Energy release rate. Fracture energy

Fracture of steel. Following Griffith, you are performing the same experiment using steel rather than glass. Let's say you have several bodies of the steel. Using a diamond saw, you cut each body with a crack of length $2a$. The lengths of the cracks are different in different bodies. You load each body in tension up to fracture, and record the applied stress at fracture, σ_c. Many people have done such experiments and here are the basic experimental facts.

1. $\sigma_c \sqrt{a} = \text{constant}$, independent of the length of the crack.

2. The constant is orders of magnitude larger than $\sqrt{2\gamma E / \pi}$. Note that the surface energy of most solids is on the order of 1 J/m².

Thus, the Griffith theory agrees with one part of the experimental observation, but disagrees with the other. While other people complained about this large discrepancy, Irwin and Orowan did something about it. In late 1940's, they found a way to apply the Griffith theory to brittle fracture of steel.

A frame-by-frame movie. Such a movie must exist by now, but I don't have it. I'll sketch pictures in class.

Frame 1. A crack is cut into a body of steel using a diamond saw. No force is applied to the body yet.

Frame 2. Apply a small load. A small region around the tip of the crack yields. This small region is called the *plastic zone*. The body remains elastic outside the plastic zone. The front of the crack remains stationary.

Frame 3. Increase the load slightly. The plastic zone increases in size. The front of the crack still remains stationary.

Frame 4. Increase the load still more. The front of the crack starts to advance. The larger the load, the more the crack advances.

Frame 5. The load reaches a constant level, the crack advances in a steady state. Plastic deformation is confined in the thin layers beneath the crack surfaces. The thickness of the plastic layers remains constant as the crack advances.

Plastic deformation. The large discrepancy between the Griffith theory and experiments with steel had to do with plastic deformation in the steel accompanying fracture. When a crack grows in steel, many atoms off the plane of crack will change neighbors. The steel off the plane of the crack deforms plastically. After the steel fractures into two pieces, the pieces do not fit neatly. By contrast, when a crack grows in glass, atoms off the plane of the crack do not change neighbors. The glass off the plane of the crack deforms elastically. After the glass fractures into two pieces, the pieces fit together neatly.

Dissipation of energy. Let us focus on Frame 5: a crack grows in steady state. Follow a material particle near the plane of the crack. As the front of the crack passes by, the deformation of the material particle undergoes

hysteresis: elastic loading, plastic flow, and then elastic unloading. Sketch the stress-strain curve of a material particle. As the crack advances, the material particle is initially far ahead the front of the crack, and finally far behind the front of the crack. The material particle goes through a history of stress and strain. The material particle dissipates energy as heat. In this picture, the crack growing in steady state is a nonequilibrium process.

Compare this picture with the original Griffith theory. For a crack growing in steady state, the original Griffith theory regards the growth as an equilibrium process: the reduction of the elastic energy is fully stored as the surface energy. No energy is dissipated.

Small-scale yielding. For the time being we will restrict ourselves to the case that the plastic zone size is much smaller than any macroscopic length in the body, such as the length of the crack and size of the body. This condition is known as the small-scale yielding condition.

Under the small-scale yielding condition, much of the body deforms elastically. Because the size of the plastic zone is much smaller than the length of the crack, the crack can attain the steady state after extending by a length small compared to the total length of the crack.

The thickness of the plastically deformed layer in the steady state is a material property. Unless otherwise specified, we will call this steady-state thickness the plastic zone size, r_p. Let a be a representative macroscopic length by a, such as the length of the crack or the size of the body. The small-scale yielding condition requires that the plastic zone size is much smaller than the macroscopic length:

$$r_p << a.$$

The inelastic zone for silica is of atomic dimension, so that a crack beyond a few nanometers satisfies the small-scale yielding condition. By contrast, the plastic zone for a steel may be of millimeter in size, so that a crack beyond a few centimeters satisfies the small-scale yielding condition. For a particularly ductile steel, however, the plastic zone can be several centimeters in size. To test such a ductile steel under the small-scale yielding condition requires a body of a size about a file cabinet. Such a test is carried out sometimes, but is expensive. We will discuss large-scale yielding later in this course.

