\langle\boldsymbol{\xi},d\boldsymbol{\omega}\rangle\!\rangle_{L^2}&=\langle\!\langle\delta\boldsymbol{\xi},\boldsymbol{\omega}\rangle\!\rangle_{L^2}+\int_{\mathcal{S}_1}\mathsf{t}\boldsymbol{\omega}\wedge*\mathsf{n}\boldsymbol{\xi}+\int_{\mathcal{S}_2}\mathsf{t}\boldsymbol{\omega}\wedge*\mathsf{n}\boldsymbol{\xi}=0,\\
\langle\!\langle d\boldsymbol{\omega},\delta\boldsymbol{\zeta}\rangle\!\rangle_{L^2}&=\langle\!\langle\boldsymbol{\omega},\delta\delta\boldsymbol{\zeta}\rangle\!\rangle_{L^2}+\int_{\mathcal{S}_1}\mathsf{t}\boldsymbol{\omega}\wedge*\mathsf{n}(\delta\boldsymbol{\zeta})+\int_{\mathcal{S}_2}\mathsf{t}\boldsymbol{\omega}\wedge*\mathsf{n}(\delta\boldsymbol{\zeta})=0,
\end{aligned}$$

where in the last relation, we used the fact that $\mathsf{n}(\delta\boldsymbol{\zeta})=\delta(\mathsf{n}\boldsymbol{\zeta})=0$, on $\mathcal{S}_2$. This means that the components of (2.5a) are mutually L^2-orthogonal. Since $\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$ is finite-dimensional, we have the L^2-orthogonal decomposition $\overline{\mathcal{H}^k(\bar{\mathcal{B}})}=\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)\oplus\mathbb{D}$, where $\overline{\mathcal{H}^k(\bar{\mathcal{B}})}$ is the L^2-closure of $\mathcal{H}^k(\bar{\mathcal{B}})$, and $\mathbb{D}:=\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp\cap\overline{\mathcal{H}^k(\bar{\mathcal{B}})}$. Since $\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$ contains only smooth forms, one can also write the L^2-orthogonal decomposition

$$\mathcal{H}^k(\bar{\mathcal{B}})=\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)\oplus\widetilde{\mathbb{D}}, \tag{2.6}$$

with

$$\widetilde{\mathbb{D}}:=\mathbb{D}\cap H^1(\Lambda^kT^*\mathcal{B})=\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp\cap\mathcal{H}^k(\bar{\mathcal{B}}).$$

Let $\boldsymbol{\nu}\in\widetilde{\mathbb{D}}$ and suppose $\boldsymbol{\mu}_{\boldsymbol{\nu}}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp$ is the DN-potential of $\boldsymbol{\nu}$. We define $\boldsymbol{\zeta}_{\boldsymbol{\nu}}:=d\boldsymbol{\mu}_{\boldsymbol{\nu}}$. The boundary conditions of $\boldsymbol{\mu}_{\boldsymbol{\nu}}$ imply that $\mathsf{n}\boldsymbol{\zeta}_{\boldsymbol{\nu}}|_{\mathcal{S}_2}=\mathsf{n}(d\boldsymbol{\mu}_{\boldsymbol{\nu}})|_{\mathcal{S}_2}=0$. Moreover, for any $\boldsymbol{\xi}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$, we have

$$\langle\!\langle\boldsymbol{\nu}-\delta\boldsymbol{\zeta}_{\boldsymbol{\nu}},\boldsymbol{\xi}\rangle\!\rangle_{L^2}=-\langle\!\langle\boldsymbol{\zeta}_{\boldsymbol{\nu}},d\boldsymbol{\xi}\rangle\!\rangle_{L^2}+\int_{\mathcal{S}_1}\mathsf{t}\boldsymbol{\xi}\wedge*\mathsf{n}\boldsymbol{\zeta}_{\boldsymbol{\nu}}+\int_{\mathcal{S}_2}\mathsf{t}\boldsymbol{\xi}\wedge*\mathsf{n}\boldsymbol{\zeta}_{\boldsymbol{\nu}}=0,$$

and hence $\boldsymbol{\nu}-\delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}\in\mathcal{H}^k(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)^\perp$. We also have $\boldsymbol{\nu}-\delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}\in\mathcal{H}^k(\bar{\mathcal{B}})$, since

$$\begin{aligned}
d\left(\boldsymbol{\nu}-\delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}\right)&=d\circ d\circ\delta\boldsymbol{\mu}_{\boldsymbol{\nu}}=0,\\
\delta\left(\boldsymbol{\nu}-\delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}\right)&=\delta\boldsymbol{\nu}-\delta\circ\delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}=0.
\end{aligned}$$

Note that $\boldsymbol{\nu} - \delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}$ satisfies the following boundary conditions:

$$\begin{aligned}
&\mathsf{t}\,(\boldsymbol{\nu} - \delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}) = \mathsf{t}\,(d\circ\delta\boldsymbol{\mu}_{\boldsymbol{\nu}}) = d\,(\mathsf{t}(\delta\boldsymbol{\mu}_{\boldsymbol{\nu}})) = 0, \quad &&\text{on } \mathcal{S}_1,\\
&\mathsf{n}\,(\boldsymbol{\nu} - \delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}) = \mathsf{n}\boldsymbol{\nu}, &&\text{on } \mathcal{S}_2.
\end{aligned}$$

Let $\boldsymbol{\mu}_{\widehat{\boldsymbol{\nu}}}$ be the DN-potential of $\widehat{\boldsymbol{\nu}} := \boldsymbol{\nu} - \delta\boldsymbol{\zeta}_{\boldsymbol{\nu}}$, and let $\boldsymbol{\omega}_{\boldsymbol{\nu}} := \delta\boldsymbol{\mu}_{\widehat{\boldsymbol{\nu}}}$. Then $\mathsf{t}\boldsymbol{\omega}_{\boldsymbol{\nu}}|_{\mathcal{S}_1} = 0$, and

$$\begin{aligned}
&\langle\!\langle \widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}}, \boldsymbol{\xi}\rangle\!\rangle_{L^2} = -\int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\omega}_{\boldsymbol{\nu}} \wedge *\mathsf{n}\boldsymbol{\xi} - \int_{\mathcal{S}_2} \mathsf{t}\boldsymbol{\omega}_{\boldsymbol{\nu}} \wedge *\mathsf{n}\boldsymbol{\xi} = 0,\ \forall \boldsymbol{\xi} \in \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2),\\
&d\,(\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}}) = 0,\\
&\delta\,(\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}}) = \delta\circ\delta\circ d\boldsymbol{\mu}_{\widehat{\boldsymbol{\nu}}} = 0,
\end{aligned}$$

and therefore, $\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}} \in \widetilde{\mathbb{D}}$. However, (2.4) tells us that since $\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}} \in \mathcal{H}^k(\bar{\mathcal{B}})$ satisfies the following boundary conditions

$$\begin{aligned}
&\mathsf{t}\,(\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}}) = \mathsf{t}\widehat{\boldsymbol{\nu}} - d\,(\mathsf{t}\boldsymbol{\omega}_{\boldsymbol{\nu}}) = 0, &&\text{on } \mathcal{S}_1,\\
&\mathsf{n}\,(\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}}) = \mathsf{n}\,(\delta\circ d\boldsymbol{\mu}_{\widehat{\boldsymbol{\nu}}}) = \delta\,(\mathsf{n}(d\boldsymbol{\mu}_{\widehat{\boldsymbol{\nu}}})) = 0, \quad &&\text{on } \mathcal{S}_2.
\end{aligned}$$

We also have $\widehat{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}} \in \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$, and thus, the decomposition (2.6) implies that $\boldsymbol{\nu} - \delta\boldsymbol{\zeta}_{\boldsymbol{\nu}} - d\boldsymbol{\omega}_{\boldsymbol{\nu}} = 0$. The decomposition (2.5a) then follows from the decomposition (2.6) and the L^2-orthogonal decomposition of $\widetilde{\mathbb{D}}$ that we just established. For deriving (2.5b), we note that if $\boldsymbol{\lambda} \in \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}^k_\delta(\bar{\mathcal{B}}, \mathcal{S}_2)$, then $\boldsymbol{\lambda} \in \mathcal{H}^k_{\mathrm{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$. Conversely, if $\boldsymbol{\lambda} \in \mathcal{H}^k_{\mathrm{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$, then

$$\langle\!\langle \boldsymbol{\lambda}, d\boldsymbol{\omega}\rangle\!\rangle_{L^2} = 0,\ \forall d\boldsymbol{\omega} \in \mathcal{H}^k_d(\bar{\mathcal{B}}, \mathcal{S}_1),$$

and, therefore, the decomposition (2.5a) suggests that $\boldsymbol{\lambda} \in \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}^k_\delta(\bar{\mathcal{B}}, \mathcal{S}_2)$. Thus, $\mathcal{H}^k_{\mathrm{t}}(\bar{\mathcal{B}}, \mathcal{S}_2) = \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}^k_\delta(\bar{\mathcal{B}}, \mathcal{S}_2)$, and (2.5b) directly follows from (2.5a). Similarly, (2.5c) follows from (2.5a) and the fact that $\mathcal{H}^k_{\mathrm{n}}(\bar{\mathcal{B}}, \mathcal{S}_1) = \mathcal{H}^k(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}^k_d(\bar{\mathcal{B}}, \mathcal{S}_1)$. □

Remark 2. Theorem 1 extends the classical Friedrichs decompositions introduced by Friedrichs [8] and Duff [9] in the sense that (2.5b) with $\mathcal{S}_1 = \varnothing$, and $\mathcal{S}_2 = \partial\mathcal{B}$, and (2.5c) with $\mathcal{S}_1 = \partial\mathcal{B}$, and $\mathcal{S}_2 = \varnothing$, are the Friedrichs decompositions of harmonic fields on $\bar{\mathcal{B}}$. Note that $\mathcal{H}^k_d(\bar{\mathcal{B}}, \partial\mathcal{B}) = \mathcal{H}^k_\delta(\bar{\mathcal{B}}, \partial\mathcal{B}) = \{0\}$. If $\boldsymbol{\lambda} = \boldsymbol{\xi}_{\boldsymbol{\lambda}} + \delta\boldsymbol{\zeta}_{\boldsymbol{\lambda}} + d\boldsymbol{\omega}_{\boldsymbol{\lambda}}$ is the decomposition (2.5a) for $\boldsymbol{\lambda} \in \mathcal{H}^k(\bar{\mathcal{B}})$, then the construction introduced in the proof of Theorem 1 allows one to select $\boldsymbol{\zeta}_{\boldsymbol{\lambda}}$ and $\boldsymbol{\omega}_{\boldsymbol{\lambda}}$ to be of the Sobolev class H^2.

We are now ready to state the necessary and sufficient conditions for the existence of a solution to (2.1). The upshot is the following theorem.

