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Discrete fractal fracture mechanics

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Abstract

A modification of the classical theory of brittle fracture of solids is offered by relating discrete nature of crack propagation to the fractal geometry of the crack. The new model incorporates all previously considered theories of fracture processes, in particular the Griffith [Griffith AA. The phenomenon of rupture and flow in solids. Philos Trans Roy Soc Lond 1921;A221:163–398] theory, its contemporary extension known as LEFM and the most recently developed Quantized Fracture Mechanics (QFM) by Pugno and Ruoff [Pugno N, Ruoff RS. Quantized fracture mechanics. Philos Mag 2004;84(27):2829–45]. Using an equivalent smooth blunt crack for a given fractal crack, we find that assuming that radius of curvature of the blunt crack is a material property, the crack roughens while propagating. In other words, fractal dimension at the crack tip is a monotonically increasing function of the nominal crack length, i.e., the presence of the Mirror–Mist–Hackle phenomenon is analytically demonstrated.

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1. Introduction

In fracture mechanics, there are two fundamental failure theories proposed by Griffith [22] and Barenblatt [3]. Griffith realized that brittle fracture happens as a result of competition between strain energy release and surface energy required to create new fracture surfaces. In the Griffith's theory it is predicted that for a given crack length there is a unique critical stress above which crack grows and below which crack remains in equilibrium. Barenblatt [3] proposed the cohesive theory of fracture in which one assumes that there is a nonlinear region in the vicinity of the crack tip. The interesting thing here is that there is no stress singularity. Increasing applied loads causes the separation between crack faces to increase in the cohesive region and when the opening displacement is large enough the crack propagates.

The interest in understanding brittle fracture on a more fundamental level has led many researchers to study it in the lattice scale. Thomson et al. [55] showed that in a very simplified 1D model for a range of stresses above and below Griffith's stress a crack becomes lattice trapped. Later Hsieh and Thomson [23] extended their results to 2D. Esterling [19] using a lattice statics method, studied similar problems for three-dimensional

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cracks in a cubic lattice with nearest-neighbor interactions. Masudajindo et al. [36,37] studied fracture of crystalline materials using the lattice Green's function method. In particular, they showed that mode I and II crack problems are coupled in the lattice scale. There have been some very recent atomic-scale fracture studies in the literature (see [35,34,32,33] and references therein) and it seems that even after a few decades of research in fracture mechanics, some physics/mechanics coupled problems remain to be resolved.

Search for some novel mathematical tools that would remove the physically unacceptable singularities predicted by the classical mechanics of fracture began soon after the publication of the seminal work by Griffith [22]. Orowan [43] and Irwin [27] proposed the "plasticity correction" term that was added in the equations describing the stress intensity factors for fracture of various modes, so that for the crack length approaching zero a finite stress resulted. This outcome described the intrinsic strength of the undamaged material, while the associated stress at the crack tip could now be identified with a finite stress level corresponding to the local yield stress. Similar effort was undertaken in Russia by Novozhilov [42], who suggested that the very nature of the crack propagation is discrete, and who for the first time introduced the concept of the minimum admissible growth step a_0. According to Novozhilov [42] this entity, named by him a "fracture quantum", must be included in the energy Griffith criterion – or an equivalent local stress criterion for fracture. Initially, the fracture quantum was identified with the interatomic distance b_0 (in a cubic lattice). Similar concepts were proposed independently by Eshelby [15] (see also [10]). In recent years, similar ideas have been pursued mainly in [44], where "Quantized Fracture Mechanics" (QFM) was introduced and was later used in several applications [45,54,38,46,12,25,47].

Recent work by Ippolito et al. [25] summarizes the pertinent results of the newly developed QFM model. This model is based on the hypothesis of discrete nature of crack propagation and it is used to explain the atomistic simulations data for nano-meter size cracks in β-SiC. Here, the fracture quantum a_0 is seen to be a fraction of a nanometer. In addition to fracture quantum, another length entity has been considered; it is the finite radius of curvature ρ found at the tip of a QFM crack, which resembles a blunt crack known in LEFM. In the end, it turns out that it is not the fracture quantum alone but rather the ratio ρ/a_0 that describes the microstructural behavior of the material. The third important variable is the crack length itself, especially so in the nano-scale range, where fracture resistance is visibly sensitive to the crack length. Comparing the theoretical and atomistic simulations data, Ippolito et al. [25] found that the best fit was obtained for both a_0 and ρ being on the order of magnitude comparable with the interatomic distances in silicon carbide.

In this paper, a new model of fracture, Discrete Fractal Fracture Mechanics (DFFM), is proposed. This theory is built on the outcome of the earlier works by Wnuk and Yavari [59,60]. Here, we propose a generalized model of fracture that encompasses all the QFM results, but it adds one important characteristics of geometrical nature: fractality of the fracture surfaces. The latter is usually described either by the fractal dimension D, or the roughness exponent H. Overall, two essential parameters enter the theory: one is used to describe the micro structure when the discrete fracture is considered (like in the QFM model of fracture), while the second one is suggested by the fractal nature of fracture. In this way a new mathematical model is constructed, which in addition to the micro structural parameter, a_0, incorporates a geometrical variable, namely the fractal exponent α or the fractal dimension D (or roughness exponent H). It should be emphasized that the goal of the present theory is not to simply add a fitting parameter. Instead, it links the micro and macro phenomena of fracture through a fractal dimension (roughness exponent).

This paper is structured as follows. In Section 2, the basic ideas of discrete crack propagation are discussed. We also present a configurational force interpretation of discrete fracture. In Section 3, we introduce a theory of fractal fracture mechanics that takes into account the discrete nature of crack propagation. Given a fractal crack, assuming that the radius of curvature of its equivalent smooth blunt crack is a material property, it is shown that the fractal crack roughens while propagating, i.e. the so-called Mirror–Mist–Hackle transition phenomenon is analytically predicted. Section 4 gives a discussion of the proposed theory and its implications. Conclusions are given in Section 5.

2. Fracture criterion for discrete crack propagation

In local failure criteria, it is assumed that at a given point, a scalar-valued function of Cauchy stress reaches a critical value, i.e., $f(\boldsymbol{\sigma}(\mathbf{x})) = \sigma_c$. These types of failure criteria have been successfully used in many practical

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applications, e.g. plasticity of metals, etc. However, there are problems for which these criteria result in paradoxical predictions. An example is an infinite plate with a circular hole under uniform tension at infinity. Classical elasticity predicts a stress concentration factor 3, independent of the radius of the hole. However, in reality for small holes failure depends on the size of the hole, i.e. an appropriate failure criterion should be geometry dependent in this case [28]. In the case of stress singularities, any such failure criterion would incorrectly predict a zero resistance to failure. On the other hand, energy-based failure criteria, e.g. Griffith's fracture theory can predict a finite critical stress in the case of a finite crack, for example. However, such a failure criterion breaks down for very small cracks (see [28] for more examples). This and similar examples show the need for the so-called non-local failure criteria for design of structures with multiscale failure mechanisms.