Under the small-scale yielding condition, the deviation form elasticity is confined within thin layers of materials beneath the surfaces of the crack.

In the original Griffith theory, the deviation from elasticity is confined with a few atomic layers beneath the surfaces of the crack.

Modify the Griffith theory to account for plasticity. Griffith's picture of fracture is

Fracture = atomic bond breaking.

Griffith used the surface energy to account for the inelastic process of bond breaking, and obtained the condition for fracture:

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi a}}.$$

Irwin's and Orowan's picture is

Fracture = atomic bond breaking + plastic deformation.

They define the fracture energy Γ as the energy needed to advance a (steady state) crack by a unit area.

Fracture energy = surface energy + plastic work.

$$\Gamma = 2\gamma + w_p.$$

Here w_p is the work done to create per unit area of the plastic layers. Irwin and Orowan used the fracture energy to account for the inelastic process of bond breaking and plastic deformation, and they modified the condition for fracture as

$$\sigma_c = \sqrt{\frac{\Gamma E}{\pi a}}.$$

A few quick notes about fracture energy:
1. The fracture energy is a material property, independent of the length of the pre-crack, so long as the small-scale yielding condition applies.
2. The fracture energy is difficult to calculate from first principles, and is determined by fracture test, as described above.
3. The fracture energy is much larger than the surface energy. A lot more atoms participate in plastic deformation than in bond breaking. Some rough values. Glass: 10 J/m². Ceramics: 50 J/m². Glassy polymers: 10^3 J/m². Aluminum: 10^4 J/m². Steel: 10^5 J/m².

The above modification eliminates the discrepancy between the theory and the experiments, but is bothersome in two respects. First, the Griffith theory was developed for a small crack in a large plate. How about other configurations of crack? Second, what do we really mean by the phrase "energy needed to advance a crack by a unit area"? We need an operational definition of the fracture energy, a definition that will enable theoretical calculation and experimental measurement.

Elastic energy. Let us first consider a pre-cracked body of an arbitrary shape. Imagine many copies of the body, identical in all respect except that the sizes of the pre-cracks are different for different bodies. The body is purely elastic: no bond breaking or plastic deformation occurs. The body is loaded, say, by a force *P*. The elastic energy stored in the body *U* is a function of the displacement Δ of the weight and the area *A* of the crack, namely,

$$U = U(\Delta, A).$$

This function can be determined by solving boundary-value problems within the theory of elasticity.

Alternatively, the function $U(\Delta, A)$ can be determined by experimental measurement. For each copy of the body, we make sure that the crack is

stationary as we load the body. Consequently, the work done by the weight is fully stored as the elastic energy in the body, $Pd\Delta = dU$. We write

$$P = \frac{\partial U(\Delta, A)}{\partial \Delta}.$$

By measuring the force P as a function of Δ and A, we can integrate and obtain the function $U(\Delta, A)$.

We can even consider elastic materials of nonlinear stress-strain behavior, such as elastomers.
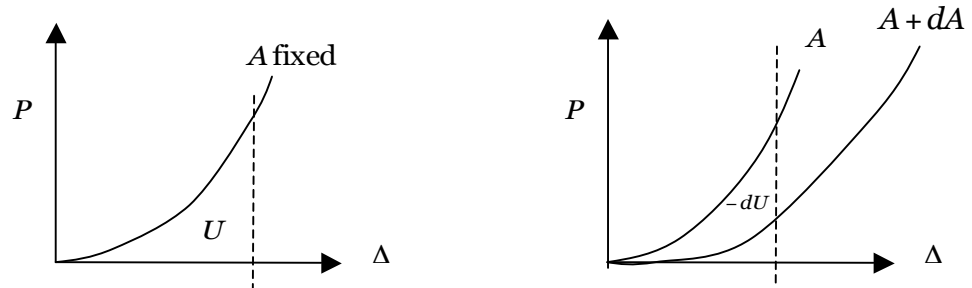
Energy Release Rate. Now consider two copies of the body: one copy has a crack of area A, and the other copy has a crack of area $A + dA$. The copy with the larger crack is more compliant—that is the load-displacement curve of the body with the larger crack is below the load-displacement curve of the body with the smaller crack. Consequently, at the same displacement, the body with a larger crack has lower elastic energy:

