Covariance in Linearized Elasticity*

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Abstract

In this paper we covariantly obtain the governing equations of linearized elasticity. Our motivation is to see if one can make a connection between (global) balance of energy in nonlinear elasticity and its counterpart in linear elasticity. We start by proving a Green-Naghdi-Rivilin theorem for linearized elasticity. We do this by first linearizing energy balance about a given reference motion and then by postulating its invariance under isometries of the Euclidean ambient space. We also investigate the possibility of covariantly deriving a linearized elasticity theory, without any reference to the local governing equations, e.g. local balance of linear momentum. In particular, we study the consequences of linearizing covariant energy balance and covariance of linearized energy balance. We show that in both cases, covariance gives all the field equations of linearized elasticity.

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1 Introduction

Linear elasticity is based on the assumption that displacement gradients are small compared to the characteristic length(s) of the deformed body. Balances of linear and angular momenta in linear elasticity have the same forms as those of nonlinear elasticity. Kinematics is described with respect to a reference state and deformed and undeformed manifolds are not distinguishable. For example, Cauchy and the first Piola-Kirchhoff stress tensors are the same. In constitutive equations, stress and strain tensors are linearly related by a fourth-order tensor of elastic constants. Governing equations of linear elasticity can be obtained by linearizing those of nonlinear elasticity about a reference motion. In the geometric theory, where a body deforms in a Riemannian ambient space, one can obtain the governing equations of linear elasticity by geometric linearization of the governing equations of nonlinear elasticity [Marsden and Hughes, 1983]. Recently, Steigmann [2007] studied frame indifference of the governing equations of linear elasticity. His main conclusion is that linearized elasticity is frame-indifferent if it is properly formulated.

It is well known that balance laws in nonlinear elasticity can be obtained by postulating an energy balance and its invariance under isometries of the ambient space if it is Euclidean [Green and Rivilin, 1964] and diffeomorphisms of the ambient space if it is Riemannian (covariance) [Marsden and Hughes, 1983; Simo and Marsden, 1984; Yavari, et al., 2006]. Now one may ask what the connection between linearized and nonlinear elasticity is in terms of energy balance and its invariance. In this paper, we make this connection in both cases of Euclidean and Riemannian ambient space manifolds. In the case of a Euclidean ambient space manifold, we first linearize/quadratize energy balance. Note that if one starts with an equilibrium configuration, the linearization of internal energy density is null and therefore one needs to look at the higher order terms, namely the quadratic approximation, besides, a calculation to linear order in the stresses requires the energy to be treated to quadratic order. After linearization/quadratization of energy balance, we find a linearized energy balance and a quadratized energy balance that are separately satisfied. We then postulate the invariance of energy balance under time-dependent rigid translations and rotations of the ambient space. We will show that invariance of the linearized energy balance gives the equations of linearized elasticity. Interestingly, the quadratized energy balance is trivially invariant under isometries of the ambient space. We show that both linearization of invariant energy balance and invariance of linearized energy balance will give all the governing equations of linearized elasticity.

In the case of a general Riemannian manifold ambient space, we study two things: (i) linearization of covariant energy balance and (ii) covariance of linearized energy balance. By "linearization of covariant energy balance" we mean linearization of difference of energy balance for a nearby motion with respect to a reference motion. We show that this linearization will give the governing equations of linearized elasticity. In the more interesting case, we first linearize energy balance with respect to a reference motion $\overset{\circ}{\varphi}_t$ and then postulate the invariance of the linearized energy balance with respect to diffeomorphisms of the ambient space. We also extend the ideas of first variation of "energy" of maps [Nishikawa, 2002] to elasticity, where energy has a more complicated form.

This paper is structured as follows. In §2 we give a brief introduction to geometric elasticity in order to make the paper self contained. In §3 we study invariance of linearized energy balance when the ambient space is Euclidean. We show the connection between energy balance in nonlinear and linear elasticity. We review Marsden and Hughes' idea of geometric linearization of elasticity in §4 and present some new results. We also revisit linearization of elasticity using variation of maps and ideas from geometric calculus of variations. In §5 we study different notions of covariance in linearized elasticity. In particular, we covariantly obtain all the governing equations of linearized elasticity. Conclusions are given in §6.

2 Linearized Elasticity in Euclidean Ambient Space

It has long been known that one can obtain all the balance laws of elasticity by postulating balance of energy and its invariance under (time-dependent) rigid translations and rotations of the current configuration [Green and Rivilin, 1964]. Here we are interested in formulating a version of the Green-Naghdi-Rivilin theorem for linearized elasticity.

Let φ_t denote a motion of a body. Energy balance for an arbitrary subbody $\mathcal{U} \subset \mathcal{B}$ is written as

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dv = \int_{\varphi_t(\mathcal{U})} \rho\left(\mathbf{b}\cdot\mathbf{v} + r\right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\mathbf{t}\cdot\mathbf{v} + h\right) da, \tag{2.1}$$

in spatial coordinates, where ρ is the density, e, r and h are the internal energy function per unit mass, the heat supply per unit mass and the heat flux, respectively, and $\mathbf{v}$, $\mathbf{b}$, and $\mathbf{t}$ are spatial velocity, body force per unit mass, and traction, respectively. In material coordinates

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0\left(\Psi + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) dV = \int_{\mathcal{U}} \rho_0\left(\mathbf{B}\cdot\mathbf{V} + R\right) dV + \int_{\partial\mathcal{U}} \left(\mathbf{T}\cdot\mathbf{V} + H\right) dA, \tag{2.2}$$

where $\Psi = \Psi(t, \mathbf{X}, \mathbf{F})$ is the free energy density per unit mass of the undeformed configuration, ρ_0 is the density per unit undeformed volume, and R, H, $\mathbf{V}$, $\mathbf{B}$ and $\mathbf{T}$ are the per unit undeformed mass versions of r, h, $\mathbf{v}$, $\mathbf{b}$ and $\mathbf{t}$, respectively. We start with material energy balance as it is written for a fixed domain and makes the calculations simpler.

Let us assume that we are given a reference motion $\overset{\circ}{\varphi}_t$. Balance of energy for this fixed motion is written as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0\left(\overset{\circ}{\Psi} + \frac{1}{2}\overset{\circ}{\mathbf{V}}\cdot\overset{\circ}{\mathbf{V}}\right) dV = \int_{\mathcal{U}} \rho_0\left(\overset{\circ}{\mathbf{B}}\cdot\overset{\circ}{\mathbf{V}} + \overset{\circ}{R}\right) dV + \int_{\partial\mathcal{U}} \left(\overset{\circ}{\mathbf{T}}\cdot\overset{\circ}{\mathbf{V}} + \overset{\circ}{H}\right) dA. \tag{2.3}$$

Now consider a C^∞ variation of this motion $\varphi_{t,s}$ parametrized by s, such that $\varphi_{t,0} = \overset{\circ}{\varphi}_t$. For each value of s, the energy balance is of the form (2.3). Now let us assume that for any value of s, energy balance is invariant under a time-dependent rigid translation of the deformed configuration $\xi_t(\mathbf{x}) = \mathbf{x} + (t - t_0)\mathbf{w}$. This would give the following two relations for $s = 0$ and $s \neq 0$ [Yavari, et al., 2006]

$$\int_{\mathcal{U}} \frac{\partial\rho_0}{\partial t}\left(\mathbf{w}\cdot\overset{\circ}{\mathbf{V}} + \frac{1}{2}\mathbf{w}\cdot\mathbf{w}\right) dV = \int_{\mathcal{U}} \rho_0(\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{A}})\cdot\mathbf{w}dV + \int_{\partial\mathcal{U}} \overset{\circ}{\mathbf{T}}\cdot\mathbf{w}dA, \tag{2.4}$$

$$\int_{\mathcal{U}} \frac{\partial\rho_0}{\partial t}\left(\mathbf{w}\cdot\mathbf{V} + \frac{1}{2}\mathbf{w}\cdot\mathbf{w}\right) dV = \int_{\mathcal{U}} \rho_0(\mathbf{B} - \mathbf{A})\cdot\mathbf{w}dV + \int_{\partial\mathcal{U}} \mathbf{T}\cdot\mathbf{w}dA, \tag{2.5}$$

where $\overset{\circ}{\mathbf{A}}$ and $\mathbf{A}$ are the material accelerations for motions $\overset{\circ}{\varphi}_t$ and φ_t, respectively. Arbitrariness of $\mathbf{w}$ gives conservation of mass $\frac{\partial\rho_0}{\partial t} = 0$ and subtracting the above two relations gives

$$\int_{\mathcal{U}} \rho_0(\mathbf{A} - \overset{\circ}{\mathbf{A}})\cdot\mathbf{w}dV = \int_{\mathcal{U}} \rho_0(\mathbf{B} - \overset{\circ}{\mathbf{B}})\cdot\mathbf{w}dV + \int_{\partial\mathcal{U}} (\mathbf{T} - \overset{\circ}{\mathbf{T}})\cdot\mathbf{w}dA = 0. \tag{2.6}$$

Linearizing the above identity about $\overset{\circ}{\varphi}_t$ gives

$$\int_{\mathcal{U}} \rho_0 \delta \mathbf{A} \cdot \mathbf{w} dV = \int_{\mathcal{U}} \rho_0 \delta \mathbf{B} \cdot \mathbf{w} dV + \int_{\partial \mathcal{U}} \delta \mathbf{T} \cdot \mathbf{w} dA. \tag{2.7}$$

Let $\mathbf{U}$ denote the vector field whose integral curves are given by $\varphi_{t=t_0,s}$, i.e., $U^i = \frac{\partial \varphi^i_{t,s}}{\partial s}\big|_{s=0}$. Then, the linearized invariance equation (2.7) can be written in terms of $\mathbf{U}$, once one observes that $\mathbf{V} - \mathbf{V_0} = \delta \mathbf{V} = \dot{\mathbf{U}}$, as

$$\int_{\mathcal{U}} \rho_0 \ddot{\mathbf{U}} \cdot \mathbf{w} dV = \int_{\mathcal{U}} \rho_0 \delta \mathbf{B} \cdot \mathbf{w} dV + \int_{\partial \mathcal{U}} \delta \mathbf{T} \cdot \mathbf{w} dA. \tag{2.8}$$

Since $\delta \mathbf{T} = \operatorname{Div} \delta \mathbf{P} \cdot \hat{\mathbf{N}}$, where $\hat{\mathbf{N}}$ is the unit normal vector to $\partial \mathcal{U}$ at $\mathbf{X} \in \partial \mathcal{U}$, arbitrariness of $\mathbf{w}$ and $\mathcal{U}$ will imply the following.

$$\rho_0 \ddot{\mathbf{U}} = \rho_0 \delta \mathbf{B} + \operatorname{Div} \delta \mathbf{P}, \tag{2.9}$$

or

$$\rho_0 \delta \mathbf{A} = \rho_0 \delta \mathbf{B} + \operatorname{Div} \delta \mathbf{P}, \tag{2.10}$$

which is nothing but linearization of the local balance of linear momentum. Similarly, assuming invariance of energy balance under rotations with constant velocity, linearization of energy balance difference will give linearization of balance of angular momentum.

Next, let us linearize the balance of energy about a reference motion first, and then postulate its invariance under isometries of the Euclidean ambient space. This turns out to be the more interesting case.