Theorem 3. *The following conditions are necessary and sufficient for the existence of a solution $\boldsymbol{\alpha}$ to the boundary-value problem (2.1):*

$$\langle\!\langle \boldsymbol{\gamma}, \delta\boldsymbol{\psi}\rangle\!\rangle_{L^2} = 0, \qquad \forall \boldsymbol{\psi} \in H^1_{\mathrm{t}}(\Lambda^{k+2}T^*\mathcal{B}, \partial\mathcal{B}), \tag{2.7a}$$

$$\langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\kappa}\rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\eta} \wedge *\mathsf{n}\boldsymbol{\kappa}, \quad \forall \boldsymbol{\kappa} \in \mathcal{H}^{k+1}_{\mathrm{t}}(\bar{\mathcal{B}}, \mathcal{S}_2). \tag{2.7b}$$

A solution $\boldsymbol{\alpha}$ can be chosen such that $\delta\boldsymbol{\alpha} = 0$. Moreover, it is also possible to choose $\boldsymbol{\alpha}$ such that the linear mapping $(\boldsymbol{\gamma}, \mathsf{t}\boldsymbol{\eta}) \mapsto \boldsymbol{\alpha}$ becomes a continuous mapping, that is, there exists a real constant $c > 0$ such that

$$\|\boldsymbol{\alpha}\|_{H^2} \leq c\,(\|\boldsymbol{\gamma}\|_{H^1} + \|\mathsf{t}\boldsymbol{\eta}\|_{H^{3/2}})\,. \tag{2.8}$$

Proof. Using Green's formula, it is straightforward to show the necessity of (2.7). To show the sufficiency of (2.7), note that using the Hodge-Morrey decomposition (2.2) and the decomposition (2.5b), one can decompose $\boldsymbol{\gamma}$ as

$$\boldsymbol{\gamma} = d\boldsymbol{\phi}_{\boldsymbol{\gamma}} + \delta\boldsymbol{\psi}_{\boldsymbol{\gamma}} + \boldsymbol{\kappa}_{\boldsymbol{\gamma}} + d\boldsymbol{\omega}_{\boldsymbol{\gamma}}. \tag{2.9}$$

The condition (2.7a) then implies that $\delta\boldsymbol{\psi}_{\boldsymbol{\gamma}} = 0$. Let $\boldsymbol{\eta}_0 \in H^{\frac{3}{2}}(\Lambda^k T^*\mathcal{B}|_{\partial\mathcal{B}})$ be the zero extension of $\mathsf{t}\boldsymbol{\eta}$ to $\partial\mathcal{B}$, i.e. $\boldsymbol{\eta}_0|_{\mathcal{S}_1} = \mathsf{t}\boldsymbol{\eta}$, and $\boldsymbol{\eta}_0|_{\mathcal{S}_2} = 0$. The trace theorem for sections of vector bundles (e.g. see [4, Theorem

1.3.7]) tells us that there is a k-form $\widetilde{\boldsymbol{\eta}} \in H^2(\Lambda^k T^*\mathcal{B})$ such that $\widetilde{\boldsymbol{\eta}}|_{\partial\mathcal{B}} = \boldsymbol{\eta}_0$. Similar to (2.9), we can write $\widetilde{\boldsymbol{\eta}} = d\boldsymbol{\phi}_{\widetilde{\boldsymbol{\eta}}} + \delta\boldsymbol{\psi}_{\widetilde{\boldsymbol{\eta}}} + \boldsymbol{\kappa}_{\widetilde{\boldsymbol{\eta}}} + d\boldsymbol{\omega}_{\widetilde{\boldsymbol{\eta}}}$. Let $\boldsymbol{\theta} = d\widehat{\boldsymbol{\eta}}$, where $\widehat{\boldsymbol{\eta}} := \delta\boldsymbol{\psi}_{\widetilde{\boldsymbol{\eta}}} + \boldsymbol{\kappa}_{\widetilde{\boldsymbol{\eta}}} + d\boldsymbol{\omega}_{\widetilde{\boldsymbol{\eta}}}$. Note that $\mathsf{t}\widehat{\boldsymbol{\eta}} = \mathsf{t}\widetilde{\boldsymbol{\eta}} = \mathsf{t}\boldsymbol{\eta}_0$, and since $\boldsymbol{\theta}$ is exact, the analogue of the decomposition (2.9) for $\boldsymbol{\theta}$ reads $\boldsymbol{\theta} = d\boldsymbol{\phi}_{\boldsymbol{\theta}} + \boldsymbol{\kappa}_{\boldsymbol{\theta}} + d\boldsymbol{\omega}_{\boldsymbol{\theta}}$. Now, we define

$$\boldsymbol{\alpha} := \boldsymbol{\phi}_{\boldsymbol{\gamma}} + \boldsymbol{\omega}_{\boldsymbol{\gamma}} + \widehat{\boldsymbol{\eta}} - \boldsymbol{\phi}_{\boldsymbol{\theta}} - \boldsymbol{\omega}_{\boldsymbol{\theta}} \in H^2(\Lambda^k T^*\mathcal{B}). \tag{2.10}$$

We have $d\boldsymbol{\alpha} = \boldsymbol{\gamma} + \boldsymbol{\kappa}_{\boldsymbol{\theta}} - \boldsymbol{\kappa}_{\boldsymbol{\gamma}}$, and $\mathsf{t}\boldsymbol{\alpha} = \mathsf{t}\boldsymbol{\eta}$, on $\mathcal{S}_1$. From (2.5b), we know that $\boldsymbol{\kappa}_{\boldsymbol{\theta}} - \boldsymbol{\kappa}_{\boldsymbol{\gamma}} \in \mathcal{H}_{\mathrm{t}}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2)$. On the other hand, for any $\boldsymbol{\kappa} \in \mathcal{H}_{\mathrm{t}}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2)$ one can write

$$\langle\!\langle \boldsymbol{\kappa}_{\boldsymbol{\theta}} - \boldsymbol{\kappa}_{\boldsymbol{\gamma}}, \boldsymbol{\kappa} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\eta} \wedge *\mathsf{n}\boldsymbol{\kappa} - \langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\kappa} \rangle\!\rangle_{L^2} = 0, \quad \forall \boldsymbol{\kappa} \in \mathcal{H}_{\mathrm{t}}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2),$$

where the last equality follows from (2.7b). This means that $\boldsymbol{\kappa}_{\boldsymbol{\theta}} - \boldsymbol{\kappa}_{\boldsymbol{\gamma}}$ is also normal to $\mathcal{H}_{\mathrm{t}}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2)$, and consequently $\boldsymbol{\kappa}_{\boldsymbol{\theta}} - \boldsymbol{\kappa}_{\boldsymbol{\gamma}} = 0$. Therefore, $\boldsymbol{\alpha}$ is a solution of (2.1). This proves the sufficiency of (2.7). The component $d\boldsymbol{\phi}$ in the Hodge-Morrey decomposition (2.2) can be chosen such that $\delta\boldsymbol{\phi} = 0$, e.g. see [4, Lemma 2.4.7]. Moreover, the proof of Theorem 1 shows that the component $d\boldsymbol{\omega}$ in the decomposition (2.5b) can be chosen such that $\delta\boldsymbol{\omega} = 0$. Since $\delta\widehat{\boldsymbol{\eta}} = 0$, the solution $\boldsymbol{\alpha}$ given in (2.10) can be chosen such that $\delta\boldsymbol{\alpha} = 0$. The trace theorem, Lemma 2.4.11 of [4], and the proof of Theorem 1 suggest that $\widehat{\boldsymbol{\eta}}$, $\boldsymbol{\phi}_{\boldsymbol{\gamma}}$, $\boldsymbol{\phi}_{\boldsymbol{\theta}}$, $\boldsymbol{\omega}_{\boldsymbol{\gamma}}$, and $\boldsymbol{\omega}_{\boldsymbol{\theta}}$ can be chosen to be of H^2-class with

$$\begin{aligned}
&\|\widehat{\boldsymbol{\eta}}\|_{H^2} \leq c_1 \|\mathsf{t}\boldsymbol{\eta}\|_{H^{3/2}}, \\
&\|\boldsymbol{\phi}_{\boldsymbol{\gamma}} + \boldsymbol{\omega}_{\boldsymbol{\gamma}}\|_{H^2} \leq c_2 \|\boldsymbol{\gamma}\|_{H^1}, \\
&\|\boldsymbol{\phi}_{\boldsymbol{\theta}} + \boldsymbol{\omega}_{\boldsymbol{\theta}}\|_{H^2} \leq \bar{c}_3 \|\boldsymbol{\theta}\|_{H^1} = \bar{c}_3 \|d\widehat{\boldsymbol{\eta}}\|_{H^1} \leq c_3 \|\widehat{\boldsymbol{\eta}}\|_{H^2}.
\end{aligned}$$

The Sobolev estimate (2.8) then follows from (2.10) and the above inequalities. □

Alternatively, one can write the following necessary and sufficient integrability conditions for (2.1).

Theorem 4. *The integrability conditions (2.7) are equivalent to the following conditions:*

$$d\boldsymbol{\gamma} = 0, \tag{2.11a}$$

$$\mathsf{t}\boldsymbol{\gamma}|_{\mathcal{S}_1} = \mathsf{t}(d\boldsymbol{\eta}), \tag{2.11b}$$

$$\langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\xi} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\eta} \wedge *\mathsf{n}\boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathcal{H}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2). \tag{2.11c}$$

Proof. First, we show that (2.7) ⇒ (2.11). Suppose (2.7a) holds. Then, for any $\boldsymbol{\psi} \in H_{\mathrm{t}}^1(\Lambda^{k+2}T^*\mathcal{B}, \partial\mathcal{B})$ we have

$$\langle\!\langle d\boldsymbol{\gamma}, \boldsymbol{\psi} \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{\gamma}, \delta\boldsymbol{\psi} \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \mathsf{t}\boldsymbol{\gamma} \wedge *\mathsf{n}\boldsymbol{\psi} = 0,$$

and since $H_{\mathrm{t}}^1(\Lambda^{k+2}T^*\mathcal{B}, \partial\mathcal{B})$ is dense in $L^2(\Lambda^{k+2}T^*\mathcal{B})$, we conclude that $d\boldsymbol{\gamma} = 0$. Let $\widetilde{\boldsymbol{\eta}} \in H^2(\Lambda^k T^*\mathcal{B})$ be any extension of $\boldsymbol{\eta}$ to $\mathcal{B}$. The condition (2.7b) allows one to write

$$\langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\kappa} \rangle\!\rangle_{L^2} = \int_{\partial\mathcal{B}} \mathsf{t}\widetilde{\boldsymbol{\eta}} \wedge *\mathsf{n}\boldsymbol{\kappa} + \langle\!\langle \widetilde{\boldsymbol{\eta}}, \delta\boldsymbol{\kappa} \rangle\!\rangle_{L^2} = \langle\!\langle d\widetilde{\boldsymbol{\eta}}, \boldsymbol{\kappa} \rangle\!\rangle_{L^2}, \quad \forall \boldsymbol{\kappa} \in \mathcal{H}_{\mathrm{t}}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2). \tag{2.12}$$

Since $\boldsymbol{\gamma}$ and $d\widetilde{\boldsymbol{\eta}}$ are closed, using the Hodge-Morrey decomposition and the decomposition (2.5b), one can write $\boldsymbol{\gamma} = d\boldsymbol{\phi}_{\boldsymbol{\gamma}} + \boldsymbol{\kappa}_{\boldsymbol{\gamma}} + d\boldsymbol{\omega}_{\gamma}$, and $d\widetilde{\boldsymbol{\eta}} = d\boldsymbol{\phi}_{d\widetilde{\boldsymbol{\eta}}} + \boldsymbol{\kappa}_{d\widetilde{\boldsymbol{\eta}}} + d\boldsymbol{\omega}_{d\widetilde{\boldsymbol{\eta}}}$. Then, (2.12) suggests that $\boldsymbol{\kappa}_{\boldsymbol{\gamma}} = \boldsymbol{\kappa}_{d\widetilde{\boldsymbol{\eta}}}$, and therefore

$$\mathsf{t}\boldsymbol{\gamma}|_{\mathcal{S}_1} = \mathsf{t}\boldsymbol{\kappa}_{\gamma}|_{\mathcal{S}_1} = \mathsf{t}\boldsymbol{\kappa}_{d\widetilde{\boldsymbol{\eta}}}|_{\mathcal{S}_1} = \mathsf{t}(d\widetilde{\boldsymbol{\eta}})|_{\mathcal{S}_1} = \mathsf{t}(d\boldsymbol{\eta}). \tag{2.13}$$

