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Exact solutions for the free in-plane vibrations of rectangular plates

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All classical boundary conditions including two distinct types of simple support boundary conditions are formulated by using the Rayleigh quotient variational principle for rectangular plates undergoing in-plane free vibrations. The direct separation of variables is employed to obtain the exact solutions for all possible cases. It is shown that the exact solutions of natural frequencies and mode shapes can be obtained when at least two opposite plate edges have either type of the simply-supported conditions, and some of the exact solutions were not available before. The present results agree well with FEM results, which show that the present solutions are correct and the direct separation of variables is practical. The exact solutions can be taken as the benchmarks for the validation of approximate methods.

1. Introduction

There is no doubt that transverse vibrations of plates are of great practical importance, since their natural frequencies are prone to most of the external excitation, and as such there is an extensive literature relevant to the free transverse vibrations of rectangular plates. On the contrary, only a few studies are dedicated to the free in-plane vibrations (FIV) of plates over the years, since the natural frequencies involved are much higher and beyond the level of available excitations. However, it has been found that the in-plane vibrations can be excited in the structures such as the hulls of ocean-going ships and the shells of flight vehicles, etc. Hence it is also important to study the in-plane vibrations of plates.

A significant contribution to this subject was made by Bardell et al. [1], who calculated the in-plane vibrational frequencies using the Rayleigh–Ritz method, and provided a valuable review of the related literature available up to that time, including the pioneering work of Lord Rayleigh [2] dealing with what was referred to as 'simply-supported' plates.

Gorman [3] introduced the superposition method as a means to obtain the analytical-type FIV solutions of rectangular plates with completely free boundaries, fully clamped boundaries [4] and elastic supports normal to the boundaries [5]. Du et al. [6] also analyzed the FIV of rectangular plates with elastically restrained edges by using an improved Fourier series method, in which the in-plane displacements are expressed as the superposition of a double Fourier cosine series and four supplementary functions. Additionally, Seok et al. [7] performed an FIV analysis of a cantilevered rectangular plate by using a variational approximation procedure, wherein the differential equations and traction-free conditions on two opposite edges are satisfied exactly and the remaining conditions are satisfied variationally. Singh et al. [8] investigated the FIV of isotropic non-rectangular plates according to the variational method, wherein the displacement fields are represented by much higher order polynomials than the ones used for the geometric representation. And Woodcock et al. [9] studied the effects of the ply orientation on in-plane vibrations based on the Rayleigh–Ritz formulation in conjunction with Hamilton principle.

It is noteworthy that there have been some exact solutions for the FIV of plates. Park [10] derived the exact frequency equations for the FIV of the clamped circular plate by using the separation of the variables. Gorman [11] obtained the exact solutions for the FIV of rectangular plates with two opposite edges simply supported, the other opposite edges being both clamped or both free. In Gorman's elegant work, only one quarter of the rectangular plate was analyzed, and it was shown that by this approach, the interpretation of the computed mode shapes with mode family separation becomes a much more manageable task, the probability of missing an eigenvalue can be greatly reduced, and the problem of repeated eigenvalues can be avoided.

In present study, the exact solutions for the FIV of rectangular plates are attempted. There are several apparent differences between the present work and Gorman's [11] as follows.

(1) Rayleigh quotient variational principle is employed to derive the mathematical representations of all possible boundary

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conditions including clamped condition, free condition and two distinct types of simple support boundary conditions that are denoted by the symbols SS1 and SS2 [11].

(2) The direct separation of variables is used to solve the governing equations. By this approach, the exact solutions can be obtained readily. The solution procedure shows that the exact solutions are available only when at least two opposite plate edges are simply supported.

(3) The entire rectangular plate is analyzed directly, and there are no problems such as the interpretation of the computed mode shapes, the probability of missing an eigenvalue, and the repeated eigenvalues.

(4) All possible exact solutions are obtained, including the solutions for the cases SS1–C–SS1–F and SS1–SS2–SS1–F, etc., which were not available before.

The paper is organized as follows. In Section 2, the formulations of all boundary conditions are given by using the Rayleigh quotient variational principle; then in Section 3 the exact solutions are obtained through the separation of variables; finally in Section 4 the numerical experiments are conducted and the results are compared with those by FEM.