Let us consider motions that are "close" to $\overset{\circ}{\varphi}_t$ and write $\varphi_t = \overset{\circ}{\varphi}_t + \delta \varphi_t$, where $\|\delta \varphi_t\| / \| \overset{\circ}{\varphi}_t \| \ll 1$, and similarly for the spatial and time derivatives of $\delta \varphi$, where $\|.\|$ is the standard Euclidean norm. Balance of energy for the perturbed motion is written as

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{U}} \rho_0 & \left[\overset{\circ}{\Psi} + \delta \Psi + \frac{1}{2} (\overset{\circ}{\mathbf{V}} + \delta \mathbf{V}) \cdot (\overset{\circ}{\mathbf{V}} + \delta \mathbf{V}) \right] dV \\ & = \int_{\mathcal{U}} \rho_0 \left[(\overset{\circ}{\mathbf{B}} + \delta \mathbf{B}) \cdot (\overset{\circ}{\mathbf{V}} + \delta \mathbf{V}) + \overset{\circ}{R} + \delta R \right] dV \\ & \quad + \int_{\partial \mathcal{U}} \left[(\overset{\circ}{\mathbf{T}} + \delta \mathbf{T}) \cdot (\overset{\circ}{\mathbf{V}} + \delta \mathbf{V}) + \overset{\circ}{H} + \delta H \right] dA. \end{aligned} \tag{2.11}$$

Note that

$$\mathbf{B}(\mathbf{X}) = \mathbf{b}(\varphi_t(\mathbf{X})) \,, \tag{2.12}$$

and thus

$$\delta \mathbf{B} = \frac{\partial \mathbf{b}}{\partial \mathbf{x}} \cdot \delta \varphi_t \,, \tag{2.13}$$

which is shorthand for the following componentwise equations

$$\delta B^i = \sum_j \frac{\partial b^i}{\partial x^j} \delta \varphi_t^j. \tag{2.14}$$

Also

$$\Psi = \Psi(\mathbf{X}, T\varphi_t) \,, \tag{2.15}$$

where $T\varphi_t = \mathbf{F} = \frac{\partial \varphi_t}{\partial \mathbf{X}}$ denotes the deformation gradient. Therefore, to first order in $\delta\varphi_t$

$$\delta\Psi = \frac{\partial \Psi}{\partial \mathbf{F}} \cdot \delta\mathbf{F} = \frac{\partial \Psi}{\partial \mathbf{F}} \cdot \frac{\partial \delta\varphi_t}{\partial \mathbf{X}} = \overset{\circ}{\mathbf{P}} \cdot \frac{\partial \delta\varphi_t}{\partial \mathbf{X}}\,, \tag{2.16}$$

which, when written out in components, reads[1]

$$\delta\Psi = \frac{\partial \Psi}{\partial F^i{}_j} \delta F^i{}_j = \frac{\partial \Psi}{\partial F^i{}_j} \frac{\partial \delta\varphi_t^i}{\partial X^j} = \overset{\circ}{P}_i{}^j \, \frac{\partial \delta\varphi_t^i}{\partial X^j}. \tag{2.17}$$

Velocity variation is calculated as follows.

$$\mathbf{V} = \mathbf{V}(\mathbf{X}) = \frac{\partial \varphi_t(\mathbf{X})}{\partial t} = \frac{\partial \left(\overset{\circ}{\varphi}_t + \delta\varphi_t\right)}{\partial t} = \overset{\circ}{\mathbf{V}} + \frac{\partial \delta\varphi_t(\mathbf{X})}{\partial t}. \tag{2.18}$$

Thus

$$\delta\mathbf{V} = \frac{\partial \delta\varphi_t(\mathbf{X})}{\partial t}. \tag{2.19}$$

Similarly

$$\delta\mathbf{A} = \frac{\partial^2 \delta\varphi_t(\mathbf{X})}{\partial t^2}. \tag{2.20}$$

Subtracting (2.3) from (2.11) yields

$$\begin{aligned} &\int_{\mathcal{U}} \rho_0 \left(\dot{\overline{\delta\Psi}} + \overset{\circ}{\mathbf{V}} \cdot \delta\mathbf{A} + \overset{\circ}{\mathbf{A}} \cdot \delta\mathbf{V} + \delta\mathbf{V} \cdot \delta\mathbf{A}\right) dV \\ &\quad = \int_{\mathcal{U}} \rho_0 \left(\overset{\circ}{\mathbf{B}} \cdot \delta\mathbf{V} + \overset{\circ}{\mathbf{V}} \cdot \delta\mathbf{B} + \delta\mathbf{B} \cdot \delta\mathbf{V} + \delta R\right) dV \\ &\quad + \int_{\partial\mathcal{U}} \left(\overset{\circ}{\mathbf{T}} \cdot \delta\mathbf{V} + \delta\mathbf{T} \cdot \overset{\circ}{\mathbf{V}} + \delta\mathbf{T} \cdot \delta\mathbf{V} + \delta H\right) dA, \end{aligned} \tag{2.21}$$

where we have used conservation of mass $\dot{\rho}_0 = 0$. Note that

$$\delta\mathbf{V} = \dot{\mathbf{U}}, \quad \delta\mathbf{A} = \ddot{\mathbf{U}}. \tag{2.22}$$

Note also that

$$\begin{aligned} \rho_0 \delta\Psi &= \rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} \cdot \delta\mathbf{F} + \frac{1}{2} \delta\mathbf{F} \cdot \rho_0 \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \mathbf{F}} \cdot \delta\mathbf{F} + o\left(\|\delta\mathbf{F}\|^2\right) \\ &= \overset{\circ}{\mathbf{P}} \cdot \nabla\mathbf{U} + \frac{1}{2} \nabla\mathbf{U} \cdot \overset{\circ}{\mathsf{C}} \cdot \nabla\mathbf{U} + o\left(\|\nabla\mathbf{U}\|^2\right), \end{aligned} \tag{2.23}$$

where $\overset{\circ}{\mathsf{C}}$ is the elasticity tensor of the reference motion. We keep the quadratic term too because we want to work to linear order in force, which means quadratic order in energy.[2] Therefore

$$\rho_0 \dot{\overline{\delta\Psi}} = \dot{\overline{\overset{\circ}{\mathbf{P}}}} \cdot \nabla\mathbf{U} + \overset{\circ}{\mathbf{P}} \cdot \nabla\dot{\mathbf{U}} + \frac{1}{2} \nabla\mathbf{U} \cdot \dot{\overline{\overset{\circ}{\mathsf{C}}}} \cdot \nabla\mathbf{U} + \nabla\mathbf{U} \cdot \overset{\circ}{\mathsf{C}} \cdot \nabla\dot{\mathbf{U}}. \tag{2.24}$$

[1] From now on, we will mostly skip the component form of the equations, which will hopefully be apparent from the context.

[2] Besides, if the reference motion is a static equilibrium configuration the first term would vanish.

Variation of traction can be simplified to read

$$\delta\mathbf{T} = \delta\mathbf{P}\cdot\hat{\mathbf{N}} = \left(\frac{\partial\mathbf{P}}{\partial\mathbf{F}}\cdot\delta\mathbf{F}\right)\cdot\hat{\mathbf{N}} = \overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\cdot\hat{\mathbf{N}}. \tag{2.25}$$

Now (2.21) can be simplified to read

$$\begin{aligned}\int_{\mathcal{U}}\Bigg[&\dot{\overset{\circ}{\mathbf{P}}}\cdot\nabla\mathbf{U}+\overset{\circ}{\mathbf{P}}\cdot\nabla\dot{\mathbf{U}}+\frac{1}{2}\nabla\mathbf{U}\cdot\dot{\overset{\circ}{\mathsf{C}}}\cdot\nabla\mathbf{U}\\&+\nabla\mathbf{U}\cdot\overset{\circ}{\mathsf{C}}\cdot\nabla\dot{\mathbf{U}}+\rho_0\left(\overset{\circ}{\mathbf{V}}\cdot\ddot{\mathbf{U}}+\overset{\circ}{\mathsf{C}}\cdot\dot{\mathbf{U}}+\dot{\mathbf{U}}\cdot\ddot{\mathbf{U}}\right)\Bigg]dV\\&=\int_{\mathcal{U}}\rho_0\left(\overset{\circ}{\mathbf{B}}\cdot\dot{\mathbf{U}}+\overset{\circ}{\mathbf{V}}\cdot\delta\mathbf{B}+\delta\mathbf{B}\cdot\dot{\mathbf{U}}+\delta R\right)dV\\&\quad+\int_{\partial\mathcal{U}}\left[\left(\overset{\circ}{\mathbf{P}}\cdot\dot{\mathbf{U}}+\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\cdot\overset{\circ}{\mathbf{V}}+\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\cdot\dot{\mathbf{U}}\right)\cdot\hat{\mathbf{N}}+\delta H\right]dA.\end{aligned} \tag{2.26}$$

Or

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}}&\left[\overset{\circ}{\mathbf{P}}\cdot\nabla\mathbf{U}+\frac{1}{2}\nabla\mathbf{U}\cdot\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}+\rho_0\left(\overset{\circ}{\mathbf{V}}\cdot\dot{\mathbf{U}}+\tfrac{1}{2}\dot{\mathbf{U}}\cdot\dot{\mathbf{U}}\right)\right]dV\\&=\int_{\mathcal{U}}\rho_0\left(\overset{\circ}{\mathbf{B}}\cdot\dot{\mathbf{U}}+\overset{\circ}{\mathbf{V}}\cdot\delta\mathbf{B}+\delta\mathbf{B}\cdot\dot{\mathbf{U}}+\delta R\right)dV\\&\quad+\int_{\partial\mathcal{U}}\left[\left(\overset{\circ}{\mathbf{P}}\cdot\dot{\mathbf{U}}+\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\cdot\overset{\circ}{\mathbf{V}}+\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\cdot\dot{\mathbf{U}}\right)\cdot\hat{\mathbf{N}}+\delta H\right]dA.\end{aligned} \tag{2.27}$$

We call (2.27) the ***perturbed energy balance***. Note that in (2.27) there are terms linear and terms quadratic in $\mathbf{U}$. Rescaling $\mathbf{U}$ by an arbitrary value, e.g. $\mathbf{U}\to\epsilon\mathbf{U}$, one can conclude that the sum of linear and quadratic terms should be zero separately, i.e.

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}}\left[\overset{\circ}{\mathbf{P}}\cdot\nabla\mathbf{U}+\rho_0\,\overset{\circ}{\mathbf{V}}\cdot\dot{\mathbf{U}}\right]dV&=\int_{\mathcal{U}}\rho_0\left(\overset{\circ}{\mathbf{B}}\cdot\dot{\mathbf{U}}+\overset{\circ}{\mathbf{V}}\cdot\delta\mathbf{B}+\delta\mathbf{B}\cdot\dot{\mathbf{U}}+\delta R\right)dV\\&\quad+\int_{\partial\mathcal{U}}\left[\overset{\circ}{\mathbf{T}}\cdot\dot{\mathbf{U}}+\delta\mathbf{T}\cdot\overset{\circ}{\mathbf{V}}+\delta H\right]dA,\end{aligned} \tag{2.28}$$

$$\frac{d}{dt}\int_{\mathcal{U}}\left[\frac{1}{2}\nabla\mathbf{U}\cdot\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}+\frac{1}{2}\rho_0\dot{\mathbf{U}}\cdot\dot{\mathbf{U}}\right]dV=\int_{\mathcal{U}}\rho_0\delta\mathbf{B}\cdot\dot{\mathbf{U}}dV+\int_{\partial\mathcal{U}}\delta\mathbf{T}\cdot\dot{\mathbf{U}}dA. \tag{2.29}$$

We call (2.28) the ***linearized energy balance*** and (2.29) the ***quadratized energy balance***.

In classical linear elasticity, the initial motion is a stress-free static configuration, i.e. $\overset{\circ}{\mathbf{V}}=\mathbf{0}$, $\overset{\circ}{\mathbf{P}}=\mathbf{0}$, and $\overset{\circ}{\mathbf{B}}=\mathbf{0}$ and it is assumed that there are no heat sources and fluxes, i.e., $\delta R=\delta H=0$. In this case, the perturbed energy balance is identical to the quadratized energy balance (2.29). This is the so-called ***Power Theorem*** in classical linear elasticity [Fosdick and Truskinovsky, 2003]. In other words, in classical linear elasticity the linearized energy balance is identically zero and one needs to look at the quadratized energy balance.

Postulating invariance of the perturbed energy balance under isometries of the Euclidean ambient space one can consider both the linearized and the quadratized energy balances. It can be shown that postulating invariance of the quadratized energy balance does not give any new governing equations, i.e. the quadratized energy balance is trivially invariant under isometries of the ambient space. Therefore, in the following we study the consequences of postulating invariance of the linearized energy balance.

2.1 Invariance of the Linearized Energy Balance Under Isometries of the Euclidean Ambient Space

Let us first consider a rigid translation of the deformed configuration defined as

$$\mathbf{x}' = \xi_t(\mathbf{x}) = \mathbf{x} + (t - t_0)\mathbf{w}, \tag{2.30}$$

where $\mathbf{w}$ is a constant vector. Under this change of frame we have

$$\varphi_t' = \xi_t \circ \varphi_t, \qquad \overset{\circ}{\varphi}{}_t' = \xi_t \circ \overset{\circ}{\varphi}_t \,. \tag{2.31}$$

Therefore at $t = t_0$

$$\mathbf{V}' = \mathbf{V} + \mathbf{w}, \qquad \overset{\circ}{\mathbf{V}}{}' = \overset{\circ}{\mathbf{V}} + \mathbf{w}. \tag{2.32}$$

Also

$$\mathbf{U}' = \mathbf{U}, \qquad \dot{\mathbf{U}}' = \dot{\mathbf{U}}. \tag{2.33}$$

Linearized balance of energy in the new frame at $t = t_0$ is written as

$$\begin{aligned} \int_{\mathcal{U}} \rho_0 \left(\dot{\overset{\circ}{\mathbf{P}}} \cdot \boldsymbol{\nabla}\mathbf{U} + \overset{\circ}{\mathbf{P}} \cdot \boldsymbol{\nabla}\dot{\mathbf{U}} + (\overset{\circ}{\mathbf{V}} + \mathbf{w}) \cdot \ddot{\mathbf{U}} + \overset{\circ}{\mathsf{C}} \cdot \dot{\mathbf{U}} \right) dV \\ = \int_{\mathcal{U}} \rho_0 \left[\overset{\circ}{\mathbf{B}} \cdot \dot{\mathbf{U}} + (\overset{\circ}{\mathbf{V}} + \mathbf{w}) \cdot \delta\mathbf{B} + \delta R \right] dV \\ + \int_{\partial\mathcal{U}} \left[\left(\overset{\circ}{\mathbf{P}} \cdot \dot{\mathbf{U}} + \overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \cdot (\overset{\circ}{\mathbf{V}} + \mathbf{w}) \right) \cdot \hat{\mathbf{N}} + \delta H \right] dA. \end{aligned} \tag{2.34}$$

Subtracting (2.26) from (2.34) yields

$$\int_{\mathcal{U}} \rho_0 \ddot{\mathbf{U}} \cdot \mathbf{w} dV = \int_{\mathcal{U}} \rho_0 \delta\mathbf{B} \cdot \mathbf{w} dV + \int_{\mathcal{U}} \mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \cdot \mathbf{w} dV. \tag{2.35}$$

Because $\mathcal{U}$ and $\mathbf{w}$ are arbitrary, we conclude that

$$\mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) + \rho_0 \delta\mathbf{B} = \rho_0 \ddot{\mathbf{U}}. \tag{2.36}$$

Let us now consider a rigid rotation of the deformed configuration, i.e.

$$\mathbf{x}' = e^{\boldsymbol{\Omega}(t - t_0)} \mathbf{x}, \tag{2.37}$$

where $\boldsymbol{\Omega}^{\mathsf{T}} = -\boldsymbol{\Omega}$. Therefore at $t = t_0$

$$\mathbf{V}' = \mathbf{V} + \boldsymbol{\Omega}\mathbf{x}, \qquad \overset{\circ}{\mathbf{V}}{}' = \overset{\circ}{\mathbf{V}} + \boldsymbol{\Omega} \overset{\circ}{\mathbf{x}} \,. \tag{2.38}$$

This means that

$$\mathbf{U}' = \mathbf{U}, \qquad \dot{\mathbf{U}}' = \dot{\mathbf{U}}. \tag{2.39}$$

Subtracting balance of energy for $\overset{\circ}{\varphi}_t$ from that of $\overset{\circ}{\varphi}{}_t'$ and using balance of linear momentum for the perturbed motion, we obtain

$$\int_{\mathcal{U}} \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) : \boldsymbol{\Omega} dV = 0. \tag{2.40}$$

Because $\mathcal{U}$ and $\mathbf{w}$ are arbitrary, we conclude that

$$\left(\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}\right)^{\mathsf{T}}=\overset{\circ}{\mathsf{C}}\cdot\boldsymbol{\nabla}\mathbf{U}. \tag{2.41}$$

Therefore, we have proven the following proposition.