Moreover, if (2.7b) holds, then so does (2.11c). Hence (2.7) ⇒ (2.11). Conversely, suppose (2.11) holds. Then, (2.11a) implies (2.7a). The proof of Theorem 1 tells us that any $\boldsymbol{\kappa} \in \mathcal{H}_{\mathrm{t}}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2)$ can be decomposed as $\boldsymbol{\kappa} = \boldsymbol{\xi}_{\boldsymbol{\kappa}} + \delta\boldsymbol{\zeta}_{\boldsymbol{\kappa}}$, with $\boldsymbol{\xi}_{\boldsymbol{\kappa}} \in \mathcal{H}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ and $\delta\boldsymbol{\zeta}_{\boldsymbol{\kappa}} \in \mathcal{H}_{\delta}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_2)$. Thus, (2.7b) is equivalent to the following conditions:

$$\langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\xi} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\eta} \wedge *\mathsf{n}\boldsymbol{\xi}, \qquad \forall \boldsymbol{\xi} \in \mathcal{H}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2), \tag{2.14a}$$

$$\langle\!\langle \boldsymbol{\gamma}, \delta\boldsymbol{\zeta} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\eta} \wedge *\mathsf{n}(\delta\boldsymbol{\zeta}), \quad \forall \delta\boldsymbol{\zeta} \in \mathcal{H}^{k+1}_{\delta}(\bar{\mathcal{B}}, \mathcal{S}_2). \tag{2.14b}$$

The condition (2.14a) is the same as (2.11c). Using (2.11b) and an arbitrary extension $\widetilde{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$, one can write

$$\langle\!\langle \boldsymbol{\gamma}, \delta\boldsymbol{\zeta} \rangle\!\rangle_{L^2} = -\int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\gamma} \wedge *\mathsf{n}\boldsymbol{\zeta} = -\int_{\partial\mathcal{B}} \mathsf{t}(d\widetilde{\boldsymbol{\eta}}) \wedge *\mathsf{n}\boldsymbol{\zeta} = \langle\!\langle d\widetilde{\boldsymbol{\eta}}, \delta\boldsymbol{\zeta} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \mathsf{t}\boldsymbol{\eta} \wedge *\mathsf{n}(\delta\boldsymbol{\zeta}).$$

Therefore, (2.14b) holds and we conclude that (2.11) $\Rightarrow$ (2.7). □

Remark 5. By using the Hodge-Morrey decomposition for L^2-forms and slightly modifying the proof of Theorem 3, one can show that the integrability conditions (2.7) are still necessary and sufficient conditions for a weaker statement of the boundary-value problem (2.1) with $\boldsymbol{\gamma} \in L^2(\Lambda^{k+1}T^*\mathcal{B})$, $\boldsymbol{\eta} \in H^{\frac{1}{2}}(\Lambda^k T^*\mathcal{B}|_{\mathcal{S}_1})$, and $\boldsymbol{\alpha} \in H^1(\Lambda^k T^*\mathcal{B})$. However, (2.11a) and (2.11b) do not make sense for this weaker statement of (2.1), in general.[6] On the other hand, the conditions (2.11) are more useful in practice, since unlike the infinite-dimensional space $\mathcal{H}^{k+1}_{\mathsf{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$, the space $\mathcal{H}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional, and hence, one needs to verify (2.11c) only for a finite number of harmonic fields $\boldsymbol{\xi} \in \mathcal{H}^{k+1}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$.

Remark 6. The integrability conditions (2.7) and (2.11) extend the integrability conditions given in Lemmas 3.1.2 and 3.2.1 of [4], which state the integrability conditions for (2.1) with $\mathcal{S}_1 = \varnothing, \partial\mathcal{B}$. In particular, note that in the proof of Theorem 3, if $\mathcal{S}_1 = \varnothing$, then (2.10) becomes $\boldsymbol{\alpha} := \boldsymbol{\phi}_{\boldsymbol{\gamma}} + \boldsymbol{\omega}_{\boldsymbol{\gamma}}$, where $\boldsymbol{\omega}_{\boldsymbol{\gamma}} \in \mathcal{H}^k_d(\bar{\mathcal{B}}, \varnothing)$.

3 Boundary Displacements and the Compatibility of Strains

In this section, we will use the results and the arguments of the previous section to derive the compatibility equations for the displacement gradient and the linear strain in the presence of displacement boundary conditions. Note that for 3D bodies, one can define the following operators

$$\begin{aligned}
\mathbf{grad} &: H^{s+1}(T\mathcal{B}) \to H^s(\otimes^2 T\mathcal{B}), && (\mathbf{grad}\, \boldsymbol{Y})^{IJ} = Y^I{}_{,J}, \\
\mathbf{curl}^{\mathsf{T}} &: H^{s+1}(\otimes^2 T\mathcal{B}) \to H^s(\otimes^2 T\mathcal{B}), && (\mathbf{curl}^{\mathsf{T}}\boldsymbol{T})^{IJ} = \varepsilon_{JKL} T^{IL}{}_{,K}, \\
\mathbf{div} &: H^{s+1}(\otimes^2 T\mathcal{B}) \to H^s(T\mathcal{B}), && (\mathbf{div}\, \boldsymbol{T})^I = T^{IJ}{}_{,J},
\end{aligned}$$

where ε_{JKL} is the standard permutation symbol. For symmetric tensors, we also have the operators $\mathbf{grad}^{\mathsf{s}} : H^{s+1}(T\mathcal{B}) \to H^s(S^2 T\mathcal{B})$ and $\mathbf{curl} \circ \mathbf{curl} : H^{s+2}(S^2 T\mathcal{B}) \to H^s(S^2 T\mathcal{B})$, where

$$(\mathbf{grad}^{\mathsf{s}}\boldsymbol{U})^{IJ} = \frac{1}{2}\left(U^I{}_{,J} + U^J{}_{,I}\right), \quad (\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{T})^{IJ} = \varepsilon_{IKL}\varepsilon_{JMN} T^{LN}{}_{,KM}.$$

For 2D bodies, instead of $\mathbf{curl}^{\mathsf{T}}$, we have the following operators

$$\begin{aligned}
\boldsymbol{\varkappa} &: H^{s+1}(\otimes^2 T\mathcal{B}) \to H^s(T\mathcal{B}), && (\boldsymbol{\varkappa}(\boldsymbol{T}))^I = T^{I2}{}_{,1} - T^{I1}{}_{,2}, \\
\mathbf{s} &: H^{s+1}(T\mathcal{B}) \to H^s(\otimes^2 T\mathcal{B}), && (\mathbf{s}(\boldsymbol{Y}))^{IJ} = \delta^{1J} Y^I{}_{,2} - \delta^{2J} Y^I{}_{,1},
\end{aligned}$$

[6] This statement should be interpreted only in the framework of standard Sobolev spaces and their traces. By using partly Sobolev spaces induced by d and δ, it is still possible to use (2.11a) and (2.11b) for L^2 differential forms. In this case, the tangent part of an L^2 form is defined by using Green's formula and is considered to be a distribution, see [10, §3]. The boundary-value problem (2.1) for L^2 forms is useful for studying the strain compatibility on non-convex Lipschitz bodies and multi-phase bodies, where the displacement is merely of H^1-class, in general.

with δ^{IJ} being the Kronecker delta. Moreover, instead of $\mathbf{curl}\circ\mathbf{curl}$, we have the operators $\mathrm{D}_{\mathsf{c}} : H^{s+2}(S^2T\mathcal{B}) \to H^s(\mathcal{B})$ and $\mathrm{D}_{\mathsf{s}} : H^{s+2}(\mathcal{B}) \to H^s(S^2T\mathcal{B})$ defined as

$$\mathrm{D}_{\mathsf{c}}(\boldsymbol{T}) := T^{11}{}_{,22} - 2T^{12}{}_{,12} + T^{22}{}_{,11}, \quad \mathrm{D}_{\mathsf{s}}(f) := \begin{bmatrix} f_{,22} & -f_{,12} \\ -f_{,12} & f_{,11} \end{bmatrix}.$$

3.1 Nonlinear Elasticity

Let $\varphi : \mathcal{B} \to \mathbb{R}^n$ be a motion of $\mathcal{B}$ and let $\boldsymbol{U}(X) := \varphi(X) - X$, be the associated displacement field. One can assume that $\boldsymbol{U}$ is a vector field on $\mathcal{B}$. Moreover, the gradient of $\boldsymbol{U}$, which is a two-point tensor, can be identified with $\mathbf{grad}\,\boldsymbol{U}$. Now, consider the following boundary-value problem for the compatibility of the displacement gradient.

Given a $\binom{2}{0}$-tensor field $\boldsymbol{K} \in H^1(\otimes^2 T\mathcal{B})$ and an arbitrary vector field $\boldsymbol{Y} \in H^2(T\mathcal{B})$, find a displacement field $\boldsymbol{U} \in H^2(T\mathcal{B})$ such that

$$\begin{aligned} \mathbf{grad}\,\boldsymbol{U} &= \boldsymbol{K}, \quad \text{on } \mathcal{B}, \\ \boldsymbol{U} &= \boldsymbol{Y}, \quad \text{on } \mathcal{S}_1. \end{aligned} \tag{3.1}$$

We want to determine necessary and sufficient conditions for the existence of a displacement $\boldsymbol{U}$ in the above problem. Note that if $\mathcal{S}_1 = \varnothing$, (3.1) is the classical compatibility problem. Before stating the main result, we introduce some subspaces of $H^1(\otimes^2 T\mathcal{B})$. In the Cartesian coordinates $\{X^I\}$, the traction of any $\binom{2}{0}$-tensor $\boldsymbol{T}$ on a plane normal to a vector $\mathbf{Z} \in \mathbb{R}^n$ at $X \in \bar{\mathcal{B}}$ is given by $\boldsymbol{T}(\mathbf{Z}) := T^{IJ}Z^J\mathbf{E}_I$.[7] Following [11], we say that $\boldsymbol{T} \in H^1(\otimes^2 T\mathcal{B})$ is normal to an open subset $\mathcal{U} \subset \partial\mathcal{B}$ and write $\boldsymbol{T} \perp \mathcal{U}$ if $\boldsymbol{T}(\boldsymbol{Y}) = 0$, on $\mathcal{U}$, for any vector field $\boldsymbol{Y} \| \mathcal{U}$. We say that $\boldsymbol{T} \in H^1(\otimes^2 T\mathcal{B})$ is parallel to $\mathcal{U}$ and write $\boldsymbol{T} \| \mathcal{U}$ if $\boldsymbol{T}(\boldsymbol{N}) = 0$, on $\mathcal{U}$, where $\boldsymbol{N}$ is the unit outward normal vector field of $\partial\mathcal{B}$. The space $H^1_{\mathrm{n}}(T\mathcal{B}, \mathcal{S}_1)$ $\big(H^1_{\mathrm{t}}(T\mathcal{B}, \mathcal{S}_1)\big)$ is the space of H^1 vector fields normal (parallel) to $\mathcal{S}_1$. The spaces $H^1_{\mathrm{n}}(\otimes^2 T\mathcal{B}, \mathcal{S}_1)$ and $H^1_{\mathrm{t}}(\otimes^2 T\mathcal{B}, \mathcal{S}_1)$ are defined analogously. We also define the following spaces:

$$\begin{aligned} &\mathcal{H}^{\otimes}(\bar{\mathcal{B}}) := \left\{ \boldsymbol{H} \in H^1(\otimes^2 T\mathcal{B}) : \mathbf{curl}^{\mathsf{T}}\boldsymbol{H} = 0 \text{ and } \mathbf{div}\,\boldsymbol{H} = 0 \right\}, \\ &\mathcal{H}^{\otimes}_{\mathsf{t}}(\bar{\mathcal{B}}, \mathcal{S}_2) := \mathcal{H}^{\otimes}(\bar{\mathcal{B}}) \cap H^1_{\mathrm{t}}(\otimes^2 T\mathcal{B}, \mathcal{S}_2), \\ &\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) := \mathcal{H}^{\otimes}(\bar{\mathcal{B}}) \cap H^1_{\mathrm{n}}(\otimes^2 T\mathcal{B}, \mathcal{S}_1) \cap H^1_{\mathrm{t}}(\otimes^2 T\mathcal{B}, \mathcal{S}_2). \end{aligned}$$

For 2D bodies, one should replace $\mathbf{curl}^{\mathsf{T}}$ with $\boldsymbol{\varkappa}$ in the definition of $\mathcal{H}^{\otimes}(\bar{\mathcal{B}})$. One can show that $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional with $\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2) = n\, b_1(\bar{\mathcal{B}}, \mathcal{S}_1)$ [11]. Now, we can state the main result regarding the compatibility of the displacement gradient as follows.