One of the simplest and most interesting non-local failure criteria is due to Neuber [41] and Novozhilov [42], in which it is assumed that average stress reaches a critical value at the onset of fracture. This can be interpreted equivalently as assuming a minimum crack propagation length or fracture "quantum" a_0. If this is the case, then one would need a certain "critical force" on this finite region for the crack to propagate. This simple idea is the basis for the recent new attempts in modelling size-dependent failure of very small structures. See also [29,39,30] for similar discussions.

An analogous concept of finite crack step growth was proposed for elastic-plastic fracture occurring at macro-scale level by Wnuk [56,57], who instead of considering stresses ahead of the crack front analyzed the increment of the opening displacement associated with each discrete crack advance. This increment in the displacement was considered at the forward edge of the "process zone" adjacent to the crack front and embedded within a larger non-linear end zone, say the plastic zone. Wnuk's [56] results based on his "finite stretch criterion" for the subcritical crack growth were fully confirmed by independent studies of Rice and Sorensen [50] and Rice et al. [51]. Further developments of Wnuk's model were given in [58]. When one compares the discrete model, considered here, that suggests a certain "fracture quantum" (a_0) with Wnuk's model, which postulated existence of the "process zone" (Δ), one arrives at a clear conclusion that the "process zone" is an analog of the "fracture quantum". These concepts apply when two very different scale ranges are considered, namely the atomistic model for nano-fracture versus the continuum model for nonlinear macro-fracture.

In the modified energy criterion valid for discrete crack propagation model, as suggested by Pugno and Ruoff [44], the infinitesimals $\mathrm{d}\Pi$ and $\mathrm{d}A$, representing the increments of the potential energy of the system ($\mathrm{d}\Pi$) and the change in the newly created free surface area ($\mathrm{d}A$), are replaced by the finite differences $\Delta\Pi$ and ΔA, respectively, however this should be done with some care as will be explained shortly. Therefore, instead of the classic equation defining the incipient point of crack growth

$$G^{\mathrm{LEFM}} = -\frac{\mathrm{d}\Pi}{\mathrm{d}A} = G_{\mathrm{c}} \tag{2.1}$$

one applies

$$G^{\mathrm{QFM}} = -\frac{\Delta_{a_0}\Pi}{\Delta A} \geqslant G_{\mathrm{c}} \tag{2.2}$$

as will be explained in the following.

Energy failure criterion in linear elastic fracture mechanics can be stated in various mathematically equivalent forms. Consider the total potential energy of a cracked elastic body subjected to external loading that can be written as

$$\Pi_{\mathrm{tot}}(\sigma,\ell) = \Pi(\sigma,\ell) + \mathcal{S}(\ell). \tag{2.3}$$

Here σ and ℓdenote the applied stress and the crack length, say $2a$, respectively, while Π denotes the energy available for fracture, i.e. the difference between work of the external forces and the strain energy. Specifically, the potential energy of the system consisting of an elastic body loaded by the tractions t_i and containing a crack of length $2a$, is written as

$$\Pi(\sigma,\ell) = \frac{1}{2}\int_{\Omega}\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}v - \int_{\partial\Omega} t_i u_i\,\mathrm{d}s. \tag{2.4}$$

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Here σ_{ij} and ε_{ij} are the standard notations for the Cauchy stress and linearized strain tensors, u_i is the displacement vector, t_i is the vector of tractions, σ denotes the remotely applied uniform stress, while the term $\mathcal{S}(\ell)$ represents the surface energy, $\mathcal{S} = 4a\gamma$. In the classic treatment of fracture propagation, the specific energy of fracture γ is assumed to be independent of the crack length. This assumption will be challenged in the present treatment of a discrete crack growth. For a 2D pre-cracked plate with a crack oriented perpendicularly to the remote stress σ, Griffith [22] calculated all terms in the right hand side of Eq. (2.1). The first term turned out to be $2\pi\sigma^2a^2/E'$, while the second was $\pi\sigma^2a^2/E'$, in which E' equals the Young modulus E for plane stress condition and it is $E/(1-\nu^2)$ for plane strain condition.

The term $\mathcal{S}$ was introduced for the first time by Griffith. It is postulated that at the point of incipient fracture Π_{tot} attains a stationary value. Thus, at $\sigma = \text{const.}$ we have

$$\frac{d\Pi_{\text{tot}}}{d\ell} = 0 \quad \text{or} \quad -\frac{d\Pi}{d\ell} = \frac{d\mathcal{S}}{d\ell}. \tag{2.5}$$

For the classic case of the Griffith crack, we have $\ell = 2a$, $\Pi = -\pi\sigma^2a^2/E'$, $\mathcal{S} = 4a\gamma$ and thus Eq. (2.5) results in a prediction for the stress at failure

$$\sigma_{\text{crit}}^{\text{LEFM}} = \sqrt{\frac{2E'\gamma}{\pi a}}. \tag{2.6}$$

Here the symbol γ denotes the specific surface energy describing material resistance to fracture. Using the contemporary notation Eq. (2.6) is usually written as

$$\sigma_{\text{crit}}^{\text{LEFM}} = \frac{K_c}{\sqrt{\pi a}}, \tag{2.7}$$

where K_c designates the so-called material fracture toughness, which is related to the fracture energy $G_c = 2\gamma$ and the modulus E' as follows $K_c^2 = G_cE'$. The left hand side of (2.5) is readily recognized as the energy release rate G or the J-integral, c.f. [49,6]. When these quantities are used the energy fracture criterion can be written as a local criterion (rather than Griffith's global equation), namely

$$G = G_c. \tag{2.8}$$

An essential departure from the classic treatment of the fracture problem here concerns a change from the notion of continuous crack extension, for which da is an infinitesimal, to a step-wise picture of crack propagation involving a certain finite increment in the length of the crack, say $\Delta a = a_0$. The constant a_0, named by Novozhilov [42] "fracture quantum", was initially thought of as an equivalent of the interatomic distance b_0 (for a cubic lattice). Recent work of Ippolito et al. [25] indicates that a_0 is greater than all the lattice spacings; it may be several times or even an order of magnitude larger than all the lattice parameters. In fact, the "best fit" between the atomistic simulations and theoretical data pertaining to fracture at nanoscale in β-SiC simulated by Ippolito et al. [25] was obtained by choosing the appropriate value of the fracture quantum to be $a_0 = 0.25$ nm.