Proposition 2.1. *Invariance of the linearized balance of energy under time-dependent rigid translations and rotations of the Euclidean ambient space is equivalent to linearized balance of linear and angular momenta.*

2.2 Lagrangian Field Theory of Linearized Elasticity

Note that similar ideas can be used in obtaining equations of linear elasticity in the framework of Lagrangian mechanics. The starting point in Lagrangian field theory of elasticity is a Lagrangian density $\mathcal{L}=\mathcal{L}(\mathbf{X},t,\varphi,\dot{\varphi},\mathbf{F})$. ***Hamilton's principle of least action*** states that for the equilibrium configuration first variation of the action integral vanishes, i.e., $\delta S=0$, where

$$S=\int_{t_0}^{t_1}\int_{\mathcal{B}}\mathcal{L}dVdt. \tag{2.42}$$

Given the configurations $\overset{\circ}{\varphi}$ and φ_s, where $\varphi_0=\overset{\circ}{\varphi}$, we can write

$$\delta\int_{t_0}^{t_1}\int_{\mathcal{B}}\overset{\circ}{\mathcal{L}}\,dVdt=0 \quad\text{and}\quad \delta\int_{t_0}^{t_1}\int_{\mathcal{B}}\mathcal{L}(s)dVdt=0. \tag{2.43}$$

Or

$$\delta\int_{t_0}^{t_1}\int_{\mathcal{B}}\left[\mathcal{L}(s)-\overset{\circ}{\mathcal{L}}\right]\,dVdt=0. \tag{2.44}$$

Linearization of $\mathcal{L}(s)-\overset{\circ}{\mathcal{L}}$ will give the governing equations of linearized elasticity.

3 Geometric Elasticity

In this section, in order to make the paper self-contained, we review some notation from the geometric approach to elasticity. Refer to [Marsden and Hughes, 1983] for more details and also [Abraham, Marsden and Ratiu, 1988] and [Marsden and Ratiu, 2003].

For a smooth n-manifold M, the tangent space to M at a point $p\in M$ is denoted T_pM and the whole tangent bundle is denoted TM. We denote by $\mathcal{B}$ a reference manifold for our body and by $\mathcal{S}$ the space in which the body moves. We assume that $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with metrics $\mathbf{G}$ and $\mathbf{g}$, respectively. Local coordinates on $\mathcal{B}$ are denoted by $\{X^A\}$ and those on $\mathcal{S}$ by $\{x^a\}$.

A ***deformation*** of the body is a C^1 embedding $\varphi:\mathcal{B}\to\mathcal{S}$. The tangent map of φ is denoted $\mathbf{F}=T\varphi:T\mathcal{B}\to T\mathcal{S}$, which is often called the deformation gradient. In local charts on $\mathcal{B}$ and $\mathcal{S}$, the tangent map of φ is given by the Jacobian matrix of partial derivatives of the components of φ, as

$$\mathbf{F}=T\varphi:T\mathcal{B}\to T\mathcal{S},\quad T\varphi(\mathbf{X},\mathbf{Y})=(\varphi(\mathbf{X}),\mathbf{D}\varphi(\mathbf{X})\cdot\mathbf{Y}). \tag{3.1}$$

If $F:\mathcal{B}\to\mathbb{R}$ is a C^1 scalar function, $\mathbf{X}\in\mathcal{B}$ and $\mathbf{V_X}\in T_{\mathbf{X}}\mathcal{B}$, then $\mathbf{V_X}[F]$ denotes the derivative of F at $\mathbf{X}$ in the direction of $\mathbf{V_X}$, i.e., $\mathbf{V_X}[F]=\mathbf{D}F(\mathbf{X})\cdot\mathbf{V}$. In local coordinates

$\{X^A\}$ on $\mathcal{B}$,

$$\mathbf{V}_\mathbf{X}[F] = \frac{\partial F}{\partial X^A} V^A. \tag{3.2}$$

For $f : \mathcal{S} \to \mathbb{R}$, the ***pull-back*** of f by φ is defined by

$$\varphi^* f = f \circ \varphi. \tag{3.3}$$

If $F : \mathcal{B} \to \mathbb{R}$, the ***push-forward*** of F by φ is defined by

$$\varphi_* F = F \circ \varphi^{-1}. \tag{3.4}$$

If $\mathbf{Y}$ is a vector field on $\mathcal{B}$, then $\varphi_* \mathbf{Y} = T\varphi \circ \mathbf{Y} \circ \varphi^{-1}$, or using the $\mathbf{F}$ notation, $\varphi_* \mathbf{Y} = \mathbf{F} \circ \mathbf{Y} \circ \varphi^{-1}$ is a vector field on $\varphi(\mathcal{B})$ called the ***push-forward*** of $\mathbf{Y}$ by φ. Similarly, if $\mathbf{y}$ is a vector field on $\varphi(\mathcal{B}) \subset \mathcal{S}$, then $\varphi^* \mathbf{y} = T(\varphi^{-1}) \circ \mathbf{y} \circ \varphi$ is a vector field on $\mathcal{B}$ and is called the pull-back of $\mathbf{y}$ by φ.

The cotangent bundle of a manifold M is denoted T^*M and the fiber at a point $p \in M$ (the vector space of one-forms at p) is denoted by T_p^*M. If β is a one-form on $\mathcal{S}$, i.e., a section of the cotangent bundle $T^*\mathcal{S}$, then the one-form on $\mathcal{B}$ defined as

$$(\varphi^*\beta)_\mathbf{X} \cdot \mathbf{V}_\mathbf{X} = \beta_{\varphi(\mathbf{X})} \cdot (T\varphi \cdot \mathbf{V}_\mathbf{X}) = \beta_{\varphi(\mathbf{X})} \cdot (\mathbf{F} \cdot \mathbf{V}_\mathbf{X}) \tag{3.5}$$

for $\mathbf{X} \in \mathcal{B}$ and $\mathbf{V}_\mathbf{X} \in T_\mathbf{X}\mathcal{B}$, is called the ***pull-back*** of β by φ. Likewise, the ***push-forward*** of a one-form α on $\mathcal{B}$ is the one form on $\varphi(\mathcal{B})$ defined by $\varphi_*\alpha = (\varphi^{-1})^*\alpha$.

We can associate a vector field $\beta^\sharp$ to a one-form β on a Riemannian manifold M through the equation

$$\langle \beta_\mathbf{x}, \mathbf{v}_\mathbf{x} \rangle = \langle\!\langle \beta_\mathbf{x}^\sharp, \mathbf{v}_\mathbf{x} \rangle\!\rangle_\mathbf{x}, \tag{3.6}$$

where $\langle \, , \rangle$ denotes the natural pairing between the one form $\beta_\mathbf{x} \in T_\mathbf{x}^*M$ and the vector $\mathbf{v}_\mathbf{x} \in T_\mathbf{x}M$ and where $\langle\!\langle \beta_\mathbf{x}^\sharp, \mathbf{v}_\mathbf{x} \rangle\!\rangle_\mathbf{x}$ denotes the inner product between $\beta_\mathbf{x}^\sharp \in T_\mathbf{x}M$ and $\mathbf{v}_\mathbf{x} \in T_\mathbf{x}M$ induced by the metric $\mathbf{g}$. In coordinates, the components of $\beta^\sharp$ are given by $\beta^a = g^{ab}\beta_b$.

A type $\begin{pmatrix} m \\ n \end{pmatrix}$-tensor at $\mathbf{X} \in \mathcal{B}$ is a multilinear map

$$\mathbf{T} : \underbrace{T_\mathbf{X}^*\mathcal{B} \times ... \times T_\mathbf{X}^*\mathcal{B}}_{m \text{ copies}} \times \underbrace{T_\mathbf{X}\mathcal{B} \times ... \times T_\mathbf{X}\mathcal{B}}_{n \text{ copies}} \to \mathbb{R}. \tag{3.7}$$

$\mathbf{T}$ is said to be contravariant of order m and covariant of order n. In a local coordinate chart

$$\mathbf{T}(\alpha^1, ..., \alpha^m, \mathbf{V}_1, ..., \mathbf{V}_n) = T^{\,i_1...i_m}{}_{j_1...j_n} \alpha_{i_1}^1 ... \alpha_{i_m}^m V_1^{j_1} ... V_n^{j_n}, \tag{3.8}$$

where $\alpha^k \in T_\mathbf{X}^*\mathcal{B}$ and $\mathbf{V}^k \in T_\mathbf{X}\mathcal{B}$.

Suppose $\varphi : \mathcal{B} \to \mathcal{S}$ is a regular map and $\mathbf{T}$ is a tensor of type $\begin{pmatrix} m \\ n \end{pmatrix}$. Push-forward of $\mathbf{T}$ by φ is denoted $\varphi_*\mathbf{T}$ and is a $\begin{pmatrix} m \\ n \end{pmatrix}$-tensor on $\varphi(\mathcal{B})$ defined by

$$(\varphi_*\mathbf{T})(\mathbf{x})(\alpha^1, ..., \alpha^m, \mathbf{v}_1, ..., \mathbf{v}_n) = \mathbf{T}(\mathbf{X})(\varphi_*\alpha^1, ..., \varphi_*\alpha^m, \varphi_*\mathbf{v}_1, ..., \varphi_*\mathbf{v}_n), \tag{3.9}$$

where $\alpha^k \in T_\mathbf{x}^*\mathcal{S}, \mathbf{v}_k \in T_\mathbf{x}\mathcal{S}, \mathbf{X} = \varphi^{-1}(\mathbf{x}), \varphi^*(\alpha^k) \cdot \mathbf{v}_l = \alpha^k \cdot (T\varphi \cdot \mathbf{v}_l)$ and $\varphi^*(\mathbf{v}_l) = T(\varphi^{-1})\mathbf{v}_l$. Similarly, pull-back of a tensor $\mathbf{t}$ defined on $\varphi(\mathcal{B})$ is given by $\varphi^*\mathbf{t} = (\varphi^{-1})_*\mathbf{t}$.

A ***two-point tensor*** $\mathbf{T}$ of type $\begin{pmatrix} m & r \\ n & s \end{pmatrix}$ at $\mathbf{X} \in \mathcal{B}$ over a map $\varphi : \mathcal{B} \to \mathcal{S}$ is a multilinear map

$$T : \underbrace{T^*_{\mathbf{X}}\mathcal{B} \times ... \times T^*_{\mathbf{X}}\mathcal{B}}_{m \text{ copies}} \times \underbrace{T_{\mathbf{X}}\mathcal{B} \times ... \times T_{\mathbf{X}}\mathcal{B}}_{n \text{ copies}}$$
$$\times \underbrace{T^*_{\mathbf{x}}\mathcal{S} \times ... \times T^*_{\mathbf{x}}\mathcal{S}}_{r \text{ copies}} \times \underbrace{T_{\mathbf{x}}\mathcal{S} \times ... \times T_{\mathbf{x}}\mathcal{S}}_{s \text{ copies}} \to \mathbb{R}, \tag{3.10}$$

where $\mathbf{x} = \varphi(\mathbf{X})$.