Theorem 7. *The following sets of conditions are equivalent and each set is necessary and sufficient for the existence of a solution to (3.1):*

i) The weak compatibility equations:

$$\langle\!\langle \boldsymbol{K}, \mathbf{curl}^{\mathsf{T}}\boldsymbol{T} \rangle\!\rangle_{L^2} = 0, \qquad \forall \boldsymbol{T} \in H^1_{\mathrm{n}}(\otimes^2 T\mathcal{B}, \partial\mathcal{B}), \tag{3.2a}$$

$$\langle\!\langle \boldsymbol{K}, \boldsymbol{H} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \langle\!\langle \boldsymbol{Y}, \boldsymbol{H}(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{H} \in \mathcal{H}^{\otimes}_{\mathsf{t}}(\bar{\mathcal{B}}, \mathcal{S}_2). \tag{3.2b}$$

For 2D bodies, the condition (3.2a) is replaced by the following condition.

$$\langle\!\langle \boldsymbol{K}, \mathrm{s}(\boldsymbol{Z}) \rangle\!\rangle_{L^2} = 0, \quad \forall \boldsymbol{Z} \in H^1_0(T\mathcal{B}).$$

[7] Clearly, $\boldsymbol{T}(\mathbf{Z})$ is not a physical traction unless $\boldsymbol{T}$ is a stress tensor.

ii) The strong compatibility equations:

$$\mathbf{curl}^{\mathsf{T}}\boldsymbol{K} = 0, \tag{3.3a}$$

$$\boldsymbol{K}(\boldsymbol{Z})|_{\mathcal{S}_1} = (\mathbf{grad}\,\boldsymbol{Y})(\boldsymbol{Z}), \qquad \forall \boldsymbol{Z} \in H^1_{\mathsf{t}}(T\mathcal{B}, \mathcal{S}_1), \tag{3.3b}$$

$$\langle\!\langle \boldsymbol{K}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \langle\!\langle \boldsymbol{Y}, \boldsymbol{Q}(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{Q} \in \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2). \tag{3.3c}$$

For 2D *bodies, the condition (3.3a) is replaced with* $\boldsymbol{\varkappa}(\boldsymbol{K}) = 0$.

Proof. We only prove the 3D case as the proof for 2D bodies is similar. For any $\binom{2}{0}$-tensor $\boldsymbol{T}$, let $\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} := T^{IJ}\mathbf{E}_J$, $I = 1, \ldots, n$, and consider the following isometric isomorphisms

$$\imath_1 : H^s(T\mathcal{B}) \to H^s(\Lambda^1 T^*\mathcal{B}), \quad \imath_1(\boldsymbol{Z}) = \boldsymbol{Z}^\flat,$$
$$\imath_2 : H^s(T\mathcal{B}) \to H^s(\Lambda^2 T^*\mathcal{B}), \quad \imath_2(\boldsymbol{Z}) = *(\boldsymbol{Z}^\flat),$$

where $\flat : H^s(T\mathcal{B}) \to H^s(\Lambda^1 T^*\mathcal{B})$ and $* : H^s(\Lambda^k T^*\mathcal{B}) \to H^s(\Lambda^{n-k}T^*\mathcal{B})$ are the flat and the Hodge-star operators, respectively. Then, one can define the following isometric isomorphisms [11, §3.1]:

$$\boldsymbol{\imath}_0 : H^s(T\mathcal{B}) \to \oplus_{i=1}^n H^s(\Lambda^0 T^*\mathcal{B}), \qquad \boldsymbol{\imath}_0(\boldsymbol{U}) = (U^1, \ldots, U^n),$$
$$\boldsymbol{\imath}_j : H^s(\otimes^2 T\mathcal{B}) \to \oplus_{i=1}^n H^s(\Lambda^j T^*\mathcal{B}), \quad \boldsymbol{\imath}_j(\boldsymbol{T}) = \big(\imath_j(\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_1}), \ldots, \imath_j(\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_n})\big), \quad j = 1, 2.$$

It is straightforward to show that $\boldsymbol{\imath}_1 \circ \mathbf{grad}(\boldsymbol{U}) = (dU^1, \ldots, dU^n)$. By using the fact that for a 0-form f, we have $\mathsf{t}f = f|_{\partial\mathcal{B}}$, one concludes that the following problem is equivalent to (3.1):

Given $\overrightarrow{\boldsymbol{K}}^\flat_{\mathbf{E}_I} \in H^1(\Lambda^1 T^*\mathcal{B})$ *and* $Y^I \in H^{\frac{3}{2}}(\Lambda^0 T^*\mathcal{B}|_{\mathcal{S}_1})$, $I = 1, \ldots, n$, *find* $U^I \in H^2(\Lambda^0 T^*\mathcal{B})$ *such that*

$$dU^I = \overrightarrow{\boldsymbol{K}}^\flat_{\mathbf{E}_I}, \quad \text{on } \mathcal{B},$$
$$\mathsf{t}U^I = \mathsf{t}Y^I, \quad \text{on } \mathcal{S}_1.$$

The isomorphism $\imath_2$ induces an isomorphism between $H^1_{\mathsf{t}}(\Lambda^2 T^*\mathcal{B}, \partial\mathcal{B})$ and $H^1_{\mathsf{n}}(T\mathcal{B}, \partial\mathcal{B})$. Hence, the condition (2.7a) can be written as

$$\begin{aligned} 0 = \langle\!\langle \overrightarrow{\boldsymbol{K}}^\flat_{\mathbf{E}_I}, \delta\boldsymbol{\psi} \rangle\!\rangle_{L^2} &= \langle\!\langle \imath_1(\overrightarrow{\boldsymbol{K}}_{\mathbf{E}_I}), \delta \circ \imath_2(\boldsymbol{Z}) \rangle\!\rangle_{L^2} \\ &= \langle\!\langle \imath_1(\overrightarrow{\boldsymbol{K}}_{\mathbf{E}_I}), \imath_1 \circ \mathrm{curl}(\boldsymbol{Z}) \rangle\!\rangle_{L^2} \\ &= \langle\!\langle \overrightarrow{\boldsymbol{K}}_{\mathbf{E}_I}, \mathrm{curl}\,\boldsymbol{Z} \rangle\!\rangle_{L^2}, \quad \forall \boldsymbol{Z} := \imath_2^{-1}(\boldsymbol{\psi}) \in H^1_{\mathsf{n}}(T\mathcal{B}, \partial\mathcal{B}), \end{aligned} \tag{3.4}$$

where curl is the standard curl operator of vector analysis. Using the relation

$$\langle\!\langle \boldsymbol{K}, \mathbf{curl}^{\mathsf{T}}\boldsymbol{T} \rangle\!\rangle_{L^2} = \sum_{I=1}^n \langle\!\langle \overrightarrow{\boldsymbol{K}}_{\mathbf{E}_I}, \mathrm{curl}\,\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} \rangle\!\rangle_{L^2},$$

it is straightforward to show that (3.4) is equivalent to (3.2a). On the other hand, $\boldsymbol{\imath}_1$ induces an isomorphism between $\mathcal{H}^{\otimes}_{\mathsf{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$ and $\oplus_{i=1}^n \mathcal{H}^1_{\mathsf{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$ and the condition (2.7b) allows one to write

$$\begin{aligned} \langle\!\langle \boldsymbol{K}, \boldsymbol{H} \rangle\!\rangle_{L^2} &= \sum_{I=1}^n \langle\!\langle \overrightarrow{\boldsymbol{K}}^\flat_{\mathbf{E}_I}, \overrightarrow{\boldsymbol{H}}^\flat_{\mathbf{E}_I} \rangle\!\rangle_{L^2} \\ &= \sum_{I=1}^n \int_{\mathcal{S}_1} Y^I \big(* \,\mathsf{n}(\overrightarrow{\boldsymbol{H}}^\flat_{\mathbf{E}_I})\big) = \sum_{I=1}^n \int_{\mathcal{S}_1} Y^I \langle\!\langle \overrightarrow{\boldsymbol{H}}_{\mathbf{E}_I}, \boldsymbol{N} \rangle\!\rangle dA \\ &= \int_{\mathcal{S}_1} \langle\!\langle \boldsymbol{Y}, \boldsymbol{H}(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{H} \in \mathcal{H}^{\otimes}_{\mathsf{t}}(\bar{\mathcal{B}}, \mathcal{S}_2). \end{aligned}$$

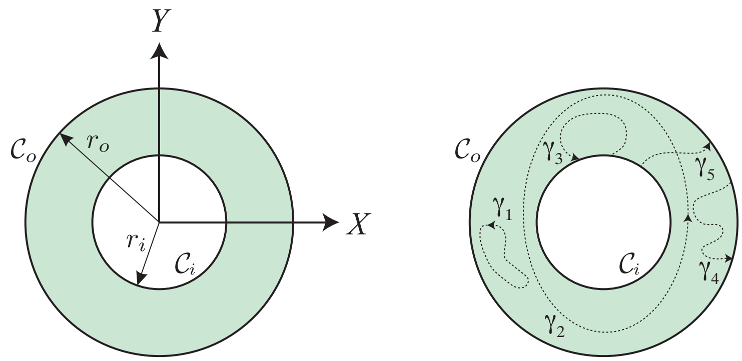

Figure 1: *The boundary of an annulus is the union of the inner circle $\mathcal{C}_i$ and the outer circle $\mathcal{C}_o$. The dimension of $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ is: (i) 2, if $\mathcal{S}_1 = \varnothing$, (ii) 0, if either $\mathcal{S}_1 = \mathcal{C}_i$ or $\mathcal{S}_1 = \mathcal{C}_o$, and (iii) 2, if $\mathcal{S}_1 = \mathcal{C}_i \cup \mathcal{C}_o$.*

Therefore, the condition (3.2b) follows from (2.7b). Similarly, one can also show that the integrability conditions (3.3) follow from the integrability conditions (2.11). □

Remark 8. The special case of the strong conditions (3.3) for zero displacement $\boldsymbol{Y} = 0$, on $\mathcal{S}_1$, was derived in [11]. The discussion of Remark 5 also applies to the integrability conditions (3.2) and (3.3), that is, the weak conditions (3.2) are still meaningful for less-smooth data $\boldsymbol{K} \in L^2(\otimes^2 T\mathcal{B})$ and $\boldsymbol{Y} \in H^1(T\mathcal{B})$. Theorem 3 implies that one can choose solutions of (3.1) such that the linear mapping $(\boldsymbol{K}, \boldsymbol{Y}|_{\mathcal{S}_1}) \mapsto \boldsymbol{U}$ is continuous. Similar to the discussions of [12, Remark 15], note that Theorem 7 does not guarantee that $\boldsymbol{U}$ corresponds to a motion of $\mathcal{B}$, that is, the mapping $\varphi : \mathcal{B} \to \mathbb{R}^n$ associated to $\boldsymbol{U}$ is not an embedding, in general. Also note that one only needs values of $\boldsymbol{Y}$ on $\mathcal{S}_1$ to calculate the right-hand side of (3.3b).