Now, in their Quantized Fracture Mechanics, Pugno and Ruoff [44] replaced Eq. (2.5) by an expression that replaces differentials with finite differences, namely

$$-\frac{\Delta_{a_0}\Pi}{\Delta\ell} = \frac{\Delta_{a_0}\mathcal{S}}{\Delta\ell}, \tag{2.9}$$

where

$$\frac{\Delta_{a_0}f}{\Delta\ell} = \frac{f(\ell + a_0) - f(\ell)}{a_0}. \tag{2.10}$$

Note that this is not quite correct because if at the point of incipient fracture Π_{tot} attains a maximum value (unstable crack growth) then

$$\frac{\Delta_{a_0}\Pi_{\text{tot}}}{\Delta\ell} = \frac{\Pi_{\text{tot}}(\ell + \Delta\ell) - \Pi_{\text{tot}}(\ell)}{\Delta\ell} \leqslant 0. \tag{2.11}$$

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Thus, instead of (2.9) one has

$$-\frac{\Delta_{a_0}\Pi}{\Delta\ell} \geqslant \frac{\Delta_{a_0}\mathcal{S}}{\Delta\ell}. \tag{2.12}$$

Let us briefly review the derivation process of the expression analogous to (2.7), but for discrete crack growth. With $\Pi = -\pi\,\sigma^2 a^2/E'$, we have

$$-\frac{\Delta_{a_0}\Pi}{\Delta\ell} = \frac{1}{2a_0}\left(\frac{\pi\sigma^2}{E'}\right)[(a+a_0)^2 - a^2] = \frac{\pi\sigma^2}{E'}(a + a_0/2). \tag{2.13}$$

This is the energy release rate $G(\sigma,a)$, which at failure is set equal to the fracture energy G_c. Thus we obtain

$$\frac{\pi\sigma^2}{E'}\left(a+\frac{a_0}{2}\right) \geqslant G_c. \tag{2.14}$$

The critical stress is defined to be the minimum stress that satisfies the above inequality. Hence it follows that

$$\sigma_{\text{crit}}^{\text{QFM}} = \sqrt{\frac{E'G_c}{\pi\left(a+\frac{a_0}{2}\right)}} = \frac{K_c}{\sqrt{\pi\left(a+\frac{a_0}{2}\right)}}. \tag{2.15}$$

Contrary to the Griffith–Orowan–Irwin theory of fracture, Eq. (2.15) predicts a finite stress for a vanishing crack length, namely

$$\sigma_0 = \lim_{a\to 0}\sigma_{\text{crit}}^{\text{QFM}} = \sqrt{\frac{2}{\pi a_0}}K_c = \sqrt{\frac{2E'G_c}{\pi a_0}}. \tag{2.16}$$

This is a very interesting way of representing the intrinsic strength of an undamaged material through the entities, which define its elastic modulus, its resistance to a propagating crack and the characteristic microstructural constant a_0.

Let us attempt to estimate the size of the fracture quantum for fracture occurring in brittle solids. The estimate will be expressed in terms of the interatomic distance b_0 (in a cubic lattice). We know that the intrinsic strength of an undamaged material (implying a zero crack length) equals $\sqrt{\frac{2}{\pi a_0}}K_c$, and hence the size of the fracture quantum can be calculated as $2K_c^2/\pi\sigma_0^2$. If the intrinsic strength of an undamaged material is identified as the molecular strength σ_{mol}, then we arrive at the following predictions: Replace K_c^2 by EG_c, or by $2E\gamma$, then the fracture quantum a_0 can be estimated as follows:

$$a_0 = \frac{2}{\pi}\frac{K_c^2}{\sigma_{\text{mol}}^2} = \frac{2}{\pi}\left(\frac{2Eb_0\sigma_{\text{mol}}}{\sigma_{\text{mol}}^2}\right) = \frac{4}{\pi}\frac{Eb_0}{\sigma_{\text{mol}}}. \tag{2.17}$$

Now, assuming that the magnitude of the molecular strength σ_0 is on the order of magnitude of $E/10$, one obtains an estimate $a_0 \sim 12b_0$. Since b_0 is about 2×10^{-10} m, one gets an estimate for the quantum fracture in brittle solids as 2.4 nm. In principle, if the quantities σ_0 and K_c are obtained in a careful laboratory test, then the fracture quantum a_0 can be estimated as follows

$$a_0 = \frac{2}{\pi}\left(\frac{K_c}{\sigma_0}\right)^2. \tag{2.18}$$

This quantity is an atomistic analog of the plastic zone size suggested by the early research of Orowan [43] and Irwin [26] and then Barenblatt [3] and Dugdale [14] for elastic–plastic fracture occurring at macro-scale level that reads

$$r_p = \frac{\pi}{8}\left(\frac{K_c}{\sigma_Y}\right)^2. \tag{2.19}$$

For a low carbon structural steel such as ASTM-A36 the toughness K_c is on the order of magnitude of 100 MPa, while the local yield stress measured at the crack tip equals the constraint factor of about three times the standard yield point of 250 MPa, which gives 750 MPa. Therefore, the estimate of the plastic zone size is

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about 3.42 mm. This is seven orders of magnitude larger than the fracture quantum estimated above! It is noteworthy, though, that in elasto-plastic fracture the length of the plastic zone *per se* should not be considered a fracture quantum. It is rather the size of the process zone Δ embedded within the plastic zone and adjacent to the crack front that could be viewed as a finite growth step (or fracture quantum), cf. [56–58,6]. For ductile materials Δ is much smaller than r_p, while in the brittle limit Δ roughly equals r_p as then they both approach the fracture quantum a_0. This distinction between Δ and r_p clarifies the diversity of the ranges of the observed quanta a_0 for various materials and specimen geometries as reported in the literature, cf. [52,9].

2.1. Configurational-force interpretation of discrete fracture

The idea of a driving force in continuum mechanics goes back to Eshelby [16–18] and this notion is important in developing evolution laws for the movement of defects, including dislocations, vacancies, interfaces, cavities, cracks, etc. Driving forces on these defects cause climb and glide of dislocations, diffusion of point defects, migration of interfaces, changing the shape of cavities and propagation of cracks, to mention a few examples. Eshelby defined the force on a defect as the generalized force corresponding to position of the defect (in the reference configuration), which is thought of as a generalized displacement. Eshelby studied inhomogeneities in elastostatic and elastodynamic systems by considering the explicit dependence of the elastic energy density on position in the reference configuration.

In Novozhilov's [42] approach to fracture, one assumes that a crack propagates in a continuum but in discrete steps. In other words, one can still work within the framework of continuum elasticity theory. A crack is a special case of a defect and growth of cracks can be understood as evolution of the reference configuration. In this setting, crack propagation is driven by the so-called configurational (material) force at the crack tip. It turns out that component of configurational force in the crack growth direction is the J-integral. Now if a crack propagates in a given direction and with the amount $\Delta a = a_0$, there should be a material force driving it. In continuous crack growth one has

$$f_{\text{tip}} = J\,\mathrm{d}a. \tag{2.20}$$

Note that we are working in the linearized theory of elasticity and hence material and spatial configurations are not distinguishable. Now when a crack propagates by an amount a_0, one can calculate the average material force as (for a mode I crack of initial length $2a$ in an infinite plate)[1]

$$\bar{f}_{\text{tip}} = \frac{1}{a_0}\int_0^{a_0} J\,\mathrm{d}a = \frac{\sigma^2\pi a_0}{2E'}. \tag{2.21}$$

Assuming that $\bar{f}_{\text{tip}} = G_c = K_c^2/E'$, one obtains

$$\sigma_0 = \sqrt{\frac{2}{\pi a_0}}K_c. \tag{2.22}$$

We know that the asymptotic opening stress ahead of a crack tip has the following form

$$\sigma(r) = \frac{K_I}{\sqrt{2\pi r}}. \tag{2.23}$$

[1] Pugno and Ruoff [44] define the QFM stress intensity factor as

$$K_I^{\text{QFM}} = \left(\frac{1}{a_0}\int_a^{a+a_0} K_I^2\,\mathrm{d}a\right)^{\frac{1}{2}}.$$

This means that they, implicitly, define their stress intensity factor through an average configurational force. It should also be noted that in Novozhilov's [44] criterion one needs to use the complete stress distribution and not just the asymptotic distribution. However, using average configurational force, we see in the sequel that using the asymptotic form would suffice.