Let $\mathbf{w} : \mathcal{U} \to T\mathcal{S}$ be a vector field, where $\mathcal{U} \subset \mathcal{S}$ is open. A curve $\mathbf{c} : I \to \mathcal{S}$, where I is an open interval, is an ***integral curve*** of $\mathbf{w}$ if

$$\frac{d\mathbf{c}}{dt}(r) = \mathbf{w}(\mathbf{c}(r)) \qquad \forall\ r \in I. \tag{3.11}$$

If $\mathbf{w}$ depends on the time variable explicitly, i.e., $\mathbf{w} : \mathcal{U} \times (-\epsilon, \epsilon) \to T\mathcal{S}$, an integral curve is defined by

$$\frac{d\mathbf{c}}{dt} = \mathbf{w}(\mathbf{c}(t), t). \tag{3.12}$$

Let $\mathbf{w} : \mathcal{S} \times I \to T\mathcal{S}$ be a vector field. The collection of maps $F_{t,s}$ such that for each s and $\mathbf{x}$, $t \mapsto F_{t,s}(\mathbf{x})$ is an integral curve of $\mathbf{w}$ and $F_{s,s}(\mathbf{x}) = \mathbf{x}$ is called the flow of $\mathbf{w}$. Let $\mathbf{w}$ be a C^1 vector field on $\mathcal{S}$, $F_{t,s}$ its flow, and $\mathbf{t}$ a C^1 tensor field on $\mathcal{S}$. The ***Lie derivative*** of $\mathbf{t}$ with respect to $\mathbf{w}$ is defined by

$$\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{dt}\left(F^*_{t,s}\mathbf{t}_t\right)\Big|_{t=s}. \tag{3.13}$$

If we hold t fixed in $\mathbf{t}$ then we denote

$$\pounds_{\mathbf{w}}\mathbf{t} = \frac{d}{dt}\left(F^*_{t,s}\mathbf{t}_s\right)\Big|_{t=s}, \tag{3.14}$$

which is called the ***autonomous Lie derivative***. Therefore

$$\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{\partial}{\partial t}\mathbf{t} + \pounds_{\mathbf{w}}\mathbf{t}. \tag{3.15}$$

Let $\mathbf{y}$ be a vector field on $\mathcal{S}$ and $\varphi : \mathcal{B} \to \mathcal{S}$ a regular and orientation preserving C^1 map. The ***Piola transform*** of $\mathbf{y}$ is

$$\mathbf{Y} = J\varphi^*\mathbf{y}, \tag{3.16}$$

where J is the Jacobian of φ. If $\mathbf{Y}$ is the Piola transform of $\mathbf{y}$, then the ***Piola identity*** holds:

$$\operatorname{Div}\mathbf{Y} = J(\operatorname{div}\mathbf{y}) \circ \varphi. \tag{3.17}$$

A p-form on a manifold M is a skew-symmetric $\binom{0}{p}$-tensor. The space of p-forms on M is denoted by $\Omega^p(M)$. If $\varphi : M \to N$ is a regular and orientation preserving C^1 map and $\alpha \in \Omega^p(\varphi(M))$, then

$$\int_{\varphi(M)} \alpha = \int_M \varphi^*\alpha. \tag{3.18}$$

Let $\pi : E \to \mathcal{S}$ be a vector bundle over a manifold $\mathcal{S}$ and $\mathcal{E}(\mathcal{S})$ be the space of smooth sections of E and $\mathcal{X}(\mathcal{S})$ the space of vector fields on $\mathcal{S}$. A ***connection*** on E is a map

$\boldsymbol{\nabla} : \mathcal{X}(\mathcal{S}) \times \mathcal{E}(\mathcal{S}) \to \mathcal{E}(\mathcal{S})$ such that $\forall\ f, f_1, f_2 \in C^\infty(\mathcal{S}),\ \forall\ a_1, a_2 \in \mathbb{R}$

$$i) \quad \boldsymbol{\nabla}_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y} = f_1\boldsymbol{\nabla}_{\mathbf{X}_1}\mathbf{Y} + f_2\boldsymbol{\nabla}_{\mathbf{X}_2}\mathbf{Y}, \tag{3.19}$$

$$ii) \quad \boldsymbol{\nabla}_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\boldsymbol{\nabla}_{\mathbf{X}}(\mathbf{Y}_1) + a_2\boldsymbol{\nabla}_{\mathbf{X}}(\mathbf{Y}_2), \tag{3.20}$$

$$iii) \quad \boldsymbol{\nabla}_{\mathbf{X}}(f\mathbf{Y}) = f\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}. \tag{3.21}$$

A ***linear connection*** on $\mathcal{S}$ is a connection in $T\mathcal{S}$, i.e., $\boldsymbol{\nabla} : \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S}) \to \mathcal{X}(\mathcal{S})$. In a local chart

$$\boldsymbol{\nabla}_{\partial_i}\partial_j = \gamma^k_{ij}\partial_k, \tag{3.22}$$

where γ^k_{ij} are Christoffel symbols of the connection and $\partial_i = \frac{\partial}{\partial x^i}$. A linear connection is said to be compatible with the metric of the manifold if

$$\boldsymbol{\nabla}_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle = \langle\!\langle \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle + \langle\!\langle \mathbf{Y}, \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle. \tag{3.23}$$

It can be shown that $\boldsymbol{\nabla}$ is compatible with $\mathbf{g}$ if and only if $\boldsymbol{\nabla}\mathbf{g} = \mathbf{0}$. ***Torsion*** of a connection is defined as

$$\boldsymbol{\mathfrak{T}}(\mathbf{X}, \mathbf{Y}) = \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y} - \boldsymbol{\nabla}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}], \tag{3.24}$$

where

$$[\mathbf{X}, \mathbf{Y}][F] = \mathbf{X}[\mathbf{Y}[F]] - \mathbf{Y}[\mathbf{X}[F]] \qquad \forall\ F \in C^\infty(\mathcal{S}), \tag{3.25}$$

is the ***commutator*** of $\mathbf{X}$ and $\mathbf{Y}$. $\boldsymbol{\nabla}$ is symmetric if it is torsion-free, i.e.,

$$\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y} - \boldsymbol{\nabla}_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]. \tag{3.26}$$

It can be shown that on any Riemannian manifold $(\mathcal{S}, \mathbf{g})$ there is a unique linear connection $\boldsymbol{\nabla}$ that is compatible with $\mathbf{g}$ and is torsion-free with the following Christoffel symbols

$$\gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right). \tag{3.27}$$

This is the ***Fundamental Lemma of Riemannian Geometry*** [Lee, 1997] and this connection is called the ***Levi-Civita connection***.

Curvature tensor $\boldsymbol{\mathcal{R}}$ of a Riemannian manifold $(\mathcal{S}, \mathbf{g})$ is a $\binom{1}{3}$-tensor $\boldsymbol{\mathcal{R}} : T^*_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \to \mathbb{R}$ defined as

$$\boldsymbol{\mathcal{R}}(\alpha, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \alpha\left(\boldsymbol{\nabla}_{\mathbf{w}_1}\boldsymbol{\nabla}_{\mathbf{w}_2}\mathbf{w}_3 - \boldsymbol{\nabla}_{\mathbf{w}_2}\boldsymbol{\nabla}_{\mathbf{w}_1}\mathbf{w}_3 - \boldsymbol{\nabla}_{[\mathbf{w}_1,\mathbf{w}_2]}\mathbf{w}_3\right) \tag{3.28}$$

for $\alpha \in T^*_{\mathbf{x}}S,\ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in T_{\mathbf{x}}S$. In a coordinate chart $\{x^a\}$

$$\mathcal{R}^a{}_{bcd} = \frac{\partial\gamma^a_{bd}}{\partial x^c} - \frac{\partial\gamma^a_{bc}}{\partial x^d} + \gamma^a_{ce}\gamma^e_{bd} - \gamma^a_{de}\gamma^e_{bc}. \tag{3.29}$$

Let us next review a few of the basic notions of geometric continuum mechanics.

A ***body*** $\mathcal{B}$ is identified with a Riemannian manifold $\mathcal{B}$ and a ***configuration*** of $\mathcal{B}$ is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$, where $\mathcal{S}$ is another Riemannian manifold. The set of all configurations of $\mathcal{B}$ is denoted $\mathcal{C}$. A ***motion*** is a curve $c : \mathbb{R} \to \mathcal{C}; t \mapsto \varphi_t$ in $\mathcal{C}$.

For a fixed t, $\varphi_t(\mathbf{X}) = \varphi(\mathbf{X}, t)$ and for a fixed $\mathbf{X}$, $\varphi_{\mathbf{X}}(t) = \varphi(\mathbf{X}, t)$, where $\mathbf{X}$ is position of material points in the undeformed configuration $\mathcal{B}$. The ***material velocity*** is the map $\mathbf{V}_t : \mathcal{B} \to \mathbb{R}^3$ given by

$$\mathbf{V}_t(\mathbf{X}) = \mathbf{V}(\mathbf{X}, t) = \frac{\partial\varphi(\mathbf{X}, t)}{\partial t} = \frac{d}{dt}\varphi_{\mathbf{X}}(t). \tag{3.30}$$

Similarly, the ***material acceleration*** is defined by

$$\mathbf{A}_t(\mathbf{X}) = \mathbf{A}(\mathbf{X}, t) = \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t} = \frac{d}{dt}\mathbf{V}_{\mathbf{X}}(t). \tag{3.31}$$

In components

$$A^a = \frac{\partial V^a}{\partial t} + \gamma^a_{bc} V^b V^c, \tag{3.32}$$

where γ^a_{bc} is the Christoffel symbol of the local coordinate chart $\{x^a\}$.

Here it is assumed that φ_t is invertible and regular. The ***spatial velocity*** of a regular motion φ_t is defined as

$$\mathbf{v}_t : \varphi_t(\mathcal{B}) \to \mathbb{R}^3, \qquad \mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1}, \tag{3.33}$$

and the ***spatial acceleration*** $\mathbf{a}_t$ is defined as

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{v}} \mathbf{v}. \tag{3.34}$$

In components

$$a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma^a_{bc} v^b v^c. \tag{3.35}$$

Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a C^1 configuration of $\mathcal{B}$ in $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{S}$ are manifolds. Recall that the deformation gradient is denoted by $\mathbf{F} = T\varphi$. Thus, at each point $\mathbf{X} \in \mathcal{B}$, it is a linear map

$$\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}. \tag{3.36}$$

If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ are

$$F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X}). \tag{3.37}$$

The deformation gradient may be viewed as a two-point tensor

$$\mathbf{F}(\mathbf{X}) : T^*_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{X}}\mathcal{B} \to \mathbb{R}; \quad (\alpha, \mathbf{V}) \mapsto \langle \alpha, T_{\mathbf{X}}\varphi \cdot \mathbf{V} \rangle. \tag{3.38}$$

Suppose $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with inner products $\langle\!\langle , \rangle\!\rangle_{\mathbf{X}}$ and $\langle\!\langle , \rangle\!\rangle_{\mathbf{x}}$ based at $\mathbf{X} \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{S}$, respectively. Recall that the transpose of $\mathbf{F}$ is defined by

$$\mathbf{F}^{\mathsf{T}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}, \quad \langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v} \rangle\!\rangle_{\mathbf{x}} = \langle\!\langle \mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v} \rangle\!\rangle_{\mathbf{X}} \tag{3.39}$$

for all $\mathbf{V} \in T_{\mathbf{X}}\mathcal{B}$, $\mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$. In components

$$(F^{\mathsf{T}}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X}), \tag{3.40}$$

where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively. On the other hand, the ***dual*** of $\mathbf{F}$, a metric independent notion, is defined by

$$\mathbf{F}^*(\mathbf{x}) : T^*_{\mathbf{x}}\mathcal{S} \to T^*_{\mathbf{X}}\mathcal{B}; \quad \langle \mathbf{F}^*(\mathbf{x}) \cdot \alpha, \mathbf{W} \rangle = \langle \alpha, \mathbf{F}(\mathbf{X})\mathbf{W} \rangle \tag{3.41}$$

for all $\alpha \in T^*_{\mathbf{x}}\mathcal{S}, \mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$. Considering bases $\mathbf{e}_a$ and $\mathbf{E}_A$ for $\mathcal{S}$ and $\mathcal{B}$, respectively, one can define the corresponding dual bases $\mathbf{e}^a$ and $\mathbf{E}^A$. The matrix representation of $\mathbf{F}^*$ with respect to the dual bases is the transpose of $F^a{}_A$. $\mathbf{F}$ and $\mathbf{F}^*$ have the following local representations

$$\mathbf{F} = F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A, \qquad \mathbf{F}^* = F^a{}_A dX^A \otimes \frac{\partial}{\partial x^a}. \tag{3.42}$$

The ***right Cauchy-Green deformation tensor*** is defined by

$$\mathbf{C}(X) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}, \qquad \mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^T\mathbf{F}(\mathbf{X}). \tag{3.43}$$

In components

$$C^A{}_B = (F^T)^A{}_a F^a{}_B. \tag{3.44}$$

It is straightforward to show that

$$\mathbf{C}^\flat = \varphi^*(\mathbf{g}) = \mathbf{F}^*\mathbf{g}\mathbf{F}, \text{ i.e. } \quad C_{AB} = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B. \tag{3.45}$$