Remark 9. Let $\mathbb{D}(\otimes^2 T\mathcal{B}, \mathcal{S}_2)$ be the space of divergence-free $\binom{2}{0}$-tensors with zero traction on $\mathcal{S}_2$. Using the isomorphisms introduced in the proof of Theorem 7, one can show that any $\boldsymbol{D} \in \mathbb{D}(\otimes^2 T\mathcal{B}, \mathcal{S}_2)$ admits the L^2-orthogonal decomposition $\boldsymbol{D} = \mathbf{curl}^{\mathsf{T}} \boldsymbol{T}_{\boldsymbol{D}} + \boldsymbol{H}_{\boldsymbol{D}}$, where $\boldsymbol{T}_{\boldsymbol{D}} \in H^1_{\mathrm{n}}(\otimes^2 T\mathcal{B}, \partial\mathcal{B})$, and $\boldsymbol{H}_{\boldsymbol{D}} \in \mathcal{H}^{\otimes}_{\mathrm{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$. Then, the fact that $\mathbf{curl}^{\mathsf{T}} \boldsymbol{T}_{\boldsymbol{D}}$ has zero traction on $\partial\mathcal{B}$ suggests that the weak condition (3.2) is equivalent to

$$\langle\!\langle \boldsymbol{K}, \boldsymbol{D} \rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \langle\!\langle \boldsymbol{Y}, \boldsymbol{D}(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{D} \in \mathbb{D}(\otimes^2 T\mathcal{B}, \mathcal{S}_2). \tag{3.5}$$

The above condition simply tells us that for any equilibrated *virtual* stress $\boldsymbol{D} \in \mathbb{D}(\otimes^2 T\mathcal{B}, \mathcal{S}_2)$, the virtual work done by $\boldsymbol{K}$ and $\boldsymbol{Y}$ must be equal. The integrability condition (3.5) is similar to the compatibility condition for linear strains, see [3, Corollary 3.1]. In practice, the strong integrability conditions (3.3) are more useful than (3.2) and (3.5) because unlike $\mathcal{H}^{\otimes}_{\mathrm{t}}(\bar{\mathcal{B}}, \mathcal{S}_2)$ and $\mathbb{D}(\otimes^2 T\mathcal{B}, \mathcal{S}_2)$, the space $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional.

Example 10. Let $\bar{\mathcal{B}}$ be the annulus shown in Figure 1, with $\partial\mathcal{B} = \mathcal{C}_i \cup \mathcal{C}_o$, where $\mathcal{C}_i$ and $\mathcal{C}_o$ are the inner and the outer circles of $\partial\mathcal{B}$, respectively. Suppose

$$\boldsymbol{K}(X, Y) = \begin{bmatrix} c_1\, f(X, Y) & c_1\, h(X, Y) \\ c_2\, f(X, Y) & c_2\, h(X, Y) \end{bmatrix}, \tag{3.6}$$

where $c_i \in \mathbb{R}$, $i = 1, 2$, and

$$f(X, Y) = \frac{X}{X^2 + Y^2}, \quad h(X, Y) = \frac{Y}{X^2 + Y^2}.$$

Note that if $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are the orthonormal basis corresponding to the polar coordinates (r, θ), then $\boldsymbol{K}(\mathbf{e}_\theta) = 0$, and the traction vector of $\mathbf{e}_r$ only depends on r. We have $\boldsymbol{\varkappa}(\boldsymbol{K}) = 0$. Assume that $\boldsymbol{Y}$ is an arbitrary translation, that is, $\boldsymbol{Y} = \hat{c}_1 \mathbf{E}_1 + \hat{c}_2 \mathbf{E}_2$, with $\hat{c}_i \in \mathbb{R}$, $i = 1, 2$. Now, we study the following cases:

i) $\mathcal{S}_1 = \varnothing$: In this case, the condition (3.3b) is trivial and the condition (3.3c) simply implies that $\boldsymbol{K}$ must be normal to $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \varnothing, \partial\mathcal{B})$. To verify the latter, note that $\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \varnothing, \partial\mathcal{B}) = 2b_1(\bar{\mathcal{B}}) = 2$, and that $\{\boldsymbol{Q}_1, \boldsymbol{Q}_2\}$

with

$$\boldsymbol{Q}_1=\begin{bmatrix}-h & f\\ 0 & 0\end{bmatrix},\ \text{and}\ \boldsymbol{Q}_2=\begin{bmatrix}0 & 0\\ -h & f\end{bmatrix},$$

is a basis for $\mathcal{H}^{\otimes}(\bar{\mathcal{B}},\varnothing,\partial\mathcal{B})$. Then, it is easy to show that $\langle\!\langle \boldsymbol{K},\boldsymbol{Q}_i\rangle\!\rangle_{L^2}=0$, $i=1,2$, and one concludes that (3.3c) holds. Therefore, the problem (3.1) is integrable in this case.

ii) $\mathcal{S}_1=\mathcal{C}_i$: Since $\mathbf{grad}\,\boldsymbol{Y}=0$, and $\boldsymbol{K}(\mathbf{e}_\theta)=0$, we conclude that $\boldsymbol{K}$ and $\boldsymbol{Y}$ satisfy (3.3b). Moreover, since $\dim\mathcal{H}^{\otimes}(\bar{\mathcal{B}},\mathcal{C}_i,\mathcal{C}_o)=2b_1(\bar{\mathcal{B}},\mathcal{C}_i)=0$, the condition (3.3c) is trivial and hence, the problem (3.1) is integrable.

iii) $\mathcal{S}_1=\partial\mathcal{B}$: One can show that $\boldsymbol{K}$ and $\boldsymbol{Y}$ satisfy (3.3b). We also have $\dim\mathcal{H}^{\otimes}(\bar{\mathcal{B}},\partial\mathcal{B},\varnothing)=2b_1(\bar{\mathcal{B}},\partial\mathcal{B})=2$, and $\{\boldsymbol{T}_1,\boldsymbol{T}_2\}$ with

$$\boldsymbol{T}_1=\begin{bmatrix}f & h\\ 0 & 0\end{bmatrix},\ \text{and}\ \boldsymbol{T}_2=\begin{bmatrix}0 & 0\\ f & h\end{bmatrix},$$

is a basis for $\mathcal{H}^{\otimes}(\bar{\mathcal{B}},\partial\mathcal{B},\varnothing)$. Let r_i and r_o be the radii of $\mathcal{C}_i$ and $\mathcal{C}_o$, respectively. Then, one can write

$$\left.\begin{aligned}&\langle\!\langle \boldsymbol{K},\boldsymbol{T}_j\rangle\!\rangle_{L^2}=2\pi\,c_j\ln\!\left(\tfrac{r_o}{r_i}\right),\\ &\textstyle\int_{\partial\mathcal{B}}\langle\!\langle \boldsymbol{Y},\boldsymbol{T}_j(\boldsymbol{N})\rangle\!\rangle ds=4\pi\,\hat{c}_j,\end{aligned}\right\}\quad j=1,2.$$

Therefore, (3.3c) holds if and only if $\hat{c}_j=c_j\ln\sqrt{r_o/r_i}$, $j=1,2$. In particular, note that if $\partial\mathcal{B}$ is fixed (i.e. $\hat{c}_1=\hat{c}_2=0$) and $\boldsymbol{K}\neq 0$, then the problem (3.1) does not admit any solution.

Note that the dimension of $\mathcal{H}^{\otimes}(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$ depends on the topological properties of both $\bar{\mathcal{B}}$ and $\mathcal{S}_1$. In general, unlike this example, it is not easy to explicitly obtain elements of $\mathcal{H}^{\otimes}(\bar{\mathcal{B}},\mathcal{S}_1,\mathcal{S}_2)$ and they need to be calculated numerically.

Remark 11. Let us give a more intuitive discussion on the compatibility equations of the previous example. A simple approach to formulate compatibility equations of deformation gradient $\boldsymbol{F}$ was discussed in [1]. Let us assume that one knows that a material point $\boldsymbol{X}_0$ in the reference configuration after deformation occupies the position $\boldsymbol{x}_0$ in the current configuration. Now compatibility of $\boldsymbol{F}$ is equivalent to being able to find the position of any other material point $\boldsymbol{X}$ in the current configuration. Assuming that the material manifold is path-connected, consider a path γ that connects $\boldsymbol{X}_0$ to $\boldsymbol{X}$ in the reference configuration. The position of the material point $\boldsymbol{X}$ can be calculated as

$$\boldsymbol{x}=\boldsymbol{x}_0+\int_\gamma \boldsymbol{F}ds.$$

The deformation gradient $\boldsymbol{F}$ is compatible if and only if the right hand side of the above equation is independent of the path γ. When the body is simply-connected this is equivalent to $\int_\gamma \boldsymbol{F}ds=0$, for any closed path γ. One can show that this is equivalent to the condition $\mathbf{curl}^{\mathsf{T}}\boldsymbol{F}=\mathbf{curl}^{\mathsf{T}}\boldsymbol{K}=0$ (the bulk compatibility equations).[8] For the sake of simplicity, let us assume that the prescribed displacements are zero, i.e, part of the boundary or the whole boundary is fixed. When a component of the boundary is fixed, we must have $\boldsymbol{X}=\boldsymbol{X}_0+\int_\gamma \boldsymbol{F}ds$, or equivalently $\int_\gamma \boldsymbol{K}ds=0$, where γ is a curve joining $\boldsymbol{X}_0$ and $\boldsymbol{X}$, which are located on the fixed portion of the boundary. Now let us consider the three cases separately.

i) $\mathcal{S}_1=\varnothing$: This case was discussed in [1]. In addition to the bulk compatibility equations, one needs the following condition[9]

$$\int_{\gamma_2}\boldsymbol{F}ds=\int_{\gamma_2}\boldsymbol{K}ds=0, \tag{3.7}$$

where γ_2 is the generator of the first de Rham cohomology group (see Fig. 1). Note that (3.7) is equivalent to (3.3c). For a proof see [4, Theorem 3.2.3].

[8] This condition guarantees that $\int_\gamma \boldsymbol{F}ds=0$, for any null-homotopic (contractible) path, e.g. γ_1 in Fig. 1.

[9] Note that in the nonlinear elasticity literature the integrand of the first integral is usually written as $\boldsymbol{F}d\boldsymbol{X}$. Here, we think of $\boldsymbol{F}$ as a $\binom{2}{0}$-tensor and ds as a 1-form, and hence, $\boldsymbol{F}ds$ is a vector-valued 1-form, which can be integrated in a Euclidean ambient space.

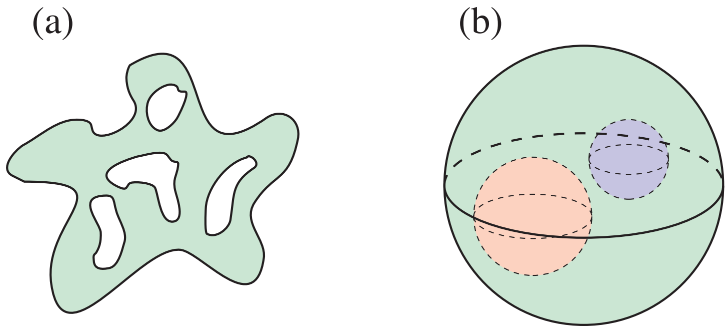

Figure 2: *(a) A non-simply-connected 2D body that has four holes for which* $\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing) = 8$. *(b) A simply-connected 3D body (a ball with two spherical holes) for which* $\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing) = 6$.

ii) $\mathcal{S}_1 = \mathcal{C}_i$ (or $\mathcal{S}_1 = \mathcal{C}_o$): In this case, we must have $\int_{\mathcal{C}_i} \boldsymbol{K} ds = 0$. Since γ_2 can be continuously shrunk to $\mathcal{C}_i$, the bulk compatibility equations tell us that $\int_{\gamma_2} \boldsymbol{K} ds = \int_{\mathcal{C}_i} \boldsymbol{K} ds = 0$. Note that this condition is also trivially satisfied on any path that starts on $\mathcal{C}_i$ and ends on $\mathcal{C}_i$, e.g. γ_3 in Fig. 1.

iii) $\mathcal{S}_1 = \partial\mathcal{B}$: Here, we must have $\int_{\mathcal{C}_0} \boldsymbol{K} ds = \int_{\mathcal{C}_i} \boldsymbol{K} ds = 0$. Similar to the previous case, $\int_{\gamma_2} \boldsymbol{K} ds = 0$, is trivial. Assuming that the bulk compatibility equations are satisfied, integral of $\boldsymbol{K}$ over curves that start on a boundary component and end on the same boundary component (e.g. paths γ_3 and γ_4 in Fig. 1) trivially vanishes. What cannot be concluded from the bulk compatibility equations is vanishing of this integral over a path that starts from a boundary component and ends on the other boundary component, e.g. the path γ_5 in Fig. 1, which is a generator of the first relative de Rham cohomology group of the pair $(\bar{\mathcal{B}}, \partial\mathcal{B})$. The simplest choice for the calculation of $\int_{\gamma_5} \boldsymbol{K} ds$ is the line segment that connects the points with coordinates $(r_i, 0)$ and $(r_o, 0)$, for which we have

$$\int_{\gamma_5} \boldsymbol{K} ds = \ln\left(\frac{r_o}{r_i}\right) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Thus, the compatibility simply implies that $\boldsymbol{K} = 0$.