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Thus averaging this stress in the interval $r \in [0, a_0]$, one obtains

$$\bar{\sigma}_{\text{tip}} = \sqrt{\frac{2}{\pi a_0}} K_{\text{I}}. \tag{2.24}$$

Now when $K_{\text{I}} = K_{\text{c}}$, $\bar{\sigma}_{\text{tip}} = \sigma_0$, i.e. averaging stress and configurational stress give the same σ_0. This is because in brittle fracture of an elastic solid, configurational force (energy release rate) is a function of only K_{I} and not the other (non-singular) parts of stress. The QFM Theory developed recently by Pugno and Ruoff [44] fully supports these conclusions.

3. Fractal cracks with discrete propagation

Fracture surfaces are usually irregular and the classical treatment of cracks, where a crack is modelled by smooth curves (surfaces), is at best an approximation. It has long been known that cracks in brittle solids have rough surfaces and this ‘‘roughness’’ can evolve in the process of crack propagation (Mirror–Mist–Hackle transition phenomenon). Very rough curves (surfaces) show up in many natural phenomena and it turns out that unlike their seemingly random forms, all these irregular (rough) objects have some hidden degree of order. A fractal is a very special case of an irregular set, which has specific properties under scaling transformations. Curiosity of some researchers and also the inability of classical fracture mechanics in explaining many interesting failure phenomena motivated several studies on modelling rough fracture surfaces with fractals. These works started in the nineties and now there is an overwhelming amount of experimental evidence that cracks in real materials are fractals in a wide range of scales. Among the important theoretical contributions we can mention Mosolov [40], Goldshtein and Mosolov [20,21], Balankin [2], Borodich [4], Cherepanov et al. [11], Xie [61], Xie and Sanderson [62], Carpinteri [7], Carpinteri and Chiaia [8], Yavari et al. [63,65], Yavari [64], Wnuk and Yavari [59,60], etc. The main results of these and similar studies were the effect of fractality on the order of stress singularity at the crack tip, existence of new modes of fracture, possibility of crack propagation in uniform compression, etc. For quite sometime there was a hope in relating toughness of a material to the fractal dimension of fractal cracks forming in it (assuming that fractal dimension is only material dependent) [31]. However, such hopes seem to be too optimistic and in our opinion not very realistic (see [5] and references therein for similar discussions).

It seems that after about two decades of research in this field, fractals do not seem to have been predictive. In this paper, we will show that in a physically important problem fractals are predictive and the predicted result agrees with experimental observations. The problem is the dependence of fractal dimension (roughness exponent) of a fractal crack on the nominal crack length. As will be shown shortly, our fractal model predicts that when a small crack starts a stable growth, its fractal dimension increases. This, to our best knowledge, is the first analytic model that predicts the well-known Mirror–Mist–Hackle transition phenomenon for fractal cracks.

Now we repeat the calculations of the previous section for an elastic body containing a fractal crack. We should mention that there has been a recent work on extending QFM to self-similar fractal cracks [47] using the analogy of a fractal crack with a re-entrant corners [9].

We begin with the expression for stress intensity factor as represented by the fractal crack geometry, c.f. [40,20,21,65,64]

$$K_{\text{I}}^{\text{f}} = \chi(\alpha)\sigma\sqrt{\pi a^{2\alpha}}. \tag{3.1}$$

The function $\chi(\alpha)$ has been calculated by Wnuk and Yavari [60] as follows

$$\chi(\alpha) = \frac{1}{\pi^{2\alpha}} \int_0^1 \frac{(1+s)^{2\alpha} + (1-s)^{2\alpha}}{(1-s^2)^{\alpha}}\, \mathrm{d}s. \tag{3.2}$$

It is interesting to check the two limits of the expression (3.1) for the stress intensity factor as predicted by the fractal crack model. One such limit is obtained for the fractal dimension D approaching one, or – equivalently – the fractal exponent α approaching the value $\frac{1}{2}$. This corresponds to a sharp non-fractal crack (Griffith case), while the other limiting case is obtained for $D = 2$ or $\alpha = 0$, which represents a 2D void filling the plane,

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similar to an elliptical cavity with the ratio of the major to minor axes equal about 1.27. For vanishing α we get $\chi = 2$, while for α approaching $\frac{1}{2}$ we obtain $\chi = 1$, as expected. Thus the Eq. (3.1) reduces as follows

$$K_{\mathrm{I}}^{\mathrm{f}} = \begin{cases} \sigma\sqrt{\pi a} & \text{for } \alpha = \frac{1}{2}, \\ 2\sqrt{\pi}\sigma & \text{for } \alpha = 0. \end{cases} \tag{3.3}$$

Physical meaning of the limiting case of $\alpha = 0$ was discussed in detail by Wnuk and Yavari [59]. In this limit the fractal crack is shown to behave as an elliptical void, for which the stress concentration can be evaluated by Inglis [24] formula

$$\sigma_{\max} = \sigma\left(1 + 2\frac{a}{b}\right). \tag{3.4}$$

With $a/b \simeq 1.27$ one recovers $\sigma_{\max}/\sigma = 2\sqrt{\pi}$ as predicted by Wnuk and Yavari's approximate model of a fractal crack of the dimension $D = 2$.

It is known that a fractal dimension (or roughness exponent) is only one measure of irregularity of a curve. In other words, a fractal dimension D does not uniquely specify a fractal curve, i.e. two different curves can have the same fractal dimension. To use this simple measure of irregularity in fracture mechanics, Wnuk and Yavari [59] made a simplifying assumption and embedded a smooth crack in the stress field generated by the given fractal crack (see Fig. 3.1) and this enabled them to find an approximation for the function $\chi(\alpha)$. Note that dimensional analysis requires that elastic energy release per unit of the D-Hausdorff measure $\mathcal{H}_{\mathrm{D}}$ of the fractal crack be [64]