Let $\varphi_t : \mathcal{B} \to \mathcal{S}$ be a regular motion of $\mathcal{B}$ in $\mathcal{S}$ and $\mathcal{P} \subset \mathcal{B}$ a p-dimensional submanifold. The ***Transport Theorem*** says that for any p-form α on $\mathcal{S}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{P})} \alpha = \int_{\varphi_t(\mathcal{P})} \mathbf{L}_{\mathbf{v}}\alpha, \tag{3.46}$$

where $\mathbf{v}$ is the spatial velocity of the motion. In a special case when $\alpha = f dv$ and $\mathcal{P} = \mathcal{U}$ is an open set

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{P})} f dv = \int_{\varphi_t(\mathcal{P})} \left[\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{v})\right] dv. \tag{3.47}$$

Balance of linear momentum for a body $\mathcal{B}$ is satisfied if for every nice open set $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\mathbf{v} dv = \int_{\varphi_t(\mathcal{U})} \rho\mathbf{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \mathbf{t} da, \tag{3.48}$$

where $\rho = \rho(\mathbf{x}, t)$ is mass density, $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ is body force vector field and $\mathbf{t} = \mathbf{t}(\mathbf{x}, \hat{\mathbf{n}}, t)$ is the traction vector. Note that according to Cauchy's stress theorem there exists a contravariant second-order tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ (Cauchy stress tensor) with components σ^{ab} such that $\mathbf{t} = \langle\!\langle \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle\!\rangle$. Note that $\langle\!\langle, \rangle\!\rangle$ is the inner product induced by the Riemmanian metric $\mathbf{g}$. Equivalently, balance of linear momentum can be written in the undeformed configuration as

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \mathbf{V} dV = \int_{\mathcal{U}} \rho_0 \mathbf{B} dV + \int_{\partial\mathcal{U}} \left\langle\!\!\left\langle \mathbf{P}, \hat{\mathbf{N}} \right\rangle\!\!\right\rangle dA, \tag{3.49}$$

where, $\mathbf{P} = J\varphi^*\sigma$ (the first Piola-Kirchhoff stress tensor) is the Piola transform of Cauchy stress tensor. Note that $\mathbf{P}$ is a two-point tensor with components P^{aA}. Note also that this is the balance of linear momentum in the deformed (physical) space written in terms of some quantities that are defined with respect to the reference configuration.

Let us emphasize that balance of linear momentum has no intrinsic meaning because integrating a vector field is geometrically meaningless, i.e., it is coordinate dependent. Geometrically, forces (interactions) take values in the cotangent bundle of the ambient space manifold (see [Kanso, et al., 2007] for a detailed discussion). The ambient space manifold is not linear in general and hence balance of forces cannot be written in an integral form, in general. In classical continuum mechanics, this balance law makes use of the linear (or affine) structure of Euclidean space.

Balance of angular momentum is satisfied for a body $\mathcal{B}$ if for every nice open set $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\mathbf{x} \times \mathbf{v} dv = \int_{\varphi_t(\mathcal{U})} \rho\mathbf{x} \times \mathbf{b} dv + \int_{\partial\varphi_t(\mathcal{U})} \mathbf{x} \times \langle\!\langle \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle\!\rangle da. \tag{3.50}$$

Balance of linear momentum, similar to balance of angular momentum, makes use of the linear structure of Euclidean space and this does *not* transform in a covariant way under a general change of coordinates.

Balance of energy holds for a body $\mathcal{B}$ if, for every nice open set $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho\left(e+\frac{1}{2}\langle\!\langle \mathbf{v},\mathbf{v}\rangle\!\rangle\right) dv = \int_{\varphi_t(\mathcal{U})} \rho\left(\langle\!\langle \mathbf{b},\mathbf{v}\rangle\!\rangle + r\right) dv + \int_{\partial\varphi_t(\mathcal{U})} \left(\langle\!\langle \mathbf{t},\mathbf{v}\rangle\!\rangle + h\right) da, \quad (3.51)$$

where $e = e(\mathbf{x},t), r = r(\mathbf{x},t)$ and $h = h(\mathbf{x},\hat{\mathbf{n}},t)$ are internal energy per unit mass, heat supply per unit mass and heat flux, respectively.

4 Geometric Linearization of Nonlinear Elasticity

Marsden and Hughes [1983] formulated the theory of linear elasticity by linearizing nonlinear elasticity assuming that reference and ambient space manifolds are Riemannian. Here we review their ideas and obtain some new results. We denote by $\mathcal{C}$ the set of all deformation mappings $\varphi : \mathcal{B} \to \mathcal{S}$. We do not discuss boundary conditions, but assume that deformation mappings satisfy all the displacement (essential) boundary conditions. One can prove that $\mathcal{C}$ is an infinite-dimensional manifold. For $\overset{\circ}{\varphi}_t \in \mathcal{C}$, an element of $T_{\overset{\circ}{\varphi}_t}\mathcal{C}$ is tangent to a curve $\varphi_{t,s} \in \mathcal{C}$ such that $\varphi_{t,0} = \overset{\circ}{\varphi}_t$. This is called variation of the configuration $\mathbf{U} = \delta\varphi_t$. Note that $\mathbf{U} = \frac{d}{ds}\big|_{s=0}\varphi_s$.

Suppose $\pi : \mathcal{E} \to \mathcal{C}$ is a vector bundle over $\mathcal{C}$ and let $f : \mathcal{C} \to \mathcal{E}$ be a section of this bundle. Let us assume that $\mathcal{E}$ is equipped with a connection $\boldsymbol{\nabla}$. With these assumptions, linearization of $f(\varphi)$ at $\overset{\circ}{\varphi}_t \in \mathcal{C}$ is defined as

$$\mathcal{L}\left(f;\overset{\circ}{\varphi}_t\right) := f(\overset{\circ}{\varphi}_t) + \boldsymbol{\nabla} f(\overset{\circ}{\varphi}_t)\cdot\mathbf{U}, \quad \mathbf{U} \in T_{\overset{\circ}{\varphi}_t}\mathcal{C}, \quad (4.1)$$

where

$$\boldsymbol{\nabla} f(\overset{\circ}{\varphi}_t)\cdot\mathbf{U} = \frac{d}{ds}\,\boldsymbol{\alpha}_s \cdot f(\varphi_{t,s})\Big|_{s=0} \quad (4.2)$$

and $\boldsymbol{\alpha}_s$ is parallel transport of members of $\mathcal{E}_{\varphi_{t,s}}$ to $\mathcal{E}_{\overset{\circ}{\varphi}_t}$ along a curve $\varphi_{t,s}$ tangent to $\mathbf{U}$ at $\overset{\circ}{\varphi}_t$.

In [Marsden and Hughes, 1983] it is shown that deformation gradient has the following linearization about $\overset{\circ}{\varphi}_t$.

$$\mathcal{L}\left(\mathbf{F};\overset{\circ}{\varphi}\right) = \overset{\circ}{\mathbf{F}} + \boldsymbol{\nabla}\mathbf{U}, \quad (4.3)$$

where $\overset{\circ}{\mathbf{F}} = T\,\overset{\circ}{\varphi}_t$. One can think of $\mathbf{F}$ as a vector-valued one-form with the local representation

$$\mathbf{F} = F^a{}_A\, \mathbf{e}_a \otimes dX^A. \quad (4.4)$$

Thus

$$\boldsymbol{\varepsilon} := \mathcal{L}\left(\mathbf{F};\overset{\circ}{\varphi}\right) - \overset{\circ}{\mathbf{F}} = U^a{}_{|A}\, \mathbf{e}_a \otimes dX^A \quad (4.5)$$

can be thought of as a geometric linearized strain, which is a vector-valued one form. See [Yavari, 2007] for more discussion on this geometric strain and constitutive equations written in terms of it. Material velocity is linearized at follows.

$$\mathcal{L}\left(\mathbf{V};\overset{\circ}{\varphi}\right) = \overset{\circ}{\mathbf{V}} + \dot{\mathbf{U}}, \quad (4.6)$$

where $\dot{\mathbf{U}}$ is the covariant time derivative of $\mathbf{U}$, i.e.

$$\dot{U}^a = \frac{\partial U^a}{\partial t} + \gamma^a_{bc} \overset{\circ}{V}{}^b U^c. \tag{4.7}$$

This can be rewritten as

$$\dot{U}^a = \left(\frac{\partial U^a}{\partial X^b} + \gamma^a_{bc} U^c \right) \overset{\circ}{V}{}^b . \tag{4.8}$$

Or $\dot{\mathbf{U}} = \boldsymbol{\nabla}_{\overset{\circ}{\mathbf{V}}} \mathbf{U}$.

Material acceleration is linearized as follows.

$$\mathcal{L}\left(\mathbf{A}; \overset{\circ}{\varphi}\right) = \overset{\circ}{\mathbf{A}} + \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \tag{4.9}$$

where $\boldsymbol{\mathcal{R}}$ is the curvature tensor of $(\mathcal{S}, \mathbf{g})$. In components, the linearized acceleration has the following form

$$\ddot{U}^a + \mathcal{R}^a{}_{bcd} V^b U^c V^d. \tag{4.10}$$

Proof of this result is lengthy but straightforward. Note that this is a generalization of Jacobi equation. Note also that in [Marsden and Hughes, 1983] it is implicitly assumed that $\boldsymbol{\mathcal{R}} = \mathbf{0}$.

We know that transpose of deformation gradient is defined as

$$\langle\!\langle \mathbf{F}\mathbf{W}, \mathbf{z} \rangle\!\rangle_{\mathbf{g}} = \left\langle\!\left\langle \mathbf{W}, \mathbf{F}^{\mathsf{T}}\mathbf{z} \right\rangle\!\right\rangle_{\mathbf{G}} \qquad \forall\ \mathbf{W} \in T_{\mathbf{X}}\mathcal{B},\ \mathbf{z} \in T_{\mathbf{x}}\mathcal{S}. \tag{4.11}$$

Thus

$$\mathcal{L}\left(\mathbf{F}^{\mathsf{T}}; \overset{\circ}{\varphi}\right) = \overset{\circ}{\mathbf{F}}{}^{\mathsf{T}} + (\boldsymbol{\nabla}\mathbf{U})^{\mathsf{T}}. \tag{4.12}$$

The right Cauchy-Green strain tensor has the following linearization

$$\mathcal{L}\left(\mathbf{C}; \overset{\circ}{\varphi}_t\right) = \overset{\circ}{\mathbf{C}} + \overset{\circ}{\mathbf{F}}{}^{\mathsf{T}} \boldsymbol{\nabla}\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^{\mathsf{T}} \overset{\circ}{\mathbf{F}} . \tag{4.13}$$

Or in component form

$$\mathcal{L}\left(\mathbf{C}; \overset{\circ}{\varphi}_t\right)_{AB} = \overset{\circ}{C}_{AB} + g_{ab} \overset{\circ}{F}{}^a{}_A U^b{}_{|B} + g_{ab} \overset{\circ}{F}{}^b{}_B U^a{}_{|A}. \tag{4.14}$$

Balance of angular momentum in component form reads

$$P^{aA} F^b{}_A = P^{bA} F^a{}_A. \tag{4.15}$$

This also implies that

$$\overset{\circ}{P}{}^{aA} \overset{\circ}{F}{}^b{}_A = \overset{\circ}{P}{}^{bA} \overset{\circ}{F}{}^a{}_A. \tag{4.16}$$

Linearization of this relation about $\overset{\circ}{\varphi}$ reads

$$\begin{aligned} &\overset{\circ}{P}{}^{aA} \overset{\circ}{F}{}^b{}_A + \overset{\circ}{P}{}^{aA} U^b{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{aA}{}_c{}^B\right) \overset{\circ}{F}{}^b{}_A U^c{}_{|B} \\ &= \overset{\circ}{P}{}^{bA} \overset{\circ}{F}{}^a{}_A + \overset{\circ}{P}{}^{bA} U^a{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{bA}{}_c{}^B\right) \overset{\circ}{F}{}^a{}_A U^c{}_{|B}, \end{aligned} \tag{4.17}$$

Using (4.16) this is simplified to read

$$\overset{\circ}{P}{}^{aA}\ U^b{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{aA}{}_c{}^B\right) \overset{\circ}{F}{}^b{}_A\ U^c{}_{|B} = \overset{\circ}{P}{}^{bA}\ U^a{}_{|A} + \left(\overset{\circ}{\mathsf{C}}{}^{bA}{}_c{}^B\right) \overset{\circ}{F}{}^a{}_A\ U^c{}_{|B}. \tag{4.18}$$

In terms of Cauchy stress this reads

$$\overset{\circ}{\boldsymbol{\sigma}}: \nabla \boldsymbol{u} + \overset{\circ}{\boldsymbol{a}}: \nabla \boldsymbol{u} = \nabla \boldsymbol{u} : \overset{\circ}{\boldsymbol{\sigma}} + \nabla \boldsymbol{u} : \overset{\circ}{\boldsymbol{a}}\ . \tag{4.19}$$

Or in components

$$\overset{\circ}{\sigma}{}^{ac}\ u^b{}_{|c} + \overset{\circ}{a}{}^{ab}{}_c{}^d\ u^c{}_{|d} = \overset{\circ}{\sigma}{}^{bc}\ u^a{}_{|c} + \overset{\circ}{a}{}^{ba}{}_c{}^d\ u^c{}_{|d}, \tag{4.20}$$

where [Marsden and Hughes, 1983]

$$\overset{\circ}{a}{}^{ac}{}_b{}^d = \frac{1}{J}\ F^c{}_A F^d{}_B\ \overset{\circ}{A}{}^{aA}{}_b{}^B \quad \text{and} \quad \mathbf{u} = \mathbf{U} \circ \varphi^{-1}. \tag{4.21}$$

Independent works have been done in the literature of geometric calculus of variations (see [Nishikawa, 2002] and [Baird, et al., 2004] and references therein) on similar problems. There, the idea is to obtain the first and second variations of "energy" of maps between two given Riemannian manifolds. In the following, we make a connection between these efforts and geometric elasticity.