Note that for sufficiently smooth displacement gradients and boundary displacements and for the special cases $\mathcal{S}_1 = \varnothing, \partial\mathcal{B}$, the equivalence of the approach of this remark and that of Theorem 7 is a consequence of Theorem 3.2.3 of Schwarz [4] and Theorem 6 of Duff [13]. In particular, Duff [13, Theorem 6] implies that the condition (3.3c) with $\mathcal{S}_1 = \partial\mathcal{B}$ is equivalent to

$$\int_{C_i} \boldsymbol{K} ds = \int_{\partial C_i} \boldsymbol{Y} = \boldsymbol{Y}(\boldsymbol{X}_2^i) - \boldsymbol{Y}(\boldsymbol{X}_1^i), \quad i = 1, ..., b_1(\bar{\mathcal{B}}, \partial\mathcal{B}),$$

where C_i's are the generators of the first relative singular homology group $H_1(\bar{\mathcal{B}}, \partial\mathcal{B}; \mathbb{R})$. Note that each $\partial C_i = [\boldsymbol{X}_1^i, \boldsymbol{X}_2^i]$ is an oriented pair of points $(\boldsymbol{X}_1^i, \boldsymbol{X}_2^i)$ such that $\boldsymbol{X}_1^i$ and $\boldsymbol{X}_2^i$ lie on $\partial\mathcal{B}$.

Example 12. Our conclusion in the part (iii) of the previous example can be extended in the following sense: Let $\bar{\mathcal{B}}$ be a 2D non-simply-connected body containing a finite number of holes, see Figure 2(a). We have

$$\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing) = \dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \varnothing, \partial\mathcal{B}) = 2b_1(\bar{\mathcal{B}}) = 2(\#\text{ of holes}).$$

Let $\boldsymbol{K} \neq 0$ be an element of $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing)$. Since $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing)$ and $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \varnothing, \partial\mathcal{B})$ are L^2-orthogonal, one concludes that $\boldsymbol{K}$ satisfies the integrability conditions (3.3) with $\mathcal{S}_1 = \varnothing$, and therefore, $\boldsymbol{K}$ is compatible. However, if we impose zero displacement on $\partial\mathcal{B}$, then it is impossible to satisfy (3.3c), which states that $\boldsymbol{K}$ must be orthogonal to $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing)$. This means that $\boldsymbol{K}$ is not compatible if we fix the whole boundary. For

3D bodies, we have (e.g. see [14, page 410])

$$\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \varnothing, \partial\mathcal{B}) = 3b_1(\bar{\mathcal{B}}) = 3(\text{genus of } \partial\mathcal{B}),$$
$$\dim \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing) = 3b_2(\bar{\mathcal{B}}) = 3\big((\#\text{ of components of } \partial\mathcal{B}) - 1\big).$$

The above discussion for non-zero $\boldsymbol{K} \in \mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing)$ also holds for 3D bodies with $b_2(\bar{\mathcal{B}}) \neq 0$. Note that in 2D, the dimension of $\mathcal{H}^{\otimes}(\bar{\mathcal{B}}, \partial\mathcal{B}, \varnothing)$ is non-zero if and only if $\bar{\mathcal{B}}$ is non-simply-connected. However, in 3D, this dimension is determined by $b_2(\bar{\mathcal{B}})$, which is not related to simply-connectedness, see Figure 2(b).

3.2 Linear Elasticity

The effects of boundary conditions on the compatibility of the linear strain can be studied using arguments similar to those of §2 together with the Hodge-Morrey-Friedrichs-type L^2-orthogonal decompositions introduced in [15]. More specifically, consider the following spaces for 3D bodies:

$$\mathcal{H}^S(\bar{\mathcal{B}}) := \{\boldsymbol{H} \in H^2(S^2T\mathcal{B}) : \boldsymbol{H} \in \ker \mathbf{curl} \circ \mathbf{curl} \cap \ker \mathbf{div}\},$$
$$\mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}}) := \{\boldsymbol{H} \in \mathcal{H}^S(\bar{\mathcal{B}}) : \boldsymbol{H}(\boldsymbol{N}) = 0, \text{ on } \partial\mathcal{B}\},$$
$$\mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}}) := \{\boldsymbol{H} \in \mathcal{H}^S(\bar{\mathcal{B}}) : \boldsymbol{H} = \mathbf{grad}^{\mathsf{s}}\boldsymbol{Z},\ \boldsymbol{Z}|_{\partial\mathcal{B}} \in \mathsf{RIG}\},$$

where $\mathsf{RIG} := \ker \mathbf{grad}^{\mathsf{s}}$, is the space of infinitesimal rigid motions. The spaces $\mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}})$ and $\mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}})$ are finite-dimensional with

$$\dim \mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}}) = 6b_1(\bar{\mathcal{B}}), \text{ and } \dim \mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}}) = 6b_1(\bar{\mathcal{B}}, \partial\mathcal{B}) = 6b_2(\bar{\mathcal{B}}).$$

For any $\boldsymbol{S} \in H^2(S^2T\mathcal{B})$, one can write the following L^2-orthogonal decompositions:

$$\boldsymbol{S} = \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C}_{\boldsymbol{S}} + \mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{S}} + \boldsymbol{Q}_{\boldsymbol{S}} + \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{A}_{\boldsymbol{S}}, \tag{3.8a}$$
$$\boldsymbol{S} = \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C}_{\boldsymbol{S}} + \mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{S}} + \boldsymbol{R}_{\boldsymbol{S}} + \mathbf{grad}^{\mathsf{s}}\boldsymbol{Z}_{\boldsymbol{S}}, \tag{3.8b}$$

where $\boldsymbol{C}_{\boldsymbol{S}} \in H^2_0(S^2T\mathcal{B})$, $\boldsymbol{V}_{\boldsymbol{S}} \in H^1_0(T\mathcal{B})$, $\boldsymbol{Q}_{\boldsymbol{S}} \in \mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}})$, $\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{A}_{\boldsymbol{S}} \in \mathcal{H}^S(\bar{\mathcal{B}})$, $\boldsymbol{R}_{\boldsymbol{S}} \in \mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}})$, and $\mathbf{grad}^{\mathsf{s}}\boldsymbol{Z}_{\boldsymbol{S}} \in \mathcal{H}^S(\bar{\mathcal{B}})$. By suitably replacing $\mathbf{curl} \circ \mathbf{curl}$ with D_{c} or D_{s}, one can write similar decompositions for 2D bodies. In particular, we have $\mathcal{H}^S(\bar{\mathcal{B}}) = H^2(S^2T\mathcal{B}) \cap \ker \mathrm{D}_{\mathsf{c}} \cap \ker \mathbf{div}$, and

$$\dim \mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}}) = \dim \mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}}) = 3b_1(\bar{\mathcal{B}}) = 3b_1(\bar{\mathcal{B}}, \partial\mathcal{B}).$$

The 2D analogues of the decompositions (3.8) read

$$\boldsymbol{S} = \mathrm{D}_{\mathsf{s}}(f_{\boldsymbol{S}}) + \mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{S}} + \boldsymbol{Q}_{\boldsymbol{S}} + \mathrm{D}_{\mathsf{s}}(h_{\boldsymbol{S}}), \tag{3.9a}$$
$$\boldsymbol{S} = \mathrm{D}_{\mathsf{s}}(f_{\boldsymbol{S}}) + \mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{S}} + \boldsymbol{R}_{\boldsymbol{S}} + \mathbf{grad}^{\mathsf{s}}\boldsymbol{Z}_{\boldsymbol{S}}, \tag{3.9b}$$

with $f_{\boldsymbol{S}} \in H^2_0(\mathcal{B})$ and $\mathrm{D}_{\mathsf{s}}(h_{\boldsymbol{S}}) \in \mathcal{H}^S(\bar{\mathcal{B}})$. The following Green's formula holds for any $\boldsymbol{Y} \in H^1(T\mathcal{B})$ and $\boldsymbol{S} \in H^1(S^2T\mathcal{B})$:

$$\langle\!\langle \mathbf{grad}^{\mathsf{s}}\boldsymbol{Y}, \boldsymbol{S} \rangle\!\rangle_{L^2} = -\langle\!\langle \boldsymbol{Y}, \mathbf{div}\, \boldsymbol{S} \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, \boldsymbol{S}(\boldsymbol{N}) \rangle\!\rangle dA. \tag{3.10}$$

One can also write Green's formula for $\mathbf{curl} \circ \mathbf{curl}$ as follows [11]: For any $\boldsymbol{S}, \boldsymbol{T} \in H^2(S^2T\mathcal{B})$, we have

$$\begin{aligned} \langle\!\langle \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{T}, \boldsymbol{S} \rangle\!\rangle_{L^2} &= \langle\!\langle \boldsymbol{T}, \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{S} \rangle\!\rangle_{L^2} \\ &\quad + \sum_{I=1}^{3} \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{N}, \overrightarrow{\mathbf{curl}\, \boldsymbol{T}}_{\mathbf{E}_I} \times \overrightarrow{\boldsymbol{S}}_{\mathbf{E}_I} + \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} \times \overrightarrow{\mathbf{curl}\, \boldsymbol{S}}_{\mathbf{E}_I} \rangle\!\rangle dA, \end{aligned} \tag{3.11}$$

where $\times$ is the standard cross product of vectors and $(\mathbf{curl}\, \boldsymbol{T})^{IJ} = (\mathbf{curl}^{\mathsf{T}} \boldsymbol{T})^{JI}$. The 2D counterpart of (3.11) reads as follows: Let $\boldsymbol{u}_{\partial}$ be the (oriented) unit vector field along $\partial\mathcal{B}$. Then, for any $\boldsymbol{T} \in H^2(S^2T\mathcal{B})$ and

$f \in H^2(\mathcal{B})$, one can write

$$\langle\!\langle \mathrm{D}_{\mathsf{c}}(\boldsymbol{T}), f \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{T}, \mathrm{D}_{\mathsf{s}}(f) \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{u}_\partial, f\, \boldsymbol{\varkappa}(\boldsymbol{T}) + f_{,2} \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_1} - f_{,1} \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_2} \rangle\!\rangle ds. \tag{3.12}$$

The classical compatibility problem for the linear strain can be stated as follows.

$$\textit{Given } \boldsymbol{\epsilon} \in H^2(S^2T\mathcal{B}), \textit{ find a displacement field } \boldsymbol{U} \in H^1(T\mathcal{B}) \textit{ such that } \boldsymbol{\epsilon} = \mathbf{grad}^{\mathsf{s}}\boldsymbol{U}. \tag{3.13}$$

The decompositions (3.8b) and (3.9b) allow one to easily prove the following theorem.

Theorem 13. *The compatibility problem (3.13) admits a solution if and only if*

$$\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{\epsilon} = 0 \ (\textit{or } \mathrm{D}_{\mathsf{c}}(\boldsymbol{\epsilon}) = 0, \ \textit{if } n = 2), \tag{3.14a}$$

$$\langle\!\langle \boldsymbol{\epsilon}, \boldsymbol{R} \rangle\!\rangle_{L^2} = 0, \quad \forall \boldsymbol{R} \in \mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}}). \tag{3.14b}$$

Proof. The necessity of these conditions simply follows from the relation $\mathbf{curl} \circ \mathbf{curl} \circ \mathbf{grad}^{\mathsf{s}} = 0$, and Green's formula (3.10). On the other hand, if $\boldsymbol{\epsilon} \in \ker \mathbf{curl} \circ \mathbf{curl}$, then (3.8b) and (3.11) imply that $\boldsymbol{\epsilon}$ can be decomposed as $\boldsymbol{\epsilon} = \mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{\epsilon}} + \boldsymbol{R}_{\boldsymbol{\epsilon}} + \mathbf{grad}^{\mathsf{s}}\boldsymbol{Z}_{\boldsymbol{\epsilon}}$. Then, (3.14b) suggests that $\boldsymbol{R}_{\boldsymbol{\epsilon}} = 0$, and therefore $\boldsymbol{\epsilon} = \mathbf{grad}^{\mathsf{s}}(\boldsymbol{V}_{\boldsymbol{\epsilon}} + \boldsymbol{Z}_{\boldsymbol{\epsilon}})$. Similar arguments hold for 2D bodies as well. □

Remark 14. Ting [16] obtained a weak formulation for the compatibility of the linear strain by using a Helmholtz-type decomposition for symmetric tensors. He wrote a condition similar to (3.14b) with $\boldsymbol{R}$ belonging to the infinite-dimensional space of divergence-free symmetric tensors with zero boundary tractions. Georgescu [17, Theorem 5.3] derived conditions equivalent to (3.14). In his formulation, instead of the harmonic space $\mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}})$, he used the tensor product of infinitesimal rigid motions and some specific harmonic vector fields. It is straightforward to show that $\mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}})$ is equal to that tensor product.