2. Differential equations and boundary conditions

Consider the harmonic normal vibrations of a rectangular plate as shown in Fig. 1. The maximum strain energy $\Pi_{\max}$ and the maximum kinetic energy $T_{\max}$ can be represented, respectively, as

$$\Pi_{\max} = \frac{1}{2}\iint_A \boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon}\,\mathrm{d}x\,\mathrm{d}y$$
$$T_{\max} = \omega^2 T_0, \quad T_0 = \frac{1}{2}\iint_A \rho(u^2+v^2)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{E}$ are the strain vector and elastic matrix, respectively. By means of Rayleigh quotient variational principle $\delta\Pi_{\max} = \omega^2\delta T_0$, one can obtain

$$\iint_A \left[\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau}{\partial y} + \rho\omega^2 u\right)\delta u + \left(\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau}{\partial x} + \rho\omega^2 v\right)\delta v\right]\mathrm{d}x\,\mathrm{d}y$$
$$- \int_\Gamma [(\sigma_x l + \tau m)\delta u + (\sigma_y m + \tau l)\delta v]\,\mathrm{d}s = 0 \tag{2}$$

where $l = \cos(n,x) = \cos\theta$, and $m = \cos(n,y) = \sin\theta$, n is the outer normal direction of the boundary. Due to the necessary and sufficient conditions of Rayleigh quotient principle, the governing equations and the homogeneous boundary conditions can be obtained. The latter are given in Table 1, and the governing

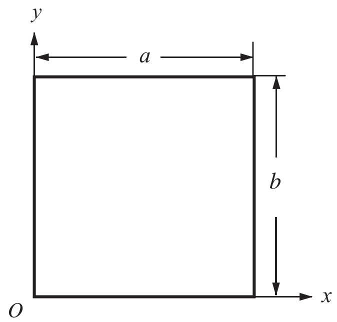

Fig. 1. Plate and coordinates.

Table 1
The classic boundary conditions of rectangular plate.

B.C.	$x = 0$ or a	$y = 0$ or b
Clamped (C)	$u = v = 0$	$u = v = 0$
Free (F)	$\sigma_x = 0,\ \tau = 0$	$\sigma_y = 0,\ \tau = 0$
Simply supported	$u = 0,\ \tau = 0$ (SS2)	$u = 0,\ \sigma_y = 0$ (SS1)
	$v = 0,\ \sigma_x = 0$ (SS1)	$v = 0,\ \tau = 0$ (SS2)

equations are

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau}{\partial y} + \rho\omega^2 u = 0$$
$$\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau}{\partial x} + \rho\omega^2 v = 0 \tag{3}$$

or

$$\frac{\partial^2 u}{\partial x^2} + \upsilon_1\frac{\partial^2 u}{\partial y^2} + \upsilon_2\frac{\partial^2 v}{\partial x\partial y} + \upsilon_1\left(\frac{\omega}{c}\right)^2 u = 0$$
$$\frac{\partial^2 v}{\partial y^2} + \upsilon_1\frac{\partial^2 v}{\partial x^2} + \upsilon_2\frac{\partial^2 u}{\partial x\partial y} + \upsilon_1\left(\frac{\omega}{c}\right)^2 v = 0 \tag{4}$$

where u and v are the in-plane displacements in x and y directions, respectively, $c = \sqrt{G/\rho}$ is the shear wave velocity, and

$$\upsilon_1 = \frac{1-\upsilon}{2}, \quad \upsilon_2 = \frac{1+\upsilon}{2} \tag{5}$$

By solving Eq. (4), one can obtain the modes and natural frequencies. Further, it is not difficult to solve Eq. (4) when the four plate edges are simply supported through the inverse method.