4.1 Linearization of Elasticity Using Variation of Maps

Here we follow Nishikawa [2002] but with a notation closer to ours. The main motivation for studying variational problems in [Nishikawa, 2002] is to understand geodesics in Riemannian manifolds as minimization problems. Interestingly, these studies are closely related to elasticity. Let us consider two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ and a time-dependent motion $\varphi_t : \mathcal{B} \to \mathcal{S}$. What Nishikawa denotes by $d\varphi$ is $\mathbf{F}$ in our notation, which is an element of $T^*_{\mathbf{X}}\mathcal{B} \otimes T_{\mathbf{x}}\mathcal{S}$, i.e. a vector-valued one-form.[3] One can then define an inner product on $T^*_{\mathbf{X}}\mathcal{B} \otimes T_{\mathbf{x}}\mathcal{S}$ such that in this inner product

$$|\mathbf{F}|^2 = \mathrm{tr}(\mathbf{C}). \tag{4.22}$$

Energy density of the map φ is defined as

$$e(\varphi, \mathbf{X}) = \frac{1}{2}\, \mathrm{tr}(\mathbf{C}(\mathbf{X})). \tag{4.23}$$

Note that this is a very special case of energy density in elasticity. Energy of the map φ is then defined as

$$E(\varphi) = \int_{\mathcal{B}} e(\varphi, \mathbf{X})\ dV(\mathbf{X}). \tag{4.24}$$

Let us denote the Levi-Civita connections induced by $\mathbf{G}$ and $\mathbf{g}$ by ∇_0 and ∇, respectively. One can define a connection $\widetilde{\nabla} = \varphi^* \nabla$ by

$$\widetilde{\nabla}_{\mathbf{V}}(\mathbf{y} \circ \varphi) = \nabla_{(\varphi_* \mathbf{V})}\mathbf{y} \qquad \forall\ \mathbf{y} \in T_{\varphi(\mathbf{X})}\mathcal{S},\ \mathbf{V} \in T_{\mathbf{X}}\mathcal{B}. \tag{4.25}$$

Consider a reference deformation map $\overset{\circ}{\varphi}_t$ and a C^∞ variation of it $\varphi_{t,s}$ such that $s \in I =$

[3] For a discussion on reformulating continuum mechanics using bundle-valued forms see [Kanso, et al., 2007].

$(-\epsilon,\epsilon)$ and $\varphi_{t,0} = \overset{\circ}{\varphi}_t$. Let us define

$$\mathbf{U} = \mathbf{U}_t(\mathbf{X}) = \frac{d}{ds}\Big|_{s=0} \varphi_{t,s}(\mathbf{X}). \tag{4.26}$$

First variation of deformation gradient is defined as

$$\nabla_{\frac{\partial}{\partial s}} \mathbf{F}(s)\Big|_{s=0} = \nabla_{\frac{\partial}{\partial s}} \left(\frac{\partial \varphi_{t,s}}{\partial \mathbf{X}}\right)\Bigg|_{s=0} = \nabla \mathbf{U}. \tag{4.27}$$

Note that for each $s \in I$ and $\mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$, $\mathbf{F}(s)\mathbf{W} \in T_{\varphi_{t,s}(\mathbf{X})}\mathcal{S}$, i.e. $\mathbf{F}(s)\mathbf{W}$ lies in different tangent spaces for different values of s and this is why covariant derivative with respect to s is used.

Tension field of φ is defined as

$$\boldsymbol{\tau}(\varphi) = tr(\nabla \mathbf{F}), \tag{4.28}$$

or in components

$$\tau^a(\varphi) = F^a{}_{A|B} G^{AB}. \tag{4.29}$$

It can be shown that [Nishikawa, 2002]

$$\frac{d}{ds} E(\varphi_{t,s})\Big|_{s=0} = \int_{\mathcal{B}} \langle \mathbf{F}^{\mathsf{T}}, \nabla \mathbf{U} \rangle dV = -\int_{\mathcal{B}} \langle\!\langle \boldsymbol{\tau}(\varphi), \mathbf{U} \rangle\!\rangle dV, \tag{4.30}$$

where the first integrand on the right-hand side in components reads $\left(F^{\mathsf{T}}\right)^B{}_b U^b{}_{|B}$. A C^∞ map $\varphi_t \in C^\infty(\mathcal{B},\mathcal{S})$ is called a ***harmonic map*** if its tension field $\boldsymbol{\tau}(\varphi)$ vanishes identically. In other words, φ_t is a harmonic map if for any variation $\varphi_{t,s}$

$$\frac{d}{ds} E(\varphi_{t,s})\Big|_{s=0} = 0. \tag{4.31}$$

In elasticity, this corresponds to an equilibrium configuration in the absence of body and inertial forces.

The left Cauchy-Green strain tensor for the perturbed motion $\varphi_{t,s}$ is defined as

$$C_{AB}(s) = F^a{}_A(s) F^b{}_B(s) g_{ab}(s). \tag{4.32}$$

Note that for any $s \in I$ and $\mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$, $\mathbf{C}(s)\mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$, i.e., $\mathbf{C}(s)\mathbf{W}$ lies in the same linear space for all $s \in I$. Thus, the first variation of $\mathbf{C}$ can be calculated as

$$\frac{d}{ds} C_{AB}(s) = \nabla_{\frac{\partial}{\partial s}} F^a{}_A(s) F^b{}_B(s) g_{ab}(s) + F^a{}_A(s) \nabla_{\frac{\partial}{\partial s}} F^b{}_B(s) g_{ab}(s). \tag{4.33}$$

Note also that [Nishikawa, 2002]

$$\nabla_{\frac{\partial}{\partial s}} \left(\frac{\partial \varphi_{t,s}}{\partial X^A}\right) = \nabla_{\frac{\partial}{\partial X^A}} \left(\frac{\partial \varphi_{t,s}}{\partial s}\right). \tag{4.34}$$

Therefore

$$\frac{d}{ds}\Big|_{s=0} C_{AB}(s) = U^a{}_{|A} \overset{\circ}{F}{}^b{}_B \, g_{ab} + \overset{\circ}{F}{}^a{}_A \, U^b{}_{|B} \, g_{ab}, \tag{4.35}$$

which is identical to (4.13).

We know that material free energy density has the following form [Marsden and Hughes, 1983]

$$\Psi = \Psi(\mathbf{X}, t, \mathbf{C}). \tag{4.36}$$

Thus

$$\Psi(s) = \Psi(\mathbf{X}, t, \mathbf{C}(s)). \tag{4.37}$$

We also know that

$$\Psi(s) = \Psi(0) + \left[\frac{d}{ds}\Big|_{s=0} \Psi(s) \right] s + o(s). \tag{4.38}$$

Thus

$$\frac{d}{ds}\Big|_{s=0} \Psi(s) = \frac{\partial \overset{\circ}{\Psi}}{\partial \overset{\circ}{\mathbf{C}}} \cdot \left(\boldsymbol{\nabla}\mathbf{U} \cdot \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} + \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) = \frac{1}{2} \overset{\circ}{\mathbf{S}} \cdot \left(\boldsymbol{\nabla}\mathbf{U} \cdot \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} + \mathbf{G} \overset{\circ}{\mathbf{F}}^{\mathsf{T}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \tag{4.39}$$

is the linearization of Ψ and where $\overset{\circ}{\mathbf{S}}$ is the second Piola-Kirchhoff stress. Using such ideas one can linearize all the governing equations of nonlinear elasticity about a given reference motion $\overset{\circ}{\varphi}_t$. In this work, we are interested in obtaining the governing equations of linearized elasticity covariantly using energy balance and its symmetry properties.

5 A Covariant Formulation of Linearized Elasticity

There are two possibilities for postulating covariance in linearized elasticity: (i) To postulate invariance of energy balance under spatial diffeomorphisims of the ambient space and then linearizing the energy balance about a given motion (Linearization of Covariance), and (ii) To first write energy balance for a *perturbed* motion and then postulate its invariance under spatial diffeomorphisms of the ambient space (Covariance of Linearized Energy Balance). In this section, we study the consequences of both postulates.

5.1 Linearization of Covariant Energy Balance

For the sake of simplicity, we use the material energy balance. Let us first define

$$E(\mathbf{X}, t, \mathbf{g}) = e(\varphi_t(\mathbf{X}), t, \mathbf{g}(\varphi_t(\mathbf{X}))). \tag{5.1}$$

We know that under a spatial diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$ [Yavari, et al., 2006]

$$E'(\mathbf{X}, t, \mathbf{g}) = E(\mathbf{X}, t, \xi^* \mathbf{g}), \tag{5.2}$$

where

$$\xi^* \mathbf{g} = (T\xi)^* \cdot \mathbf{g} \cdot T\xi, \tag{5.3}$$

is the pull-back of $\mathbf{g}$ by ξ_t. Therefore, at time $t = t_0$

$$\dot{\overline{E'}} = \dot{E} + \frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{g}} \mathbf{W}, \tag{5.4}$$

where $\mathbf{W} = \mathbf{w} \circ \varphi_t$ and $\mathbf{w} = \frac{\partial}{\partial t} \xi_t$. Material balance of energy for the motions $\overset{\circ}{\varphi}_t$ and $\overset{\circ}{\varphi}{}'_t$ (at time $t = t_0$), respectively, reads

$$\int_{\mathcal{U}} \rho_0 \left(\dot{\overline{\overset{\circ}{E}}} + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{V}}, \overset{\circ}{\mathbf{A}} \right\rangle\!\!\right\rangle \right) dV = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{V}} \right\rangle\!\!\right\rangle + \overset{\circ}{R} \right) dV + \int_{\partial \mathcal{U}} \left(\left\langle\!\!\left\langle \overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{V}} \right\rangle\!\!\right\rangle + \overset{\circ}{H} \right) dA, \tag{5.5}$$

and

$$\int_{\mathcal{U}} \rho_0 \left(\dot{\overset{\circ}{E}} + \frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{g}} \mathbf{W} + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{V}} + \mathbf{W} \right\rangle\!\!\right\rangle \right) dV = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{V}} + \mathbf{W} \right\rangle\!\!\right\rangle + \overset{\circ}{R} \right) dV + \int_{\partial \mathcal{U}} \left(\left\langle\!\!\left\langle \overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{V}} + \mathbf{w} \right\rangle\!\!\right\rangle + \overset{\circ}{H} \right) dA. \quad (5.6)$$

Note that $\overset{\circ}{\varphi}{}_t'$ was defined in (2.31). Subtracting (5.5) from (5.6), one obtains

$$\int_{\mathcal{U}} \rho_0 \left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \mathbf{W} \right\rangle\!\!\right\rangle \right) dV = \int_{\mathcal{U}} \left\langle\!\!\left\langle \rho_0 \, \overset{\circ}{\mathbf{B}}, \mathbf{W} \right\rangle\!\!\right\rangle + \int_{\partial \mathcal{U}} \left\langle\!\!\left\langle \overset{\circ}{\mathbf{T}}, \mathbf{W} \right\rangle\!\!\right\rangle dA. \quad (5.7)$$

Similarly, for the motion $\varphi_{t,s}$

$$\int_{\mathcal{U}} \rho_0 \left(\frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}_s} \mathbf{g} + \langle\!\langle \mathbf{A}, \mathbf{W}_s \rangle\!\rangle \right) dV = \int_{\mathcal{U}} \langle\!\langle \rho_0 \mathbf{B}, \mathbf{W}_s \rangle\!\rangle + \int_{\partial \mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{W}_s \rangle\!\rangle \, dA, \quad (5.8)$$

where $\mathbf{W}_s = \mathbf{w} \circ \varphi_{t,s}$. Therefore, for an arbitrary vector field $\mathbf{w}$ we have

$$\int_{\mathcal{U}} \rho_0 \left[\frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}_s} \mathbf{g} - \frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}} : \mathfrak{L}_{\mathbf{W}} \mathbf{g} + \langle\!\langle \mathbf{A}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \mathbf{W} \right\rangle\!\!\right\rangle \right] dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \mathbf{W} \right\rangle\!\!\right\rangle \right) + \int_{\partial \mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{T}}, \mathbf{W} \right\rangle\!\!\right\rangle \right) dA. \quad (5.9)$$