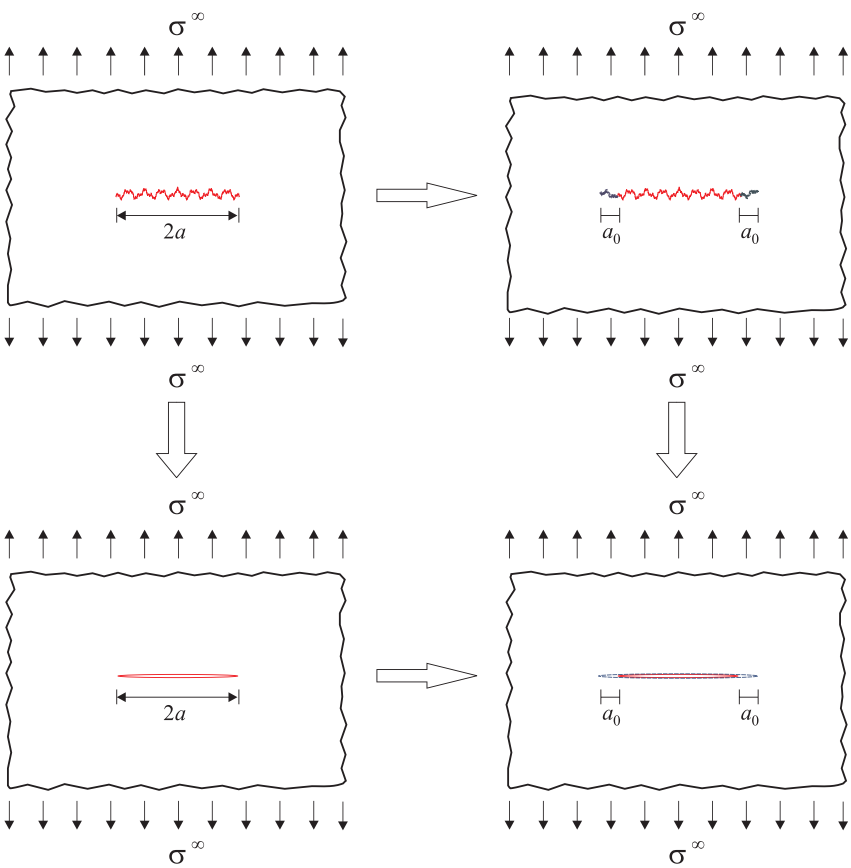

Fig. 3.1. Discrete growth of a fractal crack and its auxiliary smooth crack.

$$G^{\mathrm{f}} = \Psi(D)\frac{(K_{\mathrm{I}}^{\mathrm{f}})^2}{E'}, \tag{3.5}$$

where $\Psi(D)$ is some scalar-valued function. The problem with Hausdorff dimension is that it is extremely difficult to calculate even for the simplest fractal sets. Also, as was mentioned fractal dimension (roughness exponent) is simply a complexity index and does not provide enough information to distinguish different irregular sets from one another, in general, and one cannot use a fractal measure unambiguously. This necessitates an approximate analysis of fractal cracks if one likes to work with a fractal dimension (or roughness exponent).

As the energy release per unit of a fractal measure (3.5) will be compared with a critical fractal energy release rate, we can equivalently consider $G^{\mathrm{f}}/\Psi(D)$ as the fractal energy release rate. Thus, we can define the following energy release rate

$$G^{\mathrm{f}} = \frac{(K_{\mathrm{I}}^{\mathrm{f}})^2}{E'} = \frac{\chi(\alpha)^2\sigma^2\pi a^{2\alpha}}{E'}. \tag{3.6}$$

For the auxiliary smooth crack the "fractal" energy available for fracture is[2]

$$-\Pi^{\mathrm{f}} = 2\int_0^a G^{\mathrm{f}}(\sigma, a)\,\mathrm{d}a = \frac{\chi(\alpha)^2\sigma^2\pi a^{2\alpha+1}}{(2\alpha+1)E'}. \tag{3.7}$$

We now set up the energy fracture criterion for finite crack extension as[3]

$$G^{\mathrm{f}} = -\frac{\Delta_{a_0}\Pi^{\mathrm{f}}}{\Delta\ell} = \frac{\chi(\alpha)^2\sigma^2\pi}{a_0(2\alpha+1)E'}[(a+a_0)^{2\alpha+1} - a^{2\alpha+1}]. \tag{3.8}$$

Thus

$$G^{\mathrm{f}} = \frac{\chi(\alpha)^2\sigma^2\pi}{a_0(2\alpha+1)E'}[(a+a_0)^{2\alpha+1} - a^{2\alpha+1}]. \tag{3.9}$$

At failure

$$\frac{\chi(\alpha)^2\sigma^2\pi}{a_0(2\alpha+1)E'}[(a+a_0)^{2\alpha+1} - a^{2\alpha+1}] \geqslant G_{\mathrm{c}}^{\mathrm{f}}. \tag{3.10}$$

It is interesting to note that the fractal energy release rate $G_{\mathrm{c}}^{\mathrm{f}}$ has the dimensions of Stress × Length$^{2\alpha}$, as this is consistent with the equation $G_{\mathrm{c}}^{\mathrm{f}} = (K_{\mathrm{c}}^{\mathrm{f}})^2/E'$, in which $K_{\mathrm{c}}^{\mathrm{f}}$ is defined by (3.1). Also note that α-dependence of the critical stress intensity factor is similar to that of notches [52]. Applying the condition $G^{\mathrm{f}} \geqslant G_{\mathrm{c}}^{\mathrm{f}}$ at the point of incipient fracture one obtains the following expression for the critical stress due to presence of a fractal crack.

$$\sigma_{\mathrm{crit}}^{\mathrm{f}} = \sqrt{\frac{(2\alpha+1)E'G_{\mathrm{c}}^{\mathrm{f}}}{\pi}}\frac{\sqrt{a_0}}{\chi(\alpha)\sqrt{(a+a_0)^{2\alpha+1} - a^{2\alpha+1}}}. \tag{3.11}$$

[2] Here a comment is in order. A fractal surface has unbounded surface area but a finite fractal measure. G^{f} has the dimension of energy per unit of a D-measure. This means that integrating this specific energy with respect to the corresponding fractal measure, the resulting quantity is energy and has a finite value. In the present model, we embed an auxiliary smooth crack in the stress field of the fractal crack. When the fractal crack propagates its nominal length increases, i.e. the auxiliary smooth crack propagates too. We integrate G^{f} with respect to the measure of the auxiliary smooth crack, i.e. the standard Lebesgue measure, and because G^{f} is finite this gives us a finite scalar that we call "fractal energy release". "Fractal potential energy" Π^{f} is defined as fractal specific energy release rate G^{f} times the nominal crack length growth, i.e.

$$-\frac{\mathrm{d}\Pi^{\mathrm{f}}}{\mathrm{d}\ell} = G^{\mathrm{f}}.$$

[3] Note that there is always a lower cut-off ϵ for fractality of a rough crack, i.e. a rough crack should be modelled as a physical fractal [4]. In this case, in the present formulation it is implicitly assumed that $a_0 > \epsilon$. However, a rough crack can be idealized as a mathematical fractal and in that case there is no restriction on a_0.