3. The exact eigensolutions

For the free transverse vibrations of rectangular thin plate, the authors have obtained the exact results for the cases SSCC, SCCC and CCCC by means of the direct separation of variables [12]. The direct separation of variables is employed again to obtain the exact solutions for the FIV of plates. The separation-of-variable solution of Eq. (4) can be written as

$$u(x,y) = Ae^{\mu x}e^{\lambda y}$$
$$v(x,y) = Be^{\mu x}e^{\lambda y} \tag{6}$$

Substitution of Eq. (6) into Eq. (4) leads to

$$\begin{bmatrix} \mu^2 + \upsilon_1\lambda^2 + \upsilon_1\left(\frac{\omega}{c}\right) & \lambda\mu\upsilon_2 \\ \lambda\mu\upsilon_2 & \lambda^2 + \upsilon_1\mu^2 + \upsilon_1\left(\frac{\omega}{c}\right)^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

The existence of nontrivial solutions of A and B requires that

$$(\lambda^2+\mu^2)^2 + \frac{3-\upsilon}{2}\left(\frac{\omega}{c}\right)^2(\lambda^2+\mu^2) + \upsilon_1\left(\frac{\omega}{c}\right)^4 = 0 \tag{8}$$

or

$$\left[\lambda^2 + \mu^2 + \left(\frac{\omega}{c}\right)^2\right]\left[\lambda^2 + \mu^2 + \upsilon_1\left(\frac{\omega}{c}\right)^2\right] = 0 \tag{9}$$

From Eq. (9), one can have

$$\mu_{1,3} = \pm \mathrm{i}\Omega, \quad \mu_{2,4} = \pm \mathrm{i}\Lambda \tag{10}$$

$$\lambda_{1,3} = \pm \mathrm{i}T, \quad \lambda_{2,4} = \pm \mathrm{i}Z \tag{11}$$

where

$$\Omega = \sqrt{\lambda^2 + \left(\frac{\omega}{c}\right)^2}, \quad \Lambda = \sqrt{\lambda^2 + \upsilon_1\left(\frac{\omega}{c}\right)^2} \tag{12}$$

$$T = \sqrt{\mu^2 + \left(\frac{\omega}{c}\right)^2}, \quad Z = \sqrt{\mu^2 + \upsilon_1 \left(\frac{\omega}{c}\right)^2} \tag{13}$$

Therefore, the solutions of Eq. (4) can be written as

$$\begin{aligned} u(x,y) &= \phi_1(x)\phi_2(y) \\ v(x,y) &= \psi_1(x)\psi_2(y) \end{aligned} \tag{14}$$

where

$$\begin{aligned} \phi_1(x) &= A_1 \cos \Omega x + A_2 \sin \Omega x + A_3 \cos \Lambda x + A_4 \sin \Lambda x \\ \psi_1(x) &= C_1 \cos \Omega x + C_2 \sin \Omega x + C_3 \cos \Lambda x + C_4 \sin \Lambda x \\ \phi_2(y) &= B_1 \cos Ty + B_2 \sin Ty + B_3 \cos Zy + B_4 \sin Zy \\ \psi_2(y) &= D_1 \cos Ty + D_2 \sin Ty + D_3 \cos Zy + D_4 \sin Zy \end{aligned} \tag{15}$$

By substituting Eq. (14) into Eq. (4), one can find that the solutions are meaningless except for the following forms:

$$\begin{matrix} u(x,y) = \phi_1(x)e^{\lambda y} \\ v(x,y) = \psi_1(x)e^{\lambda y} \end{matrix} \quad \text{or} \quad \begin{matrix} u(x,y) = \phi_2(y)e^{\mu x} \\ v(x,y) = \psi_2(y)e^{\mu x} \end{matrix} \tag{16a,b}$$

It is shown below that Eqs. (16) can be satisfied only when at least two opposite plate edges are simply supported. The relations of $\phi_1(x)$ and $\psi_1(x)$ should be determined prior to the derivation of the exact solutions, which can be obtained by inserting Eq. (16a) into Eq. (4) as

$$\left.\begin{matrix} C_2 = \frac{\Omega}{\lambda} A_1 = k_1 A_1 \\ C_1 = -\frac{\Omega}{\lambda} A_2 = -k_1 A_2 \end{matrix}\right\} \quad \text{where} \quad \begin{matrix} k_1 = \frac{\Omega}{\lambda} \\ \lambda^2 = \Omega^2 - \left(\frac{\omega}{c}\right)^2 \end{matrix} \tag{17}$$

$$\left.\begin{matrix} C_4 = \frac{\lambda}{\Lambda} A_3 = k_2 A_3 \\ C_3 = -\frac{\lambda}{\Lambda} A_4 = -k_2 A_4 \end{matrix}\right\} \quad \text{where} \quad \begin{matrix} k_2 = \frac{\lambda}{\Lambda} \\ \lambda^2 = \Lambda^2 - \upsilon_1 \left(\frac{\omega}{c}\right)^2 \end{matrix} \tag{18}$$