Note that

$$\int_{\partial \mathcal{U}} \left\langle\!\!\left\langle \overset{\circ}{\mathbf{T}}, \mathbf{W} \right\rangle\!\!\right\rangle dA = \int_{\mathcal{U}} \left(\left\langle\!\!\left\langle \mathrm{Div}\, \overset{\circ}{\mathbf{P}}, \mathbf{W} \right\rangle\!\!\right\rangle + \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\omega} + \overset{\circ}{\boldsymbol{\tau}} : \mathbf{k} \right) dV, \quad (5.10)$$

$$\int_{\partial \mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{W}_s \rangle\!\rangle \, dA = \int_{\mathcal{U}} \left(\langle\!\langle \mathrm{Div}\, \mathbf{P}, \mathbf{W}_s \rangle\!\rangle + \boldsymbol{\tau} : \boldsymbol{\omega}_s + \boldsymbol{\tau} : \mathbf{k}_s \right) dV, \quad (5.11)$$

where, $\overset{\circ}{\boldsymbol{\tau}} = \overset{\circ}{\mathbf{P}} \overset{\circ}{\mathbf{F}}$ and $\boldsymbol{\tau} = \mathbf{P}\mathbf{F}$ are Kirchhoff stresses and $\boldsymbol{\omega}$ and $\mathbf{k}$ have the coordinate representations $k_{ab} = \frac{1}{2} \left(W_{a|b} + W_{b|a} \right)$ and $\omega_{ab} = \frac{1}{2} \left(W_{a|b} - W_{b|a} \right)$ with similar representations for $\boldsymbol{\omega}_s$ and $\mathbf{k}_s$. Note also that

$$\mathcal{L} \left(\langle\!\langle \mathbf{A}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \mathbf{W} \right\rangle\!\!\right\rangle ; \overset{\circ}{\varphi}_t \right) = \left\langle\!\!\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle, \quad (5.12)$$

$$\mathcal{L} \left(\langle\!\langle \mathbf{B}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \mathbf{W} \right\rangle\!\!\right\rangle ; \overset{\circ}{\varphi}_t \right) = \left\langle\!\!\left\langle \boldsymbol{\nabla}_{\mathbf{U}} \, \overset{\circ}{\mathbf{B}}, \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle, \quad (5.13)$$

$$\mathcal{L} \left(\boldsymbol{\tau} : (\boldsymbol{\omega}_s + \mathbf{k}_s) - \overset{\circ}{\boldsymbol{\tau}} : (\boldsymbol{\omega} + \mathbf{k}) ; \overset{\circ}{\varphi}_t \right) = \left[\overset{\circ}{\mathbf{P}} \, \boldsymbol{\nabla} \mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla} \mathbf{U} \right) \overset{\circ}{\mathbf{F}} \right] : (\boldsymbol{\omega} + \mathbf{k}) + \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\nabla}_{\mathbf{U}} \boldsymbol{\nabla} \mathbf{W}, \quad (5.14)$$

$$\mathcal{L} \left(\langle\!\langle \mathrm{Div}\, \mathbf{P}, \mathbf{W}_s \rangle\!\rangle - \left\langle\!\!\left\langle \mathrm{Div}\, \overset{\circ}{\mathbf{P}}, \mathbf{W} \right\rangle\!\!\right\rangle ; \overset{\circ}{\varphi}_t \right) = \left\langle\!\!\left\langle \mathrm{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla} \mathbf{U} \right), \mathbf{W} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathrm{Div}\, \overset{\circ}{\mathbf{P}}, \boldsymbol{\nabla}_{\mathbf{U}} \mathbf{W} \right\rangle\!\!\right\rangle. \quad (5.15)$$

For the derivative of the internal energy we have the following linearization.

$$\mathcal{L}\left(\frac{\partial E}{\partial \mathbf{g}}:\mathfrak{L}_{\mathbf{W}_s}\mathbf{g}-\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}:\mathfrak{L}_{\mathbf{W}}\mathbf{g};\overset{\circ}{\varphi}_t\right)=\nabla_{\mathbf{U}}\left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right):\mathfrak{L}_{\mathbf{W}}\mathbf{g}+\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}:\nabla_{\mathbf{U}}(\mathfrak{L}_{\mathbf{W}}\mathbf{g}). \tag{5.16}$$

Thus, (5.9) is now simplified to read

$$\begin{aligned}\int_{\mathcal{U}}\rho_0\Bigg[\nabla_{\mathbf{U}}\left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right)&:\mathfrak{L}_{\mathbf{W}}\mathbf{g}+\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}:\nabla_{\mathbf{U}}(\mathfrak{L}_{\mathbf{W}}\mathbf{g})+\left\langle\!\!\left\langle \ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}},\mathbf{U},\overset{\circ}{\mathbf{V}}),\mathbf{W}\right\rangle\!\!\right\rangle\\ &+\left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}},\nabla_{\mathbf{U}}\mathbf{W}\right\rangle\!\!\right\rangle\Bigg]dV=\int_{\mathcal{U}}\rho_0\left(\left\langle\!\!\left\langle \nabla_{\mathbf{U}}\overset{\circ}{\mathbf{B}},\mathbf{W}\right\rangle\!\!\right\rangle+\left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}},\nabla_{\mathbf{U}}\mathbf{W}\right\rangle\!\!\right\rangle\right)\\ &+\int_{\mathcal{U}}\Bigg\{\left\langle\!\!\left\langle \mathrm{Div}\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right),\mathbf{W}\right\rangle\!\!\right\rangle+\left\langle\!\!\left\langle \mathrm{Div}\,\overset{\circ}{\mathbf{P}},\nabla_{\mathbf{U}}\mathbf{W}\right\rangle\!\!\right\rangle\\ &\quad+\left[\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\right]:(\boldsymbol{\omega}+\mathbf{k})+\overset{\circ}{\boldsymbol{\tau}}:\nabla_{\mathbf{U}}(\boldsymbol{\omega}+\mathbf{k})\Bigg\}dV.\end{aligned} \tag{5.17}$$

We can choose $\mathbf{W}$ such that $\nabla_{\mathbf{U}}\mathbf{W}$, $\nabla_{\mathbf{U}}\boldsymbol{\omega}$, and $\nabla_{\mathbf{U}}\mathbf{k}$ are all zero. In this case arbitrariness of $\mathbf{W}$ and $\mathcal{U}$ implies that

$$\mathrm{Div}\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)+\rho_0\nabla_{\mathbf{U}}\overset{\circ}{\mathbf{B}}=\rho_0\left(\ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}},\mathbf{U},\overset{\circ}{\mathbf{V}})\right), \tag{5.18}$$

$$\nabla_{\mathbf{U}}\overset{\circ}{\boldsymbol{\tau}}=2\rho_0\nabla_{\mathbf{U}}\left(\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right)=\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}, \tag{5.19}$$

$$\left[\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\right]^{\mathsf{T}}=\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}. \tag{5.20}$$

Now substituting these back into (5.17) one obtains

$$\begin{aligned}\int_{\mathcal{U}}\rho_0\left[\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}:\nabla_{\mathbf{U}}\mathbf{k}+\left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}},\nabla_{\mathbf{U}}\mathbf{W}\right\rangle\!\!\right\rangle\right]dV&=\int_{\mathcal{U}}\rho_0\left\langle\!\!\left\langle \overset{\circ}{\mathbf{B}},\nabla_{\mathbf{U}}\mathbf{W}\right\rangle\!\!\right\rangle\\ &+\int_{\mathcal{U}}\left\{\left\langle\!\!\left\langle \mathrm{Div}\,\overset{\circ}{\mathbf{P}},\nabla_{\mathbf{U}}\mathbf{W}\right\rangle\!\!\right\rangle+\overset{\circ}{\boldsymbol{\tau}}:\nabla_{\mathbf{U}}(\boldsymbol{\omega}+\mathbf{k})\right\}dV.\end{aligned} \tag{5.21}$$

Note that, as a consequence of covariance of energy balance for the motion $\overset{\circ}{\varphi}_t$, (5.21) is trivially satisfied. Therefore, we have proven the following proposition.

Proposition 5.1. *Linearization of covariant energy balance is equivalent to linearization of all the field equations of elasticity.*

5.2 Covariance of Linearized Energy Balance

The more interesting case is when one first linearizes energy balance about a reference motion and then postulates its invariance under arbitrary spatial diffeomorphisms. Subtracting the balance of energy for the motion $\overset{\circ}{\varphi}_t$ from that of $\varphi_{t,s}$ yields

$$\begin{aligned}\int_{\mathcal{U}} \rho_0 \left(\dot{E} - \dot{\overset{\circ}{E}} + \langle\langle \mathbf{A}, \mathbf{V} \rangle\rangle - \left\langle\left\langle \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle \right) dV &= \int_{\mathcal{U}} \rho_0 \left(\langle\langle \mathbf{B}, \mathbf{V} \rangle\rangle - \left\langle\left\langle \overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle \right) dV \\ &+ \int_{\mathcal{U}} \rho_0 \left(R - \overset{\circ}{R} \right) dV + \int_{\partial\mathcal{U}} \left(\langle\langle \operatorname{Div} \mathbf{P}, \mathbf{V} \rangle\rangle - \left\langle\left\langle \operatorname{Div} \overset{\circ}{\mathbf{P}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle \right) dV \\ &+ \int_{\mathcal{U}} \left(\boldsymbol{\tau} : \boldsymbol{\nabla} \mathbf{V} - \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\nabla} \overset{\circ}{\mathbf{V}} \right) dV + \int_{\partial\mathcal{U}} (H - \overset{\circ}{H}) dA = 0. \qquad (5.22)\end{aligned}$$

Now let us linearize the integrands. Body force power has the following linearization

$$\begin{aligned}\mathfrak{L} \left(\langle\langle \mathbf{B}, \mathbf{V} \rangle\rangle - \left\langle\left\langle \overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle ; \overset{\circ}{\varphi}_t \right) &= \frac{d}{ds}\Big|_{s=0} \langle\langle \boldsymbol{\alpha}_s \cdot \mathbf{B}, \boldsymbol{\alpha}_s \cdot \mathbf{V} \rangle\rangle \\ &= \left\langle\left\langle \frac{d}{ds}\Big|_{s=0} \boldsymbol{\alpha}_s \cdot \mathbf{B}, \mathbf{V} \right\rangle\right\rangle + \left\langle\left\langle \overset{\circ}{\mathbf{B}}, \frac{d}{ds}\Big|_{s=0} \boldsymbol{\alpha}_s \cdot \mathbf{V} \right\rangle\right\rangle \\ &= \left\langle\left\langle \boldsymbol{\nabla}_{\mathbf{U}} \overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle + \left\langle\left\langle \overset{\circ}{\mathbf{B}}, \dot{\mathbf{U}} \right\rangle\right\rangle . \qquad (5.23)\end{aligned}$$

Similarly, inertial force power has the following linearization

$$\begin{aligned}\mathfrak{L} \left(\langle\langle \mathbf{A}, \mathbf{V} \rangle\rangle - \left\langle\left\langle \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle ; \overset{\circ}{\varphi}_t \right) &= \frac{d}{ds}\Big|_{s=0} \langle\langle \boldsymbol{\alpha}_s \cdot \mathbf{A}, \boldsymbol{\alpha}_s \cdot \mathbf{V} \rangle\rangle \\ &= \left\langle\left\langle \frac{d}{ds}\Big|_{s=0} \boldsymbol{\alpha}_s \cdot \mathbf{A}, \mathbf{V} \right\rangle\right\rangle + \left\langle\left\langle \mathbf{A}, \frac{d}{ds}\Big|_{s=0} \boldsymbol{\alpha}_s \cdot \mathbf{V} \right\rangle\right\rangle \\ &= \left\langle\left\langle \ddot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}}, \mathbf{U}, \overset{\circ}{\mathbf{V}}), \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle + \left\langle\left\langle \overset{\circ}{\mathbf{A}}, \dot{\mathbf{U}} \right\rangle\right\rangle , \qquad (5.24)\end{aligned}$$

where $\boldsymbol{\mathcal{R}}$ is the curvature tensor of the ambient space manifold. Traction power is linearized as follows

$$\mathcal{L} \left(\boldsymbol{\tau} : \boldsymbol{\nabla} \mathbf{V} - \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\nabla} \overset{\circ}{\mathbf{V}} ; \overset{\circ}{\varphi}_t \right) = \left[\overset{\circ}{\mathbf{P}} \, \boldsymbol{\nabla} \mathbf{U} + \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla} \mathbf{U} \right) \overset{\circ}{\mathbf{F}} \right] : \boldsymbol{\nabla} \overset{\circ}{\mathbf{V}} + \overset{\circ}{\boldsymbol{\tau}} : \boldsymbol{\nabla} \dot{\mathbf{U}}, \qquad (5.25)$$

$$\mathcal{L} \left(\langle\langle \operatorname{Div} \mathbf{P}, \mathbf{V} \rangle\rangle - \left\langle\left\langle \operatorname{Div} \overset{\circ}{\mathbf{P}}, \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle ; \overset{\circ}{\varphi}_t \right) = \left\langle\left\langle \operatorname{Div} \left(\overset{\circ}{\mathsf{C}} \cdot \boldsymbol{\nabla} \mathbf{U} \right), \overset{\circ}{\mathbf{V}} \right\rangle\right\rangle + \left\langle\left\langle \operatorname{Div} \overset{\circ}{\mathbf{P}}, \dot{\mathbf{U}} \right\rangle\right\rangle . \qquad (5.26)$$

Internal energy part of energy balance is linearized as follows. Note that

$$\begin{aligned}\frac{d}{dt} \int_{\mathcal{U}} \rho_0 E dV = \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{C}} : \varphi^* (\mathbf{L}_{\mathbf{V}} \mathbf{g}) dV &= \int_{\mathcal{U}} \rho_0 \varphi^* \left(\frac{\partial E}{\partial \mathbf{g}} \right) : \varphi^* (\mathbf{L}_{\mathbf{V}} \mathbf{g}) dV \\ &= \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{g}} : \mathbf{L}_{\mathbf{V}} \mathbf{g} dV. \qquad (5.27)\end{aligned}$$

Thus[4]

$$\begin{aligned}\mathcal{L} \left(\frac{d}{dt} \int_{\mathcal{U}} \rho_0 E dV ; \overset{\circ}{\varphi}_t \right) &= \frac{d}{dt} \int_{\mathcal{U}} \rho_0 \, \overset{\circ}{E} \, dV \\ &+ \int_{\mathcal{U}} \left[\boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial E}{\partial \mathbf{g}} \right) : \mathfrak{L}_{\mathbf{V}} \mathbf{g} + \rho_0 \frac{\partial E}{\partial \mathbf{g}} : \boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{V}} \mathbf{g}) \right] dV. \qquad (5.28)\end{aligned}$$

[4]Note that because $\mathbf{g}$ is time independent $\mathbf{L}_{\mathbf{V}} \mathbf{g} = \mathfrak{L}_{\mathbf{V}} \mathbf{g}$.