$$U(\Delta, A + dA) < U(\Delta, A).$$

Define energy release rate, G, as the reduction of the elastic energy associated with the crack increasing per unit area:

$$G = -\frac{U(\Delta, A + dA) - U(\Delta, A)}{dA}.$$

The displacement Δ is held fixed as the area of the crack changes. Write the above definition using the notation of calculus:

$$G = -\frac{\partial U(\Delta, A)}{\partial A}.$$

The partial derivative signifies that the displacement Δ is held fixed when the area of the crack A varies.

Once we know the function $U(\Delta, A)$, the above definition gives the energy release rate G. Thus, G is purely an elastic quantity, and you need to know nothing about the process of fracture to obtain G.

When both the displacement of the applied force and the area of the crack vary, the elastic energy of the body varies according to

$$dU = Pd\Delta - GdA.$$

Just as P is the thermodynamic force conjugate to the displacement Δ, the energy release rate G is the thermodynamic force conjugate to the area A.

Fracture energy. Consider a pre-cracked body loaded by a weight P. Under the small-scale yielding condition, we can still obtain the function $U(\Delta, A)$ as if the entire body were purely elastic, either by solving a boundary-value problem with the theory of elasticity, or by the load-displacement curves determined experimentally with bodies containing cracks of different sizes.

When the weight drops by distance $d\Delta$, the weight does work $Pd\Delta$. Under the small-scale yielding condition, much of the work done by the weight is stored in the body as elastic energy, and only a small fraction of the work done by the weight goes to inelastic processes such as breaking atomic bonds and plastic deformation. We will use this small fraction to define the fracture energy. That is, the fracture energy Γ is defined as an excess, according to

$$Pd\Delta = dU + \Gamma dA.$$

This definition of the fracture energy is independent of microscopic processes, be they bond breaking or plasticity.

Fracture criterion. Now compare the two definitions: the energy release rate G, and the fracture energy Γ. The crack will grow if the energy release rate equals the fracture energy:

$$G = \Gamma.$$

The energy release rate is the driving force for the extension of the crack, and represents an applied load. The fracture energy is the resistance to the extension of the crack, and represents a material property. The relation between G and Γ is analogous to the relation between stress and strength.

The above discussions complete the modifications of the Griffith theory to deal with
1. cracked bodies of any configuration, and
2. materials capable of plastic deformation.

In what follows we collect a few useful mathematical refinements. These refinements often confuse students, but contain no new information.

Potential energy. View the body and the weight together as a system, and lump their energy together:

$$\Pi = U - P\Delta.$$

This quantity is called the potential energy in mechanics, and is called the Gibbs free energy in thermodynamics. This definition, in combination with $dU = Pd\Delta - GdA$, gives
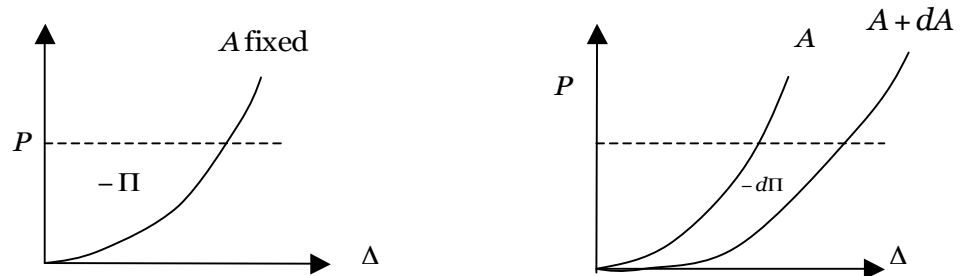
$$d\Pi = -\Delta dP - GdA.$$