Next, we study the compatibility of the linear strain with prescribed values for the displacement on $\partial\mathcal{B}$. More specifically, we consider the following boundary-value problem:

Given $\boldsymbol{\epsilon} \in H^2(S^2T\mathcal{B})$ *and a vector field* $\boldsymbol{Y} \in H^3(T\mathcal{B})$*, find a displacement* $\boldsymbol{U} \in H^3(T\mathcal{B})$ *such that*

$$\begin{aligned} \mathbf{grad}^{\mathsf{s}}\boldsymbol{U} &= \boldsymbol{\epsilon}, \quad \text{on } \mathcal{B}, \\ \boldsymbol{U} &= \boldsymbol{Y}, \quad \text{on } \partial\mathcal{B}. \end{aligned} \tag{3.15}$$

Theorem 15. *The following sets of conditions are equivalent and each set is necessary and sufficient for the existence of a solution to (3.15):*

i) The weak compatibility equations:

$$\langle\!\langle \boldsymbol{\epsilon}, \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C} \rangle\!\rangle_{L^2} = 0, \qquad \forall \boldsymbol{C} \in H^2_0(S^2T\mathcal{B}), \tag{3.16a}$$

$$\langle\!\langle \boldsymbol{\epsilon}, \boldsymbol{H} \rangle\!\rangle_{L^2} = \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, \boldsymbol{H}(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{H} \in \mathcal{H}^S(\bar{\mathcal{B}}). \tag{3.16b}$$

In 2D, the condition (3.16a) is replaced with

$$\langle\!\langle \boldsymbol{\epsilon}, \mathrm{D}_{\mathsf{s}}(f) \rangle\!\rangle_{L^2} = 0, \quad \forall f \in H^2_0(\mathcal{B}).$$

ii) The strong compatibility equations:

$$\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{\epsilon}=0, \tag{3.17a}$$

$$\int_{\partial\mathcal{B}}\sum_{I=1}^{3}\Big\{\langle\!\langle \boldsymbol{N},\overrightarrow{\mathbf{curl}\,\boldsymbol{A}}_{\mathbf{E}_I}\times\overrightarrow{(\boldsymbol{\epsilon}-\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y})}_{\mathbf{E}_I}\rangle\!\rangle+ \langle\!\langle \boldsymbol{N},\overrightarrow{\boldsymbol{A}}_{\mathbf{E}_I}\times\overrightarrow{(\mathbf{curl}(\boldsymbol{\epsilon}-\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y}))}_{\mathbf{E}_I}\rangle\!\rangle\Big\}dA=0,\quad \forall\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}\in\mathcal{H}^{S}(\bar{\mathcal{B}}) \tag{3.17b}$$

$$\langle\!\langle\boldsymbol{\epsilon},\boldsymbol{Q}\rangle\!\rangle_{L^2}=\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{Y},\boldsymbol{Q}(\boldsymbol{N})\rangle\!\rangle dA,\qquad \forall\boldsymbol{Q}\in\mathcal{H}^{S}_{\mathbf{g}}(\bar{\mathcal{B}}), \tag{3.17c}$$

For 2D bodies, the condition (3.17a) is replaced with $\mathrm{D}_{\mathsf{c}}(\boldsymbol{\epsilon})=0$ *and (3.17b) is replaced with*

$$\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{u}_{\partial},h\,\mathbf{c}(\boldsymbol{\epsilon}-\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y})+h_{,2}\overrightarrow{(\boldsymbol{\epsilon}-\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y})}_{\mathbf{E}_1}-h_{,1}\overrightarrow{(\boldsymbol{\epsilon}-\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y})}_{\mathbf{E}_2}\rangle\!\rangle ds=0,\quad \forall\mathrm{D}_{\mathsf{s}}(h)\in\mathcal{H}^{S}(\bar{\mathcal{B}}).$$

Proof. Suppose $\bar{\mathcal{B}}$ is 3D. By using (3.10) and (3.11), it is straightforward to show that the integrability conditions (3.16) are necessary. For proving the sufficiency of (3.16), note that (3.8a) and (3.16a) suggest that $\boldsymbol{\epsilon}$ can be decomposed as $\boldsymbol{\epsilon}=\mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{\epsilon}}+\boldsymbol{Q}_{\boldsymbol{\epsilon}}+\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}_{\boldsymbol{\epsilon}}$. One can also write the Helmholtz-type decomposition $\boldsymbol{Y}=\mathbf{div}\,\boldsymbol{T}_{\boldsymbol{Y}}+\boldsymbol{W}_{\boldsymbol{Y}}$, where $\boldsymbol{T}_{\boldsymbol{Y}}$ has zero traction on $\partial\mathcal{B}$ and $\mathbf{grad}^{\mathsf{s}}\boldsymbol{W}_{\boldsymbol{Y}}=0$. Next, we define $\boldsymbol{D}:=\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y}$. Using (3.8a) and the relation $\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{D}=0$, one can write the decomposition $\boldsymbol{D}=\mathbf{grad}^{\mathsf{s}}\boldsymbol{V}_{\boldsymbol{D}}+\boldsymbol{Q}_{\boldsymbol{D}}+\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}_{\boldsymbol{D}}$. Let $\boldsymbol{U}:=\boldsymbol{V}_{\boldsymbol{\epsilon}}-\boldsymbol{V}_{\boldsymbol{D}}+\boldsymbol{Y}$. Then, we have $\boldsymbol{U}|_{\partial\mathcal{B}}=\boldsymbol{Y}$, and $\mathbf{grad}^{\mathsf{s}}\boldsymbol{U}=\boldsymbol{\epsilon}+\widetilde{\boldsymbol{H}}$, where

$$\widetilde{\boldsymbol{H}}:=\mathbf{curl}\circ\mathbf{curl}(\boldsymbol{A}_{\boldsymbol{D}}-\boldsymbol{A}_{\boldsymbol{\epsilon}})+\boldsymbol{Q}_{\boldsymbol{D}}-\boldsymbol{Q}_{\boldsymbol{\epsilon}}\in\mathcal{H}^{S}(\bar{\mathcal{B}}).$$

On the other hand, (3.10) and the condition (3.16b) allows one to write

$$\langle\!\langle\widetilde{\boldsymbol{H}},\boldsymbol{H}\rangle\!\rangle_{L^2}=\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{Y},\boldsymbol{H}(\boldsymbol{N})\rangle\!\rangle dA-\langle\!\langle\boldsymbol{\epsilon},\boldsymbol{H}\rangle\!\rangle_{L^2}=0,\quad \forall\boldsymbol{H}\in\mathcal{H}^{S}(\bar{\mathcal{B}}),$$

which means that $\widetilde{\boldsymbol{H}}$ is also orthogonal to $\mathcal{H}^{S}(\bar{\mathcal{B}})$. Hence, $\widetilde{\boldsymbol{H}}=0$ and $\boldsymbol{U}$ is a solution of (3.15). This shows that the integrability conditions (3.16) are also sufficient.

Next, we show that (3.16) and (3.17) are equivalent. The fact that $H^2_0(S^2T\mathcal{B})$ is dense in $L^2(S^2T\mathcal{B})$ and Green's formula (3.10) suggest that (3.16a) is equivalent to (3.17a). The decomposition (3.8a) implies that (3.16b) is equivalent to (3.17c) together with

$$\langle\!\langle\boldsymbol{\epsilon},\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}\rangle\!\rangle_{L^2}=\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{Y},(\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A})(\boldsymbol{N})\rangle\!\rangle dA,\quad \forall\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}\in\mathcal{H}^{S}(\bar{\mathcal{B}}). \tag{3.18}$$

Using (3.11) and (3.17a), the left-hand side of (3.18) can be written as

$$\langle\!\langle\boldsymbol{\epsilon},\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}\rangle\!\rangle_{L^2}=\sum_{I=1}^{3}\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{N},\overrightarrow{\mathbf{curl}\,\boldsymbol{A}}_{\mathbf{E}_I}\times\overrightarrow{\boldsymbol{\epsilon}}_{\mathbf{E}_I}+\overrightarrow{\boldsymbol{A}}_{\mathbf{E}_I}\times\overrightarrow{\mathbf{curl}\,\boldsymbol{\epsilon}}_{\mathbf{E}_I}\rangle\!\rangle dA. \tag{3.19}$$

On the other hand, (3.10) and (3.11) allow one to write the right-hand side of (3.18) as

$$\begin{aligned}\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{Y},(\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A})(\boldsymbol{N})\rangle\!\rangle dA&=\langle\!\langle\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y},\mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{A}\rangle\!\rangle_{L^2}\\&=\sum_{I=1}^{3}\int_{\partial\mathcal{B}}\langle\!\langle\boldsymbol{N},\overrightarrow{\mathbf{curl}\,\boldsymbol{A}}_{\mathbf{E}_I}\times\overrightarrow{\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y}}_{\mathbf{E}_I}+\overrightarrow{\boldsymbol{A}}_{\mathbf{E}_I}\times\overrightarrow{\mathbf{curl}(\mathbf{grad}^{\mathsf{s}}\boldsymbol{Y})}_{\mathbf{E}_I}\rangle\!\rangle dA.\end{aligned} \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18) gives the condition (3.17b), and therefore, (3.16) and (3.17) are equivalent. Note that (3.20) tells us that similar to (3.3b), the condition (3.17b) only depends on the values of

$\boldsymbol{Y}$ on $\partial\mathcal{B}$. We do not know how to further simplify (3.17b) to more clearly reflect this fact. Similar arguments together with (3.12) allow one to prove the 2D case as well. □

Remark 16. The integrability conditions (3.17) imply that a linear strain $\boldsymbol{\epsilon}$ with prescribed boundary-value $\boldsymbol{Y}$ is compatible if and only if $\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{\epsilon} = 0$ (or $\mathrm{D}_{\mathbf{c}}(\boldsymbol{\epsilon}) = 0$), $\boldsymbol{\epsilon}$ and $\boldsymbol{Y}$ satisfy (3.17b) on $\partial\mathcal{B}$, and for any equilibrated *virtual* stress $\boldsymbol{H}$ in the finite-dimensional space $\mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}})$, the work done by $\boldsymbol{\epsilon}$ must be the same as the work done by $\boldsymbol{Y}$. Rostamian [3, Corollary 3.1] showed that the compatibility problem (3.15) admits a solution if and only if

$$\langle\!\langle \boldsymbol{\epsilon}, \boldsymbol{\sigma} \rangle\!\rangle_{L^2} = \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, \boldsymbol{\sigma}(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{\sigma} \in \mathbb{D}(S^2 T\mathcal{B}), \tag{3.21}$$

where $\mathbb{D}(S^2T\mathcal{B})$ is the space of symmetric divergence-free tensors. The decompositions (3.8) tell us that any $\boldsymbol{\sigma} \in \mathbb{D}(S^2T\mathcal{B})$ can be decomposed as $\boldsymbol{\sigma} = \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C}_{\boldsymbol{\sigma}} + \boldsymbol{H}_{\boldsymbol{\sigma}}$, where $\boldsymbol{C}_{\boldsymbol{\sigma}} \in H^2_0(S^2T\mathcal{B})$, and $\boldsymbol{H}_{\boldsymbol{\sigma}} \in \mathcal{H}^S(\bar{\mathcal{B}})$. Thus, the condition (3.21) is equivalent to (3.16b) together with

$$\langle\!\langle \boldsymbol{\epsilon}, \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C} \rangle\!\rangle_{L^2} = \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, (\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C})(\boldsymbol{N}) \rangle\!\rangle dA, \quad \forall \boldsymbol{C} \in H^2_0(S^2T\mathcal{B}). \tag{3.22}$$

By using (3.10), the right-hand side of the above equation can be written as

$$\int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, (\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C})(\boldsymbol{N}) \rangle\!\rangle dA = \langle\!\langle \mathbf{grad}^{\mathsf{s}} \boldsymbol{Y}, \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{C} \rangle\!\rangle_{L^2} = 0,$$

where the last equality follows from the fact that the image of $\mathbf{grad}^{\mathsf{s}}$ and $\mathbf{curl} \circ \mathbf{curl}(H^2_0(S^2T\mathcal{B}))$ are normal to each other. Therefore, the integrability condition (3.21) is equivalent to the integrability conditions in Theorem 15. Note that in practice, it is much easier to use (3.17) instead of (3.21), because unlike $\mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}})$, the space $\mathbb{D}(S^2T\mathcal{B})$ is infinite-dimensional.