Therefore, material balance of energy for the perturbed motion reads

$$\begin{aligned}
&\int_{\mathcal{U}} \left[\boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial E}{\partial \mathbf{g}} \right) : \mathfrak{L}_{\mathbf{v}} \mathbf{g} + \rho_0 \frac{\partial E}{\partial \mathbf{g}} : \boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{v}} \mathbf{g}) \right] dV \\
&\quad + \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\left\langle \dot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \boldsymbol{\nabla}_{\mathbf{U}} \mathring{\mathbf{B}}, \mathring{\mathbf{V}} \right\rangle\!\right\rangle + \left\langle\!\left\langle \mathring{\mathbf{A}} - \mathring{\mathbf{B}}, \dot{\mathbf{U}} \right\rangle\!\right\rangle \right) dV \\
&\quad = \int_{\mathcal{U}} \rho_0 \ \mathbf{d}R \cdot \mathbf{U} \ dV + \int_{\partial\mathcal{U}} \mathbf{d}H \cdot \mathbf{U} dA \\
&\quad + \int_{\mathcal{U}} \left\{ \left[\mathring{\mathbf{P}} \ \boldsymbol{\nabla}\mathbf{U} + \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \mathring{\mathbf{F}} \right] : \boldsymbol{\nabla} \ \mathring{\mathbf{V}} + \mathring{\boldsymbol{\tau}} : \boldsymbol{\nabla}\dot{\mathbf{U}} \right\} dV \\
&\quad + \int_{\mathcal{U}} \left(\left\langle\!\left\langle \mathrm{Div} \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \mathring{\mathbf{V}} \right\rangle\!\right\rangle + \left\langle\!\left\langle \mathrm{Div} \ \mathring{\mathbf{P}}, \dot{\mathbf{U}} \right\rangle\!\right\rangle \right) dV. \qquad (5.29)
\end{aligned}$$

Under a spatial diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$, we have

$$\mathbf{U}' = \xi_{t*} \mathbf{U}. \qquad (5.30)$$

Because $\dot{\mathbf{U}} = \boldsymbol{\nabla}_{\mathring{\mathbf{V}}} \mathbf{U}$, we have

$$\dot{\mathbf{U}}' = \boldsymbol{\nabla}_{(\xi_{t*}\mathring{\mathbf{V}} + \mathbf{w})} (\xi_{t*} \mathbf{U}) = \xi_{t*} \left(\boldsymbol{\nabla}_{\mathring{\mathbf{V}}} \mathbf{U} \right) + \boldsymbol{\nabla}_{\mathbf{w}} (\xi_{t*} \mathbf{U}). \qquad (5.31)$$

Hence at $t = t_0$

$$\dot{\mathbf{U}}' = \dot{\mathbf{U}} + \mathbf{Z}, \qquad (5.32)$$

where $\mathbf{Z} = \boldsymbol{\nabla}_{\mathbf{W}} \mathbf{U} = \boldsymbol{\nabla}\mathbf{U} \cdot \mathbf{W}$. We assume that at $t = t_0$

$$\mathring{\mathbf{A}}' - \mathring{\mathbf{B}}' = \mathring{\mathbf{A}} - \mathring{\mathbf{B}}, \qquad \dot{\mathbf{U}}' + \boldsymbol{\mathcal{R}}'(\mathring{\mathbf{V}}', \mathbf{U}', \mathring{\mathbf{V}}') - \boldsymbol{\nabla}_{\mathbf{U}'} \mathring{\mathbf{B}}' = \dot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \boldsymbol{\nabla}_{\mathbf{U}} \mathring{\mathbf{B}} . \qquad (5.33)$$

Now under this spatial reframing, perturbed energy balance (5.29) at time $t = t_0$ reads

$$\begin{aligned}
&\int_{\mathcal{U}} \left[\boldsymbol{\nabla}_{\mathbf{U}} \left(\rho_0 \frac{\partial E}{\partial \mathbf{g}} \right) : (\mathfrak{L}_{\mathbf{v}} \mathbf{g} + \mathfrak{L}_{\mathbf{w}} \mathbf{g}) + \rho_0 \frac{\partial E}{\partial \mathbf{g}} : [\boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{v}} \mathbf{g}) + \boldsymbol{\nabla}_{\mathbf{U}} (\mathfrak{L}_{\mathbf{w}} \mathbf{g})] \right] dV \\
&\quad + \int_{\mathcal{U}} \rho_0 \Bigg(\left\langle\!\left\langle \dot{\mathbf{U}} + \boldsymbol{\mathcal{R}}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}) - \boldsymbol{\nabla}_{\mathbf{U}} \mathring{\mathbf{B}}, \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\right\rangle \\
&\quad + \left\langle\!\left\langle \mathring{\mathbf{A}} - \mathring{\mathbf{B}}, \dot{\mathbf{U}} + \mathbf{Z} \right\rangle\!\right\rangle \Bigg) dV \\
&= \int_{\mathcal{U}} \left\{ \left[\mathring{\mathbf{P}} \ \boldsymbol{\nabla}\mathbf{U} + \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right) \mathring{\mathbf{F}} \right] : (\boldsymbol{\nabla} \mathring{\mathbf{V}} + \boldsymbol{\nabla}\mathbf{W}) + \mathring{\boldsymbol{\tau}} : (\boldsymbol{\nabla}\dot{\mathbf{U}} + \boldsymbol{\nabla}\mathbf{Z}) \right\} dV \\
&\quad + \int_{\mathcal{U}} \left(\left\langle\!\left\langle \mathrm{Div} \left(\mathring{\mathsf{C}} \cdot \boldsymbol{\nabla}\mathbf{U} \right), \mathring{\mathbf{V}} + \mathbf{W} \right\rangle\!\right\rangle + \left\langle\!\left\langle \mathrm{Div} \ \mathring{\mathbf{P}}, \dot{\mathbf{U}} + \mathbf{Z} \right\rangle\!\right\rangle \right) dV. \qquad (5.34)
\end{aligned}$$

Subtracting (5.29) from (5.34) yields

$$\int_{\mathcal{U}}\left[\nabla_{\mathbf{U}}\left(\rho_0\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right):\mathfrak{L}_{\mathbf{w}}\mathbf{g}+\rho_0\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}:\nabla_{\mathbf{U}}(\mathfrak{L}_{\mathbf{w}}\mathbf{g})\right]dV$$
$$+\int_{\mathcal{U}}\rho_0\left(\left\langle\!\!\left\langle \ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}},\mathbf{U},\overset{\circ}{\mathbf{V}})-\nabla_{\mathbf{U}}\overset{\circ}{\mathbf{B}},\mathbf{W}\right\rangle\!\!\right\rangle+\left\langle\!\!\left\langle \overset{\circ}{\mathbf{A}}-\overset{\circ}{\mathbf{B}},\mathbf{Z}\right\rangle\!\!\right\rangle\right)dV$$
$$=\int_{\mathcal{U}}\left\{\left[\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\right]:\nabla\mathbf{W}+\overset{\circ}{\boldsymbol{\tau}}:\nabla\mathbf{Z}\right\}dV$$
$$+\int_{\mathcal{U}}\left(\left\langle\!\!\left\langle \mathrm{Div}\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right),\mathbf{W}\right\rangle\!\!\right\rangle+\left\langle\!\!\left\langle \mathrm{Div}\,\overset{\circ}{\mathbf{P}},\mathbf{Z}\right\rangle\!\!\right\rangle\right)dV. \tag{5.35}$$

Using the governing equations of the motion $\overset{\circ}{\varphi}_t$, i.e.

$$\mathrm{Div}\,\overset{\circ}{\mathbf{P}}+\rho_0\,\overset{\circ}{\mathbf{B}}=\rho_0\,\overset{\circ}{\mathbf{A}}, \tag{5.36}$$

$$\overset{\circ}{\boldsymbol{\tau}}=2\rho_0\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}, \tag{5.37}$$

$$\overset{\circ}{\boldsymbol{\tau}}=\overset{\circ}{\boldsymbol{\tau}}^{\mathsf{T}}, \tag{5.38}$$

(5.35) is simplified to read

$$\int_{\mathcal{U}}\nabla_{\mathbf{U}}\left(\rho_0\frac{\partial \overset{\circ}{E}}{\partial \mathbf{g}}\right):\mathfrak{L}_{\mathbf{w}}\mathbf{g}\,dV+\int_{\mathcal{U}}\rho_0\left\langle\!\!\left\langle \ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}},\mathbf{U},\overset{\circ}{\mathbf{V}})-\nabla_{\mathbf{U}}\overset{\circ}{\mathbf{B}},\mathbf{W}\right\rangle\!\!\right\rangle dV$$
$$=\int_{\mathcal{U}}\left[\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\right]:\nabla\mathbf{W}dV+\int_{\mathcal{U}}\left\langle\!\!\left\langle \mathrm{Div}\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right),\mathbf{W}\right\rangle\!\!\right\rangle dV. \tag{5.39}$$

Now arbitrariness of $\mathbf{W}$ and $\mathcal{U}$ implies that

$$\mathrm{Div}\left(\overset{\circ}{\mathbf{A}}\cdot\nabla\mathbf{U}\right)+\rho_0\nabla_{\mathbf{U}}\overset{\circ}{\mathbf{B}}=\rho_0\left(\ddot{\mathbf{U}}+\boldsymbol{\mathcal{R}}(\overset{\circ}{\mathbf{V}},\mathbf{U},\overset{\circ}{\mathbf{V}})\right), \tag{5.40}$$

$$\nabla_{\mathbf{U}}\overset{\circ}{\boldsymbol{\tau}}=2\rho_0\nabla_{\mathbf{U}}\left(\frac{\partial\overset{\circ}{E}}{\partial \mathbf{g}}\right)=\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}, \tag{5.41}$$

$$\left[\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}\right]^{\mathsf{T}}=\overset{\circ}{\mathbf{P}}\,\nabla\mathbf{U}+\left(\overset{\circ}{\mathsf{C}}\cdot\nabla\mathbf{U}\right)\overset{\circ}{\mathbf{F}}. \tag{5.42}$$

Thus, we have proved the following proposition.

Proposition 5.2. *Covariance of the linearized energy balance is equivalent to linearization of all the field equations of elasticity.*

In other words, one can covariantly obtain all the governing equations of linearized elasticity by postulating covariance of the linearized energy balance.

6 Conclusions

The main motivation for the present work is to understand the connection between governing equations of linearized elasticity and energy balance and its invariance (or covariance). We first looked at the case where the ambient space is Euclidean. Having a reference motion $\overset{\circ}{\varphi}_t$, we quadratized the energy balance about $\overset{\circ}{\varphi}_t$. This leads to two identities:

linearized energy balance and quadratized energy balance. We showed that postulating invariance of the linearized energy balance under isometries of the ambient space will give all the governing equations of linearized elasticity. We also showed that the quadratized energy balance is trivially invariant under isometries of the ambient space. Classical linear elasticity corresponds to choosing a stress-free reference motion. For such reference motions all the terms in the linearized energy balance are identically zero and the quadratized energy balance is identical to what is called “energy balance” or power theorem in classical linear elasticity.

We then studied the case where the ambient space is a Riemannian manifold $(\mathcal{S}, \mathbf{g})$. We first reviewed some previous ideas in geometric linearization of nonlinear elasticity and presented some new results. We also showed the close connection between these ideas and those of geometric calculus of variations. We considered two notions of covariance: (i) linearization of covariant energy balance and (ii) covariance of the linearized energy balance. We showed that postulating either (i) or (ii) will give all the governing equations of linearized elasticity. Of course, (ii) is more interesting. In other words, if one postulates invariance of the linearized energy balance about a reference motion $\mathring{\varphi}_t$ under spatial diffeomorphisms of $\mathcal{S}$ (the same diffeomorphism acts on $\mathring{\varphi}_t$ and its variations $\varphi_{t,s}$), one obtains all the governing equations of linearized elasticity. In this sense, linearized elasticity can be covariantly derived.

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