Now the potential energy is a function of the load and the crack area,

$$\Pi = \Pi(P, A).$$

The displacement Δ and the energy release rate G are the differential coefficients, namely,

$$\Delta = -\frac{\partial \Pi(P,A)}{\partial P},$$

$$G = -\frac{\partial \Pi(P,A)}{\partial A}.$$

Linear elasticity. When the body is linearly elastic, the applied force P is linear in the displacement Δ. Consequently, the elastic energy is

$$U = P\Delta/2,$$

and the potential energy is

$$\Pi = -U.$$

We can write the energy release rate as

$$G = +\frac{\partial U(P,A)}{\partial A}.$$

The partial derivative signifies that the load P is held fixed when the crack area A varies. The opposite signs in the two expressions for the energy release rate reflect a simple physical fact. When the area of the crack is larger, the body is more compliant, so that the body stores less elastic energy at a fixed displacement, but stores more elastic energy at a fixed load.
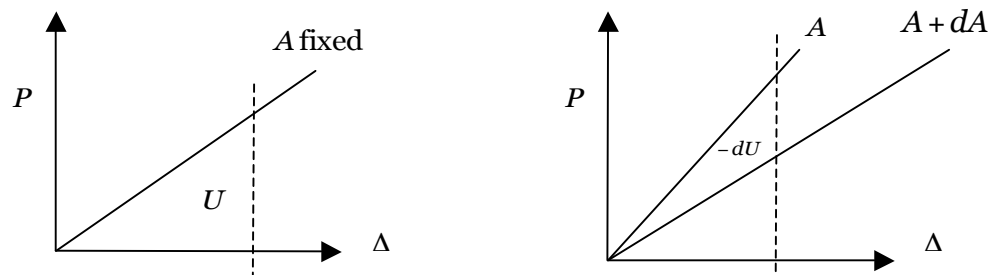
Compliance of a linearly elastic body containing a crack. For a linearly elastic body, the displacement is linear in the load. Write

$$\Delta = CP,$$

where C is the compliance of the body. For a linearly elastic body containing a crack, the compliance is independent of the load, but is a function of the area of the crack, namely,

$$C = C(A).$$

This function can be determined experimentally or calculated by solving boundary-value problems. As we said before, the compliance is an increasing function of the area of the crack.

Using the compliance, we can write the energy release rate as

$$G = \frac{P^2}{2} \frac{dC(A)}{dA}.$$
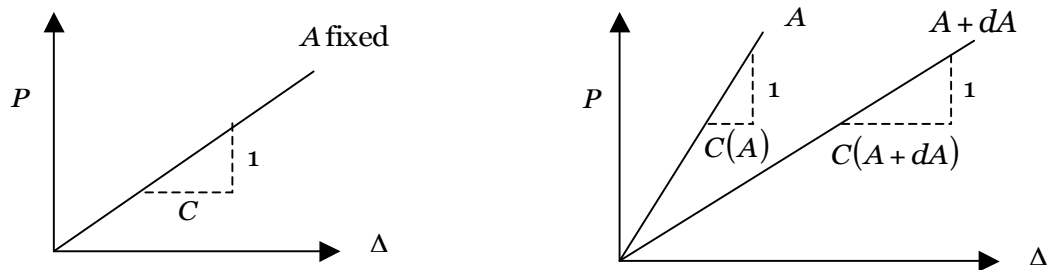
Recover the Griffith result.   The energy release rate can be determined by solving a boundary-value problem in linear elasticity. Consider the Griffith crack (i.e., a crack of length $2a$, in an infinite plate of unit thickness, subject to a remote stress σ). The boundary-value problem was solved by Inglis (1913). The crack opens to the shape of an ellipse, and the opening displacement is

$$\delta(x) = \frac{4\sigma}{E} \sqrt{a^2 - x^2} .$$

Under the stress-prescribed condition, the cracked body stores more elastic energy than the uncracked reference body. The difference in the elastic energy between the two bodies is

$$U = \frac{1}{2} \int \sigma \delta(x) dx$$

The integral extends over the length of the crack.

Integrating, we obtain that

$$U = \pi \frac{\sigma^2 a^2}{E} .$$

The form of this equation can be obtained by elementary considerations. The exact solution to the boundary-value problem gives the coefficient π. You will be asked to derive this equation from the Inglis solution.