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It is desirable to normalize this expression by dividing both sides through the intrinsic strength of undamaged material that is defined as follows.

$$\sigma_0^{\mathrm{f}} = \sqrt{\frac{(2\alpha+1)E'G_{\mathrm{c}}^{\mathrm{f}}}{\pi}\frac{1}{a_0^{\alpha}\chi(\alpha)}} = \sigma_0 \frac{K_{\mathrm{c}}^{\mathrm{f}}}{K_{\mathrm{c}}} a_0^{\frac{1}{2}-\alpha}\frac{\sqrt{\alpha+\frac{1}{2}}}{\chi(\alpha)}. \tag{3.12}$$

This formula presents a generalization of the non-fractal expression for strength given in Eq. (2.16) and it can be readily shown that for $\alpha = \frac{1}{2}$ Eq. (3.12) yields $\sigma_0^{\mathrm{f}} = \sigma_0 = \sqrt{2/\pi a_0}K_{\mathrm{c}}$ as predicted by the QFM theory, c.f. (2.16). One interesting observation is that the fracture quantum a_0 should never be assumed zero, as then the material strength would reach infinity.

If (3.11) is rewritten in a non-dimensional form, one obtains

$$s_{\mathrm{c}}^{\mathrm{f}} = \frac{1}{\sqrt{(1+X)^{2\alpha+1} - X^{2\alpha+1}}}, \tag{3.13}$$

where $X = a/a_0$ and $s_{\mathrm{c}}^{\mathrm{f}} = \sigma_{\mathrm{c}}^{\mathrm{f}}/\sigma_0^{\mathrm{f}}$. For $\alpha = \frac{1}{2}$, this formula reduces to the QFM result for a slit crack, i.e.

$$\frac{\sigma_{\mathrm{crit}}^{\mathrm{QFM}}}{\sigma_0} = \frac{1}{\sqrt{1+2X}}. \tag{3.14}$$

For α approaching zero it follows that the critical stress given by (3.13) predicts $s_{\mathrm{c}}^{\mathrm{f}} = 1$. We conclude that the discrete fractal fracture mechanics developed here encompasses all the known special theories of fracture, both the linear elastic fracture mechanics (LEFM) and the quantized fracture mechanics (QFM) theories. When the expression (3.11) is squared and divided by the square of the critical stress predicted by the LEFM, we obtain an estimate of the fracture energy γ_α/γ as

$$\frac{\gamma_\alpha}{\gamma} = \frac{(2\alpha+1)X^{2\alpha}}{(1+X)^{2\alpha+1} - X^{2\alpha+1}}. \tag{3.15}$$

Note that in this case surface energy is a function of:

(i) crack length, $X = a/a_0$,
(ii) fractal exponent α.

Note also that

$$\lim_{X\to\infty} \frac{\gamma_\alpha}{\gamma} = 1. \tag{3.16}$$

Eq. (3.15) is used to generate the family of curves shown in Fig. 3.2.

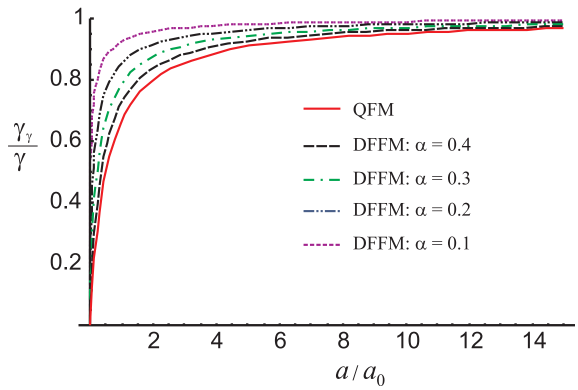

Fig. 3.2. Crack length dependent specific energy of fractal fracture shown as a function of normalized nominal crack size a/a_0 and the fractal exponent α. It is seen that for fractal exponent approaching 1/2 the present result reduces to the QFM.

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Let us examine the expression for the stress at failure due to a crack considered as a fractal. Recall that for a slit fractal crack the following stress distribution holds, c.f. [59]

$$\sigma^{\mathrm{f}}_{yy}(r) = \frac{K^{\mathrm{f}}_{\mathrm{I}}}{(2\pi r)^{\alpha}}. \tag{3.17}$$

Stress intensity factor in this equation is defined as, c.f. [59]

$$K^{\mathrm{f}}_{\mathrm{I}} = \sigma\sqrt{\pi}a^{\alpha}\chi(\alpha). \tag{3.18}$$

Letting $K^{\mathrm{f}}_{\mathrm{I}} \to K^{\mathrm{f}}_{\mathrm{c}}$ and applying the averaging technique of Novozhilov we calculate the critical stress due to a slit fractal crack

$$\sigma^{\mathrm{f}}_{\mathrm{crit}} = \left\langle \sigma^{\mathrm{f}}_{yy}(r) \right\rangle_{0,a_0} = \frac{1}{a_0}\frac{K^{\mathrm{f}}_{\mathrm{c}}}{(2\pi)^{\alpha}}\int_0^{a_0}\frac{\mathrm{d}r}{r^{\alpha}} = \frac{K^{\mathrm{f}}_{\mathrm{c}}}{(1-\alpha)(2\pi a_0)^{\alpha}}. \tag{3.19}$$

As expected this expression reduces to $\sqrt{\frac{2}{\pi a_0}}K_{\mathrm{c}}$ when α approaches $\frac{1}{2}$. Indeed, the result (3.19) describes the intrinsic material strength that now in addition to the fracture quantum a_0 depends also on the degree of fractality measured by α.

3.1. The Mist–Mirror–Hackle transition phenomenon

We know that opening stress in a blunt crack with radius of curvature ρ has the following form [13]

$$\sigma(r) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\left(1 + \frac{\rho}{2r}\right). \tag{3.20}$$

Comparison of the critical stresses obtained for a sharp and a blunt discrete crack, gives a relation between the corresponding critical stress intensity factors, K_{c} and $K^{\mathrm{b}}_{\mathrm{c}}$, i.e. [45,25]

$$K^{\mathrm{b}}_{\mathrm{c}} = K_{\mathrm{c}}\sqrt{1 + \frac{C}{2}} = K_{\mathrm{c}}\sqrt{1 + \frac{\rho}{2a_0}}, \tag{3.21}$$

where $C = \rho/a_0$. Thus, it is seen that the material toughness obtained for a blunt crack is somewhat greater than the toughness associated with the sharp crack with a zero tip radius, when $\rho = C = 0$. This is somewhat reminiscent of the Irwin's correction to the "effective stress intensity factor" valid for an elasto-plastic case. Pugno et al. [45]; Ippolito et al. [25] show that

$$\sigma^{\mathrm{QFM}}_{\mathrm{crit}} = \frac{K^{\mathrm{b}}_{\mathrm{c}}}{\sqrt{\pi\left(a + \frac{a_0}{2}\right)}} = \frac{K_{\mathrm{c}}}{\sqrt{\pi a}}\sqrt{\frac{1 + \frac{\rho}{2a_0}}{1 + \frac{a_0}{2a}}}. \tag{3.22}$$

Since the ratio $K_{\mathrm{c}}/\sqrt{\pi a}$ is recognized as $\sigma^{\mathrm{LEFM}}_{\mathrm{crit}}$ the Eq. (3.22) can be cast into the following final form

$$\sigma^{QFM}_{\mathrm{crit}} = \sigma^{\mathrm{LEFM}}_{\mathrm{crit}}\sqrt{\frac{1 + \frac{\rho}{2a_0}}{1 + \frac{a_0}{2a}}}. \tag{3.23}$$