Remark 17. By considering equilibrated virtual stresses with zero-traction on $\mathcal{S}_2$, the integrability condition (3.21) also expresses the compatibility condition for linear strains with prescribed displacements on $\mathcal{S}_1 \subset \partial\mathcal{B}$ (this extension of the result of Dorn and Schild [2] was first proved by Gurtin [18, page 188] for simply-connected bodies and then by Rostamian [3, Corollary 3.1] for arbitrary bodies). The compatibility problem (3.15) was discussed only for the case $\mathcal{S}_1 = \partial\mathcal{B}$ as the decompositions (3.8a) and (3.9a) are written by imposing boundary conditions on $\partial\mathcal{B}$. If one can find a decomposition for symmetric harmonic tensors with appropriate boundary conditions on $\mathcal{S}_1$ and $\mathcal{S}_2$ (similar to those of Theorem 1), then the arguments in the proof of Theorem 15 can be used to extend the strong integrability conditions (3.17) to the compatibility problem with prescribed displacements on $\mathcal{S}_1 \neq \partial\mathcal{B}$.

Example 18. The conclusions of Example 12 hold for linear strains as well. More specifically, suppose $\boldsymbol{\epsilon} \neq 0$ belongs to $\mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}})$. This means that $\bar{\mathcal{B}}$ is non-simply-connected in 2D or $b_2(\bar{\mathcal{B}}) \neq 0$ in 3D. Since $\mathcal{H}^S_{\mathbf{g}}(\bar{\mathcal{B}})$ and $\mathcal{H}^S_{\mathbf{c}}(\bar{\mathcal{B}})$ are orthogonal to each other, $\boldsymbol{\epsilon}$ satisfies the conditions (3.14), and therefore, it is compatible. On the other hand, if we impose zero displacement on $\partial\mathcal{B}$, then it is impossible to satisfy (3.17c), and hence, $\boldsymbol{\epsilon}$ is not compatible if we fix $\partial\mathcal{B}$.

4 Stress Tensors and Body Forces

For the sake of completeness, we show that the results of §2 are also useful for studying the existence of a tensor field that is in equilibrium with a given body force field and takes a prescribed boundary traction. Note that the linear structure of $\mathbb{R}^n$ allows one to consider body forces and first Piola-Kirchhoff stress tensors as vector fields and $\binom{2}{0}$-tensor fields on $\mathcal{B}$, respectively. Next, we consider the following boundary-value problem.

> *Given a body force $\boldsymbol{B} \in H^1(T\mathcal{B})$ and an arbitrary traction vector field $\boldsymbol{t}$ of class $H^{\frac{3}{2}}$ on $\mathcal{S}_1$, find a first Piola-Kirchhoff stress tensor $\boldsymbol{P} \in H^2(\otimes^2 T\mathcal{B})$ such that*
>
> $$\begin{aligned} \mathbf{div}\, \boldsymbol{P} + \boldsymbol{B} &= 0, \quad \text{on } \mathcal{B}, \\ \boldsymbol{P}(\boldsymbol{N}) &= \boldsymbol{t}, \quad \text{on } \mathcal{S}_1. \end{aligned} \tag{4.1}$$

Theorem 19. *If $\mathcal{S}_1 \neq \partial\mathcal{B}$, the boundary-value problem (4.1) always admits a solution. If $\mathcal{S}_1 = \partial\mathcal{B}$, (4.1) admits a solution if and only if the body $\mathcal{B}$ is in force equilibrium, that is*

$$\int_{\mathcal{B}} \boldsymbol{B} dV + \int_{\partial\mathcal{B}} \boldsymbol{t} dA = 0. \tag{4.2}$$

Proof. The Hodge-star operator allows one to rewrite the boundary-value problem (4.1) as follows:

Given $B^I dV \in H^1(\Lambda^n T^*\mathcal{B})$ and $*(t^I \boldsymbol{N})^\flat \in H^{\frac{3}{2}}(\Lambda^{n-1}T^*\mathcal{B}|_{\mathcal{S}_1})$, $I = 1, \ldots, n$, find $*\overrightarrow{\boldsymbol{P}}^\flat_{\mathbf{E}_I} \in H^2(\Lambda^{n-1}T^*\mathcal{B})$ such that

$$d(*\overrightarrow{\boldsymbol{P}}^\flat_{\mathbf{E}_I}) = -B^I dV, \quad \text{on } \mathcal{B},$$
$$\mathsf{t}(*\overrightarrow{\boldsymbol{P}}^\flat_{\mathbf{E}_I}) = *(t^I \boldsymbol{N})^\flat, \quad \text{on } \mathcal{S}_1.$$

The above problem automatically satisfies the integrability condition (2.7a). On the other hand, the condition (2.7b) can be written as

$$\begin{aligned}\langle\!\langle B^I, f \rangle\!\rangle_{L^2} = \langle\!\langle *B^I, *f \rangle\!\rangle_{L^2} &= -\int_{\mathcal{S}_1} \mathsf{t}(*t^I \boldsymbol{N})^\flat \wedge *\mathsf{n}(*f) \\ &= -\int_{\mathcal{S}_1} f\, t^I dA, \quad \forall f \in G(\mathcal{B}, \mathcal{S}_2) := \{g \in H^1(\mathcal{B}) : \operatorname{grad} g = 0 \text{ and } g|_{\mathcal{S}_2} = 0\}.\end{aligned}$$

If $\mathcal{S}_2 \neq \varnothing$, then $G(\mathcal{B}, \mathcal{S}_2)$ only contains the zero function and (2.7b) holds for any B^I and t^I. If $\mathcal{S}_2 = \varnothing$, then $G(\mathcal{B}, \mathcal{S}_2)$ contains constant functions, and (2.7b) simply reads $\int_{\mathcal{B}} B^I dV + \int_{\partial\mathcal{B}} t^I dA = 0$, $I = 1, \cdots, n$, which is equivalent to (4.2). ☐

Remark 20. Consider the boundary-value problem (4.1) for the Cauchy stress tensor, that is, instead of $\boldsymbol{P}$ consider a symmetric tensor $\boldsymbol{\sigma}$. This problem is always integrable if $\mathcal{S}_1 \neq \partial\mathcal{B}$, and if $\mathcal{S}_1 = \partial\mathcal{B}$, it is integrable if and only if

$$\langle\!\langle \boldsymbol{B}, \boldsymbol{Z} \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{t}, \boldsymbol{Z} \rangle\!\rangle dA = 0, \quad \forall \boldsymbol{Z} \in \mathsf{RIG},$$

where RIG is the space of infinitesimal rigid motions, e.g. see [3, Theorem 2.2]. This result can be proved using the Helmholtz-type decomposition mentioned in the proof of Theorem 15. The fact that any element of RIG is determined by a constant vector and a constant skew-symmetric matrix implies that the above condition is equivalent to

$$\int_{\mathcal{B}} \boldsymbol{B} dV + \int_{\partial\mathcal{B}} \boldsymbol{t} dA = 0, \text{ and } \int_{\mathcal{B}} \boldsymbol{X} \times \boldsymbol{B} dV + \int_{\partial\mathcal{B}} \boldsymbol{X} \times \boldsymbol{t} dA = 0,$$

where $\boldsymbol{X} \times \boldsymbol{B}$ and $\boldsymbol{X} \times \boldsymbol{t}$ denote the moments of $\boldsymbol{B}$ and $\boldsymbol{t}$ with respect to the origin. Therefore, the boundary-value problem (4.1) for symmetric tensors with $\mathcal{S}_1 = \partial\mathcal{B}$ is integrable if and only if the body $\mathcal{B}$ is in both force and moment equilibrium.

Acknowledgments. AA benefited from discussions with Dr. Yueh-Ju Lin. This research was partially supported by AFOSR – Grant No. FA9550-12-1-0290 and NSF – Grant No. CMMI 1042559 and CMMI 1130856.

References

[1] A. Yavari. Compatibility equations of nonlinear elasticity for non-simply-connected bodies. *Arch. Rational Mech. Anal.*, 209:237–253, 2013.

[2] W. S. Dorn and A. Schild. A converse to the virtual work theorem for deformable solids. *Q. Appl. Math.*, 14:209–213, 1956.

[3] R. Rostamian. Internal constraints in linear elasticity. *J. Elasticity*, 11:11–31, 1981.

[4] G. Schwarz. *Hodge Decomposition - A Method for Solving Boundary Value Problems (Lecture Notes in Mathematics-1607)*. Springer-Verlog, Berlin, 1995.

[5] V. Gol'dshtein, I. Mitrea, and M. Mitrea. Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds. *J. Math. Sci.*, 172:347–400, 2011.

[6] C. B. Morrey. *Multiple Integrals in the Calculus of Variations*. Springer-Verlog, New York, 1966.

[7] G. A. Baker and J. Dodziuk. Stability of spectra of Hodge-de Rham Laplacians. *Math. Z.*, 224:327–345, 1997.

[8] K. O. Friedrichs. Differential forms on Riemannian manifolds. *Comm. Pure Appl. Math.*, 8:551–590, 1955.

[9] G. F. D. Duff. On the potential theory of coclosed harmonic forms. *Can. J. Math.*, 7:126–137, 1955.

[10] T. Jakab, I. Mitrea, and M. Mitrea. On the regularity of differential forms satisfying mixed boundary conditions in a class of Lipschitz domains. *Indiana Univ. Math. J.*, 58:2043–2072, 2009.

[11] A. Angoshtari and A. Yavari. Hilbert complexes, orthogonal decompositions, and potentials for nonlinear continua. *Submitted*, 2015.

[12] A. Angoshtari and A. Yavari. Differential complexes in continuum mechanics. *Arch. Rational Mech. Anal.*, 216:193–220, 2015.

[13] G. F. D. Duff. Differential forms in manifolds with boundary. *Ann. Math.*, 56:115–127, 1952.

[14] J. Cantarella, D. De Turck, and H. Gluck. Vector calculus and the topology of domains in 3-space. *Amer. Math. Monthly*, 109:409–442, 2002.

[15] G. Geymonat and F. Krasucki. Hodge decomposition for symmetric matrix fields and the elasticity complex in Lipschitz domains. *Commun. Pure Appl. Anal.*, 8:295–309, 2009.

[16] T. W. Ting. Problem of compatibility and orthogonal decomposition of second-order symmetric tensors in a compact Riemannian manifold with boundary. *Arch. Rational Mech. Anal.*, 64:221–243, 1977.

[17] V. Georgescu. On the operator of symmetric differentiation on a compact Riemannian manifold with boundary. *Arch. Rational Mech. Anal.*, 74:143–164, 1980.

[18] M. E. Gurtin. Variational principles in the linear theory of viscoelasticity. *Arch. Rational Mech. Anal.*, 13: 179–191, 1963.