The area of the crack is $A = 2a \cdot 1$. The energy release rate is defined by

$$G = +\frac{\partial U(\sigma, A)}{\partial A} \text{ (for fixed load)},$$

giving

$$G = \pi \frac{\sigma^2 a}{E}.$$

The crack grows when $G = \Gamma$, namely,

$$\Gamma = \pi \frac{\sigma^2 a}{E}.$$

This equation recovers the Griffith condition for fracture.

Cracked body of other configurations. For cracked bodies of other configurations, the energy release rate is often written in the same form, but with a different coefficient:

$$G = \beta \frac{\sigma^2 a}{E}.$$

The coefficient β is dimensionless, and depends on the configuration of the cracked body.

For example, consider a crack of length $2a$ in an elastic strip of width $2b$, subject to a remote tensile stress σ. This boundary-value problem cannot be solved analytically. Numerical solutions are summarized in the Tada handbook. The dimensionless coefficient β is a function of a/b. A formula fits the numerical solutions is

$$\beta = \pi \frac{\left(1 - 0.5(a/b) + 0.326(a/b)^2\right)^2}{1 - (a/b)}.$$

You can comfort yourself by checking the trend and the limiting cases. The energy release rate increases with the ratio a/b. When $a/b \rightarrow 0$, the length of the crack is much smaller than the width of the strip, and the above solution recovers the result for the Griffith crack, $\beta = \pi$. When $a/b \rightarrow 1$, the small ligament carries huge stresses, so that $\beta \rightarrow \infty$.

Ways to determine energy release rate G. The energy release rate is a quantity defined within the theory of elasticity. The energy release rate is specific to the configuration of a body containing a crack, and can be determined by the following methods.

- Look it up in handbooks. Elasticity solutions to cracked bodies of many configurations can be found in handbooks, e.g., H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook,* Del Research, St. Louis, MO., 1995.
- Determine it experimentally. This method is particularly easy for linearly elastic body. For a linearly elastic body containing a crack of a fixed area, A, the displacement Δ is linear in the applied force P, namely, $\Delta = CP$. The

compliance C can be measured experimentally. Use several bodies, which are identical except for the areas of the cracks. Measure the compliance of each body, and obtain the function $C(A)$. The energy release rate is given by

$$G = \frac{P^2}{2} \frac{dC(A)}{dA}.$$

• Determine it by solving the elasticity boundary-value problem. For cracked bodies of some configurations, the boundary-value problems can be solved analytically.   For most configurations, the boundary-value problems are solved numerically by using finite element programs.

    Historically, the analytical method came first, beginning with Griffith's (1921) use of the solution obtained by Inglis (1913), and followed by Obreimoff's (1930) analysis of a splitting layer. The method of functions of a complex variable was used to great effect by Muskhelishvili and others.

    The method of using the experimentally measured compliance to determine energy release rate was probably introduced by Irwin (~1950). The method is still occasionally used today.

    The method of choice today is the finite element method.

    Ways to determine fracture energy Γ. Fracture energy is a material property. It can be determined in several ways.
• Look it up in a material data sheet. Representative values: Glass: 10 J/m². Ceramics: 50 J/m². Polymers: 10^3 J/m². Aluminum: 10^4 J/m². Steel: 10^5 J/m². Warning: The fracture energy is sensitive to the microstructure of materials; heat treatment of a steel can change the fracture energy by orders of magnitude.
• Measure it experimentally by doing a fracture test. Of course, the values on the data sheet have been determined by experimental measurement.
• Compute it by a numerical simulation of the fracture process. This is an emerging field. Exciting but immature. Not a standard engineering practice yet.

    Division of labor. The qualitative picture of the fracture may be well-understood, across disparate scales of length and time, from the distortion of electron clouds, to the jiggling of atoms, to the motion of dislocations, to the extension of a crack, to the load-carrying capacity of a structure. This statement by itself, however, is of limited value: it offers little help to the engineer trying to prevent fracture of a structure.   Hypes of multiscale computation aside, no reliable method exists today to predict fracture by computation alone.

