Direct Solution to Find the Eigenvectors of a Dynamic Modal Analysis

Let be the $(n \times n)$ system of equations:

$$[K]\{U\} + [M]\{\ddot{U}\} = \{0\}$$

And λ_i , $i = 1\ to\ n$: the corresponding Eigenvalues

$$([K] - \lambda\,[M])\{\Phi\} = \{0\}$$

Such that $[A] = [K] - \lambda\,[M]$

And $\{\Phi\}$ the Eigenvectors

The Eigenvalues are obtained from the Characteristic equation:

$$|[K] - \lambda\,[M]| = 0$$

λ_i are assumed to be known

The Eigenvectors are obtained from :

$$[A]\{\Phi\} = \{0\}$$

Such that : $[K]\{\Phi\} = \lambda\,[M]\{\Phi\}$

Let X the $(n \times 1)$ one-dimensional Vector Storing One Eigenvector of the $[\Phi]$ Modal matrix :

$$[A]\{X\} = \{0\}$$

Let be $X(1) = 1$

One has to solve the $(n-1 \times n-1)$ system of equations for each λ_i with the linear solving with a direct method like the $Gauss\ L.U$:

$$[\boldsymbol{G}]\{\mathbf{Y}\} = \{\boldsymbol{B}\}$$

Such that:

$\boldsymbol{G(i,j) = A(i+1, j+1)}$ $\quad\quad \boldsymbol{i = 1\ to\ n-1\ ,\ \ j = 1\ to\ n-1}$

$\boldsymbol{B(i) = -A(i, i+1)}$ $\quad\quad \boldsymbol{i = 1\ to\ n-1}$

$\boldsymbol{Y(i) = X(i+1)}$ $\quad\quad \boldsymbol{i = 1\ to\ n-1}$

<u>Example</u> :

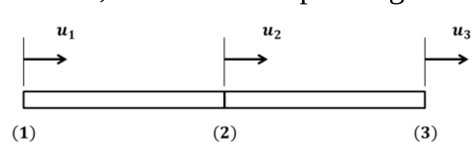
Two Bar elements subjected to a free vibration motion have the same following properties :

$k = E * S/L : stiffness$

$m = \rho * S * L : mass$

E: Longitudinal elasticity modulus, S: Transversal section, L: Length , ρ: Mass per unit volume

Considering the Lumped mass matrix, find the corresponding structural Eigenvectors.



Solution :

$$[A] = [K] - \lambda\,[M] = \begin{bmatrix} k & -k & 0 \\ -k & (k+k) & -k \\ 0 & -k & k \end{bmatrix} - \lambda \begin{bmatrix} \frac{m}{2} & 0 & 0 \\ 0 & \left(\frac{m}{2}+\frac{m}{2}\right) & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix}$$

$$\begin{vmatrix} \left(k-\lambda\frac{m}{2}\right) & -k & 0 \\ -k & (2k-\lambda m) & -k \\ 0 & -k & \left(k-\lambda\frac{m}{2}\right) \end{vmatrix} = 0$$

$$\lambda\,m\left(k-\lambda\frac{m}{2}\right)\left(\lambda\frac{m}{2}-2k\right) = 0$$

$$\boldsymbol{\lambda_1 = 0}$$

$$\boldsymbol{\lambda_2 = \frac{2k}{m}}$$

$$\boldsymbol{\lambda_3 = \frac{4k}{m}}$$

The corresponding natural Frequencies are :

$$\boldsymbol{\omega_1 = 0}$$

$$\boldsymbol{\omega_2 = \sqrt{\frac{2k}{m}}}$$

$$\boldsymbol{\omega_3 = 2\sqrt{\frac{k}{m}}}$$

The following step is searching the Eigenvectors :

$$[A] = \begin{bmatrix} \left(k-\lambda\frac{m}{2}\right) & -k & 0 \\ -k & (2k-\lambda m) & -k \\ 0 & -k & \left(k-\lambda\frac{m}{2}\right) \end{bmatrix}$$

1) $\boldsymbol{\omega_1 = 0}$

$$[A] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$[A]\{X\} = \{0\} \Longrightarrow \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\varphi_{11} = 1 \Longrightarrow \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} k \\ 0 \end{Bmatrix}$ corresponding to $[G]\{Y\} = \{B\}$

$$\begin{bmatrix} 2k & -k \\ 0 & \frac{k}{2} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} k \\ \frac{k}{2} \end{Bmatrix}$$

$$\Longrightarrow \begin{Bmatrix} \varphi_{12} \\ \varphi_{13} \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{\Phi\} = \{X\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

2) $\boldsymbol{\omega_2 = \sqrt{\frac{2k}{m}}}$

$$[A] = \begin{bmatrix} 0 & -k & 0 \\ -k & 0 & -k \\ 0 & -k & 0 \end{bmatrix}$$

$$[A]\{X\} = \{0\} \Longrightarrow \begin{bmatrix} 0 & -k & 0 \\ -k & 0 & -k \\ 0 & -k & 0 \end{bmatrix} \begin{Bmatrix} \varphi_{21} \\ \varphi_{22} \\ \varphi_{23} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\varphi_{21} = 1 \Longrightarrow \begin{bmatrix} 0 & -k \\ -k & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} k \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ k \end{Bmatrix}$$

$$\Longrightarrow \begin{Bmatrix} \varphi_{22} \\ \varphi_{23} \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$\{\Phi\} = \{X\} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

2) $\boldsymbol{\omega_3 = 2\sqrt{\frac{k}{m}}}$

$$[A] = \begin{bmatrix} -k & -k & 0 \\ -k & -2k & -k \\ 0 & -k & -k \end{bmatrix}$$

$$[A]\{X\} = \{0\} \Longrightarrow \begin{bmatrix} -k & -k & 0 \\ -k & -2k & -k \\ 0 & -k & -k \end{bmatrix} \begin{Bmatrix} \varphi_{31} \\ \varphi_{32} \\ \varphi_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\varphi_{31} = 1 \Longrightarrow \begin{bmatrix} -2k & -k \\ 0 & -\frac{k}{2} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} k \\ -\frac{k}{2} \end{Bmatrix}$$

$$\Longrightarrow \begin{Bmatrix} \varphi_{22} \\ \varphi_{23} \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\{\Phi\} = \{X\} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

The Modal matrix is then :

$$[\Phi] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Conclusion :

This method allowed to find the Eigenvectors with an Efficient Direct linear method without using any iterative process or any other mathematical combination.