Then the $\phi_1(x)$ and $\psi_1(x)$ in Eq. (15) can be rewritten as

$$\begin{aligned} \phi_1(x) &= A_1 \cos \Omega x + A_2 \sin \Omega x + A_3 \cos \Lambda x + A_4 \sin \Lambda x \\ \psi_1(x) &= -A_2 k_1 \cos \Omega x + A_1 k_1 \sin \Omega x - A_4 k_2 \cos \Lambda x + A_3 k_2 \sin \Lambda x \end{aligned} \tag{19}$$

Similarly, one can obtain the relations of $\phi_2(x)$ and $\psi_2(x)$. It follows from Eq. (19) that, if $\phi_1(x)$ is a sine function, $\psi_1(x)$ must be a cosine function, and vice versa. Therefore, Eqs. (16) can only be satisfied when at least two opposite plate edges are simply supported. This paper assumes the opposite plate edges $x = 0$ and a to be simply supported, i.e., Eq. (16b) will be used below.

3.1. The eigensolutions for the simply-supported edges x = 0 and a

There are four combinations of the simply-supported conditions for the edges $x = 0$ and a, as given in Table 2. The case SS2–SS2 is solved here for the eigenfunctions and eigenvalue equations. The SS2 conditions are

$$u = 0, \tau = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\frac{\partial v}{\partial x} = 0 \tag{20}$$

or

$$\begin{aligned} \phi_1(0) &= \phi_1(a) = 0 \\ \psi_1'(0) &= \psi_1'(a) = 0 \end{aligned} \tag{21}$$

Table 2
The four combination of SS1 and SS2 for the edges $x = 0$ and a and eigensolutions.

	Eigenvalue equations	Eigenfunctions
SS2–SS2	$\sin \Omega a = 0$	$\phi_1(x) = \sin \Omega x$, $\psi_1(x) = \cos \Omega x$
SS1–SS1	$\sin \Omega a = 0$	$\phi_1(x) = \cos \Omega x$, $\psi_1(x) = \sin \Omega x$
SS2–SS1	$\cos \Omega a = 0$	$\phi_1(x) = \sin \Omega x$, $\psi_1(x) = \cos \Omega x$
SS1–SS2	$\cos \Omega a = 0$	$\phi_1(x) = \cos \Omega x$, $\psi_1(x) = \sin \Omega x$

Substituting Eq. (19) into Eq. (21), one can obtain $A_1 = A_3 = 0$, and

$$\begin{bmatrix} \sin \Omega a & \sin \Lambda a \\ k_1 \Omega \sin \Omega a & k_2 \Lambda \sin \Lambda a \end{bmatrix} \begin{bmatrix} A_2 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

Then the eigenvalue equation is

$$(k_2 \Lambda - k_1 \Omega) \sin \Omega a \sin \Lambda a = 0 \tag{23}$$

Since $\sin \Omega a = 0$ and $\sin \Lambda a = 0$ are equivalent, here only $\sin \Omega a = 0$ is considered, and $A_4 = 0$ from Eq. (22). Thus the normal eigenfunctions in Eq. (19) can be obtained as

$$\begin{aligned} \phi_1(x) &= \sin \Omega x \\ \psi_1(x) &= -k_1 \cos \Omega x \end{aligned} \tag{24}$$

As the eigenfunctions $\phi_1(x)$ and $\psi_1(x)$ are the factors of modes, which are used below to derive the eigenfunctions $\phi_2(y)$ and $\psi_2(y)$, Eq. (24) can be rewritten as

$$\begin{aligned} \phi_1(x) &= \sin \Omega x \\ \psi_1(x) &= \cos \Omega x \end{aligned} \tag{25}$$

All possible eigenfunctions and eigenvalue equations for the simply-supported edges $x = 0$ and a can be solved in the same way. These eigensolutions are listed in Table 2 from which one can see that, although there are four combinations of SS1 and SS2, there are only two distinct types of eigenfunctions and eigenvalue equations.