If the dimensionless variables $C = \rho/a_0$ and $X = a/a_0$ are used, then the relation (3.23) reads

$$\sigma^{\mathrm{QFM}}_{\mathrm{crit}} = \sigma^{\mathrm{LEFM}}_{\mathrm{crit}}\sqrt{\frac{1 + \frac{C}{2}}{1 + \frac{1}{2X}}}. \tag{3.24}$$

It can be readily seen that the resistance to fracture $G_{\mathrm{c}} = K^2_{\mathrm{c}}/E'$ is proportional to the square of the critical stress, thus

$$\frac{G^{\mathrm{QFM}}_{\mathrm{c}}}{G^{\mathrm{LEFM}}_{\mathrm{c}}} = \left(\frac{\sigma^{\mathrm{QFM}}_{\mathrm{c}}}{\sigma^{\mathrm{LEFM}}_{\mathrm{c}}}\right)^2 = \frac{1 + \frac{C}{2}}{1 + \frac{1}{2X}}. \tag{3.25}$$

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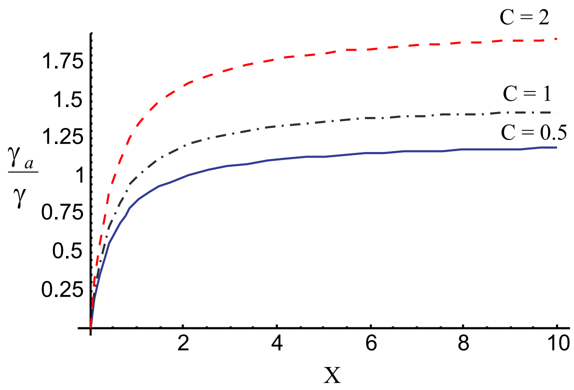

Fig. 3.3. Nondimensional specific energy of fracture shown as a function of the microstructural constant C and the crack size X.

We recall that since $G = 2\gamma$ the ratio written above can also be expressed as γ_a/γ, where γ_a denotes the crack length dependent specific energy of fracture, as is seen in Fig. 3.3. It is clearly seen that the dependence of γ_a on the non-dimensional crack length X is not significant for $X > 4$, but it is very pronounced for short cracks, i.e. for $0 < X < 4$.

Given a fractal crack, one can define an equivalent smooth blunt crack. Wnuk and Yavari [60] showed that the equivalent smooth blunt crack has the following radius of curvature.

$$\rho_\alpha = \frac{a_{\mathrm{ini}}}{\pi}\left[\frac{\chi(\alpha)}{2^{1+\alpha}(0.05)^\alpha}\right]^{\frac{2}{2\alpha-1}}, \tag{3.26}$$

where a_{ini} is the initial nominal length of the fractal crack. Thus

$$C = \frac{1}{\pi}\frac{a_{\mathrm{ini}}}{a_0}\left[\frac{\chi(\alpha)}{2^{1+\alpha}(0.05)^\alpha}\right]^{\frac{2}{2\alpha-1}}. \tag{3.27}$$

For a given α and the initial crack size expressed as a multiple of the fracture quantum, the microstructural constant C can be readily evaluated (see Fig. 3.4). Analytically the exact value of α is given as a root of the following transcendental equation

$$\frac{\rho_\alpha}{a_{\mathrm{ini}}} - \frac{1}{\pi}\left[\frac{\chi(\alpha)}{2^{1+\alpha}(0.05)^\alpha}\right]^{\frac{2}{2\alpha-1}} = 0. \tag{3.28}$$

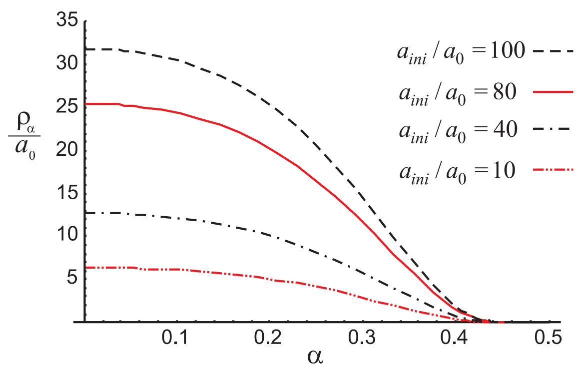

Fig. 3.4. A relationship between the microstructural constant ρ_α and the fractal exponent α. Note that for the longer cracks the radius ρ_α is larger for the same roughness. This relationship was suggested by Wnuk and Yavari [60]. A strong dependence of the finite root radius at the crack front on the degree of fractality of crack geometry is noted.

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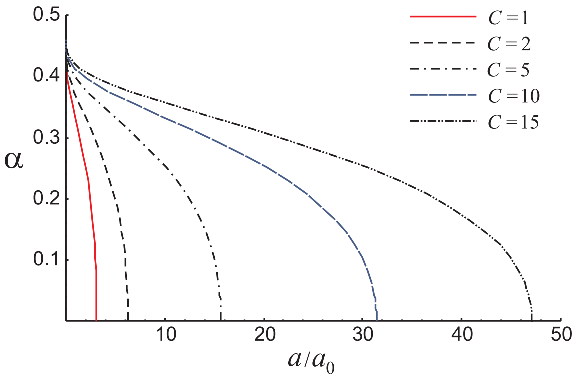

Fig. 3.5. Dependence of the fractal exponent on the current normalized crack length. The parameter that distinguishes the curves shown is the microstructural constant $C = \rho/a_0$. It is seen that for short cracks, when $C \to 0$ the function describing the Mirror–Mist–Hackle effect becomes extremely sensitive to the variations in the fractal exponent α.

The root radius ρ_α and the initial crack length for this particular problem are assumed to be known. Finally, an interesting observation needs to be made. It concerns interpretation of the constant defined above as the ratio of the initial crack size to fracture quantum, a_{ini}/a_0. As soon as the crack begins to propagate, this constant changes into a variable ratio a/a_0. And then Eq. (3.28) suggests that the fractal exponent becomes a certain function of the current crack length as it has been always suspected. The relationship must be consistent with the expression (3.28), in which a_{ini} is simply replaced by "a" implying now a function $\alpha = \alpha(a)$. A comment is in order here. As was shown in [60], stress distribution around the tip of a fractal crack depends only on the dimension (roughness exponent) at the crack tip. In other words, the method of Wnuk and Yavari [59] can be used for fractal cracks, with evolving fractal dimension in stable crack growth.

The inverse function representing the fractal exponent as a function of the current crack length $\alpha = \alpha(a)$ can be plotted using the nondimensional variables $X = a/a_0$ and α. The resulting curves are shown in Fig. 3.5. These curves reflect the Mirror–Mist–Hackle effect for the cracks described by fractals. To our knowledge, this is the first quantitative representation of the MMH effect for the fractal cracks.