    A pragmatic approach is to divide the labor between numerical computation and experimental measurement. Some quantities are easier to compute, and others easier to measure. A combination of computation and measurement solves problems economically.
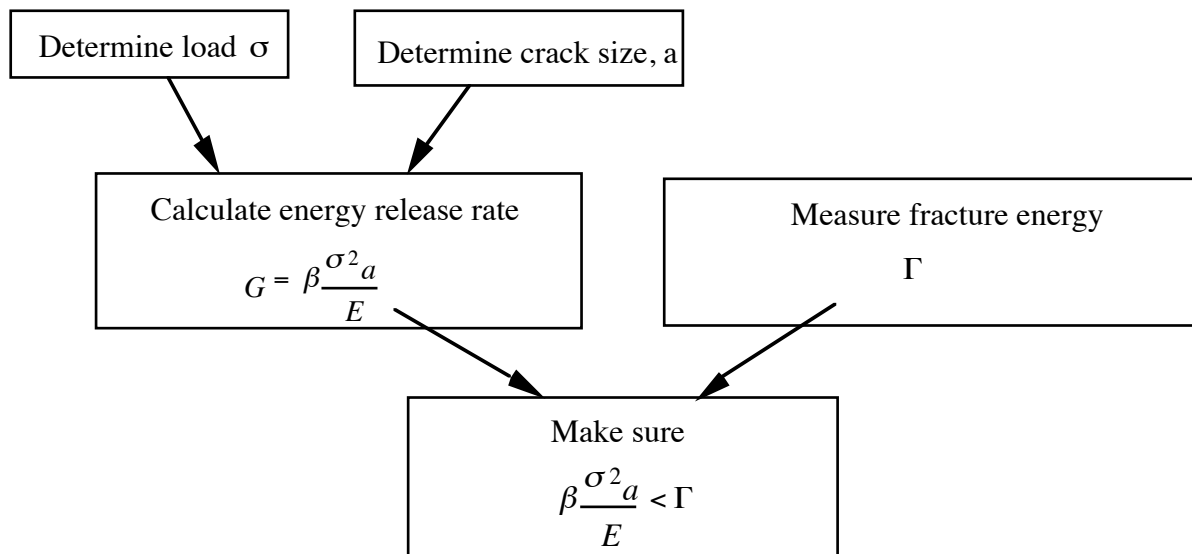
    Of course, what is easy to do changes when circumstances change. As new tools and applications emerge, it behooves us to renegotiate a more economical

division of labor. The history of fracture mechanics offers excellent lessons on such divisions and renegotiations. The evolution of fracture mechanics is better appreciated historically and economically.

Design based on fracture mechanics. Compare design based on fracture mechanics with design based on the linear elastic theory.

In the linear elastic theory of strength, the stress is the loading parameter, and the strength is the material property. We determine the stress in the body by solving a boundary-value problem, and measure the strength by doing an experiment. The structure is safe if the stress is below the strength.

In fracture mechanics, the energy release rate is the loading parameter, and the fracture energy is the material property. We determine the energy release rate of a crack in the body by solving a boundary-value problem, and measure the fracture energy by doing an experiment. The structure is safe if the energy release rate is below the fracture energy.

| Determine load σ | Determine crack size, a |

Calculate energy release rate

$$G = \beta \frac{\sigma^2 a}{E}$$

Measure fracture energy

$$\Gamma$$

Make sure

$$\beta \frac{\sigma^2 a}{E} < \Gamma$$

Historical Notes

The literature sometimes traces the modified Griffith theory to the following two papers, but these papers are not explicit enough about details for us to learn basics of the fracture mechanics. They are of historical interest.

E. Orowan, Fracture and strength of solids. Reports on Progress in Physics 12, 185-232 (1948).

G.R. Irwin, Fracture Dynamics. In Fracturing of Metals, pp. 147-166. ASM Symposium (Trans. ASM 40A), Cleveland (1948).

F.R.N. Nabarro and A.S. Argon, Egon Orowan (1901-1989). Biographical Memoirs of Fellows of the Royal Society 41, 316-340 (1995).

A.A. Wells, George Rankin Irwin. Biographical Memoirs of Fellows of the Royal Society 46, 270-283 (2000).