3.2. The eigenfunctions for arbitrary opposite edges y = 0 and b

Based on $\phi_1(x)$ and $\psi_1(x)$ obtained by using the simply-supported conditions of the edges $x = 0$ and a, the eigenfunctions $\phi_2(y)$, $\psi_2(y)$ and corresponding eigenvalue equations can be derived by using the boundary conditions of the other opposite edges $y = 0$ and b. Assume that the in-plane natural mode of the plate are given in separation of variables form as

$$\begin{aligned} u(x,y) &= \phi_2(y)\phi_1(x) \\ v(x,y) &= \psi_2(y)\psi_1(x) \end{aligned} \tag{26}$$

where $\phi_1(x)$ and $\psi_1(x)$ are given in Table 2. Let $\mu = i\Omega$, and substitute it into Eq. (13) to obtain

$$\begin{aligned} T &= \sqrt{\left(\frac{\omega}{c}\right)^2 - \Omega^2} \\ Z &= \sqrt{\upsilon_1 \left(\frac{\omega}{c}\right)^2 - \Omega^2} \end{aligned} \tag{27}$$

which show that T and Z may be real or pure imaginary. And there are three cases as follows.

Case 1: $\upsilon_1(\omega/c)^2 \geqslant \Omega^2$. The eigenvalues T and Z are real from Eq. (27). Thus the eigenfunctions in Eq. (15) can be used directly,

Table 3
The coefficient relations of the functions $\phi_2(y)$ and $\psi_2(y)$.

	$\phi_1(x) = \sin \Omega x$, $\psi_1(x) = \cos \Omega x$	$\phi_1(x) = \cos \Omega x$, $\psi_1(x) = \sin \Omega x$
Case 1	$B_1 = -k_3 D_2, B_2 = k_3 D_1$ $B_3 = k_4 D_4, B_4 = -k_4 D_3$	$B_1 = k_3 D_2, B_2 = -k_3 D_1$ $B_3 = -k_4 D_4, B_4 = k_4 D_3$
Case 2	$B_1 = -k_3 D_2, B_2 = k_3 D_1$ $B_3 = -k_4 D_4, B_4 = -k_4 D_3$	$B_1 = k_3 D_2, B_2 = -k_3 D_1$ $B_3 = k_4 D_4, B_4 = k_4 D_3$
Case 3	$B_1 = -k_3 D_2, B_2 = -k_3 D_1$ $B_3 = -k_4 D_4, B_4 = -k_4 D_3$	$B_1 = k_3 D_2, B_2 = k_3 D_1$ $B_3 = k_4 D_4, B_4 = k_4 D_3$

given by

$$\phi_2(y) = B_1 \cos Ty + B_2 \sin Ty + B_3 \cos Zy + B_4 \sin Zy$$
$$\psi_2(y) = D_1 \cos Ty + D_2 \sin Ty + D_3 \cos Zy + D_4 \sin Zy \quad (28)$$

Case 2: $(\omega/c)^2 \geqslant \Omega^2 > \upsilon_1(\omega/c)^2$. The eigenvalue T is real, and Z is pure imaginary, from Eq. (27). For simplicity, the eigenvalue Z is changed from imaginary to real, as

$$T = \sqrt{\left(\frac{\omega}{c}\right)^2 - \Omega^2}$$
$$Z = \sqrt{-\upsilon_1\left(\frac{\omega}{c}\right)^2 + \Omega^2} \quad (29)$$

and the eigenfunctions in Eq. (15) should be modified accordingly as

$$\phi_2(y) = B_1 \cos Ty + B_2 \sin Ty + B_3 \cosh Zy + B_4 \sinh Zy$$
$$\psi_2(y) = D_1 \cos Ty + D_2 \sin Ty + D_3 \cosh Zy + D_4 \sinh Zy \quad (30)$$