The functions shown in Fig. 3.5 demonstrate a rather intricate relation between fractal geometry, fracture quantum and appearance of the Mirror–Mist–Hackle effect. For the sake of argument, let us assume that the fracture quantum a_0 is allowed to approach zero, while the radius of curvature ρ remains a finite entity. In this case the microstructural constant C approaches infinity and the MMH effect disappears entirely and the crack behaves like a fractal object defined by the exponent $\alpha = 1/2$. This demonstrates that the concepts of discrete crack propagation and that of fractal nature of the crack surfaces, if incorporated into one consistent theory, produce the results, which do not exist within the framework of the classical theory of fracture.

4. Discussion

For nearly a century an apparent controversy has existed as a question without an answer built-in within the classical fracture mechanics as originated by Griffith in his seminal paper [22]. Griffith's work was inspired by an attempt to explain why the strength of solids measured in the laboratory was several orders of magnitude lower than the theoretical strength calculated based on the theoretical molecular cohesive strength. Griffith's theory provided an explanation for these discrepancies, with one caveat though, that in the limit of vanishing crack size the Griffith's theory predicted an infinite intrinsic strength of the material. Such a conclusion was physically unacceptable, but if short cracks were excluded, then the theory worked well. Thus, for nearly a century the LEFM and its variations such as the J-integral concept, c.f. [48], R-curve technique, c.f. [57,51], the EPRI estimation schemes, Shih [53] designed for the elasto-plastic range of fracture and the numerous modifications of the cohesive crack model due to Barenblatt–Dugdale–Bilby–Cottrell–Swinden research, provided remarkable analytical tools for failure prevention and engineering fracture mechanics.

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Yet, the basic controversy persisted over the past decades. The presently proposed model of Discrete Fractal Fracture Mechanics (DFFM) offers the following new features that are built into the present theory of fracture. These are: (1) quantized fracture mechanics based on an assumption of Novozhilov's "fracture quantum", and (2) fractal nature of the geometry of a crack viewed as a fractal object from its inception to the point of transition to the catastrophic fracture. It should be noted that according to our model, assuming that radius of curvature of the equivalent smooth blunt crack is a material property, the degree of fractality as measured by the exponent α or the fractal dimension D or the Hurst exponent H, becomes a certain function of the current crack length. This function turns out to be extremely sensitive to minute variations in the fractal exponent α, making the entire process highly nonlinear. The DFFM model proposed here is equipped with four essential attributes that were lacking in the previous theories of fracture, namely

- discrete rather than continuous nature of crack propagation;
- inherent roughness of the crack surface that is mathematically accounted for by the DFFM model;
- interdependence of the fractal dimension D and the current crack length;
- fractal specific energy of fracture becomes a certain function of the crack length.

Andrade and Tsien [1] in a series of experiments demonstrated that any undamaged material in its "virgin" state contains numerous microcracks, which usually remain dormant, unless the external loads are raised to the "critical" level, at which one of these pre-existing cracks begins to propagate in a catastrophic way. The onset of such propagation is predicted correctly by the Griffith's criterion. Now we submit that even in a perfect material with no initial defects, when the external loads of static or variable (fatigue) nature are applied, the cracks will eventually initiate, and once they are there they will be best represented by fractals, say $D = 0$ for an atomistic defect prior to forming a microcrack, i.e., D varies between zero and one, then a crack-like defect described by a fractal dimension of $D = 1$ (sharp initial crack), and then – as this crack propagates in a stable manner at load levels lower than the Griffith threshold, the fractal dimension of the crack will tend towards the limiting value of $D = 2$. This theoretical limit most likely is never reached with few exceptions of "forgiving" materials, which are insensitive to the existence of defects up to a certain size level. Otherwise, the onset of catastrophic fracture will interfere with the growth of subcritical crack, and the subcritical growth process ends at a certain fractal dimension that belongs to the interval (1,2), say $D_{crit} = 1.47$.

For a physicist the most interesting stage of crack development history is the stage of stable crack growth, which immediately follows the point of initiation and which can be described mathematically by use of a known function describing the dependence of the degree of fractality of fracture surfaces on the current crack length. This function is a quantitative representation of the Mirror–Mist–Hackle phenomenon and it is intimately related to another function; see Eq. (3.28) and Fig. 3.5. This function relates to the specific energy of fracture given as a function of the crack length as shown in Fig. 3.3.

5. Concluding remarks

A discrete rather than continuous model of fractal crack propagation has been proposed in this paper. This model is within the framework of the quantized fracture mechanics (QFM), but with an additional feature added to the theory of fracture process: fractal dimension (roughness exponent) of the fractal crack. Thus, certain new concepts are incorporated into the present model such as the "fracture quantum" as defined by Novozhilov [42], and a measure of "roughness" of the crack surface resulting from its fractal nature. This roughness enters all the pertinent equations, and it turns out that it influences the intrinsic strength of the material (when the crack length equals zero).

The following important limiting cases are included in the present theory of fracture:

(1) The case of "smooth" crack is obtained when the fractal dimension $D = 2(1 - \alpha)$ equals one. In this limit one recovers the quantized fracture mechanics (QFM). In this model crack propagation is viewed as a sequence of minute "jumps", each of which is proportional to a *fracture quantum*. Such notion has been discussed by Pugno and Ruoff [44], based on the earlier observations of Eshelby [15] and Novozhilov [42].

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(2) The other extreme incorporated into the present theory involves the case of $D = 2(H = 1/2)$, or $\alpha = 0$. In this limit the crack degenerates into a 2D object that behaves much like an elliptical void obeying Inglis [24] rule of stress magnification at both ends of the ellipse. Interestingly, it was exactly this formula that inspired Griffith's seminal work on Mechanics of Fracture in 1920. It appears that our model proposed here addresses this special case, too, as it results in a quite natural way when the fractal dimension D approaches 2.

Both special cases represent interesting physical realities, one is that of a 2D void described by Neuber's Notch Mechanics [41] (when $D = 2$), while the other one, when $D = 1$, corresponds to the Griffith's classic theory of sharp cracks, the well-known LEFM. The most interesting part of the theory, though, lies in between these two limits determined by the Notch Theory ($D = 2$) and the LEFM ($D = 1$). This range corresponds to ''rough'' cracks mathematically described by fractals.

Although a number of newly arising problems need to be studied in greater detail, two essential conclusions seem to transpire from the DFFM model. Our theory predicts a finite intrinsic material strength that is explicitly related to the following material properties:

(1) Fracture quantum, a_0.
(2) Fractal dimension D or the roughness exponent H.

All the pertinent equations of the present theory become substantially different from the classic theory predictions especially for short cracks. Finite radius of the curvature at the crack root turns out to be inseparably related to the fractal exponent α, or to the fractal dimension D. Wnuk and Yavari [60] postulated existence of such a relation between ρ and α by studying the thickness of a fractal boundary layer. If this thickness is known, the degree of fractality can be quantitatively established. Assuming that ρ is a material constant for the equivalent smooth blunt crack of a given fractal crack, we showed that fractal dimension at the tip of the fractal crack is a monotonically increasing function of the current length of the crack.

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