Case 3: $(\omega/c)^2 < \Omega^2$. Both T and Z are pure imaginary values, and they can also be changed to be real as

$$T = \sqrt{\Omega^2 - \left(\frac{\omega}{c}\right)^2}$$
$$Z = \sqrt{\Omega^2 - \upsilon_1\left(\frac{\omega}{c}\right)^2} \quad (31)$$

The corresponding eigenfunctions in Eq. (15) are rewritten as

$$\phi_2(y) = B_1 \cosh Ty + B_2 \sinh Ty + B_3 \cosh Zy + B_4 \sinh Zy$$
$$\psi_2(y) = D_1 \cosh Ty + D_2 \sinh Ty + D_3 \cosh Zy + D_4 \sinh Zy \quad (32)$$

To substitute Eqs. (28), (30) and (32) into Eq. (4), the coefficient relations of $\phi_2(y)$ and $\psi_2(y)$ can be determined for the above three cases that are given in Table 3, in which $k_3 = T/\Omega$, $k_4 = \Omega/Z$.

Table 4
The eigensolutions for simply-supported edges $y = 0$ and b.

	Eigenvalue equations	Eigenfunctions
SS2–SS2	$\sin Tb = 0$	$\phi_2(y) = -k_3 \cos Ty,\ \psi_2(y) = \sin Ty$
	$\sin Zb = 0$	$\phi_2(y) = k_4 \cos Zy,\ \psi_2(y) = \sin Zy$
SS1–SS1	$\sin Tb = 0$	$\phi_2(y) = k_3 \sin Ty,\ \psi_2(y) = \cos Ty$
	$\sin Zb = 0$	$\phi_2(y) = -k_4 \sin Zy,\ \psi_2(y) = \cos Zy$
SS2–SS1	$\cos Tb = 0$	$\phi_2(y) = -k_3 \cos Ty,\ \psi_2(y) = \sin Ty$
	$\cos Zb = 0$	$\phi_2(y) = k_4 \cos Zy,\ \psi_2(y) = \sin Zy$
SS1–SS2	$\cos Tb = 0$	$\phi_2(y) = k_3 \sin Ty,\ \psi_2(y) = \cos Ty$
	$\cos Zb = 0$	$\phi_2(y) = -k_4 \sin Zy,\ \psi_2(y) = \cos Zy$

Table 5
The four combinations of eigenvalue equations.

1	2	3	4
$\sin \Omega a = 0$ $\sin Tb \sin Zb = 0$	$\sin \Omega a = 0$ $\cos Tb \cos Zb = 0$	$\cos \Omega a = 0$ $\sin Tb \sin Zb = 0$	$\cos \Omega a = 0$ $\cos Tb \cos Zb = 0$

4. The eigenvalue equations and numerical results

The eigenvalue Ω and the corresponding eigenfunctions $\phi_1(x)$ and $\psi_1(x)$ for the simply-supported edges $x = 0$ and a are given in Table 2. And the eigenfunctions for another two edges $y = 0$ and b have also been derived for all three cases, see Eqs. (28), (30) and (32) and Table 3. The remaining problems are to derive the eigenvalue equations corresponding to the opposite edges $y = 0$ and b according to the relevant boundary conditions, and then to solve for the natural frequencies by using Eqs. (27), (29) and (31).

Consider a rectangular plate with in-plane dimension $a \times b = 1$ m $\times$ 1.2 m, the volume density $\rho = 2800\,\text{kg/m}^3$, Young's modulus $E = 72 \times 10^9$ Pa, and Poisson's ratio $\upsilon = 0.3$. The calculated frequencies are given in dimensionless frequency parameter $\beta = \omega a/\pi c$, and the frequencies by FEM are denoted by β^*. Except the results for the simplest cases with four simply-supported edges, all frequencies and mode shapes are compared with those by FEM, which are obtained by MSC/NASTRAN with the mesh 100×120 and using membrane elements.

Table 6
The frequency parameter β and the values of m, n and i.

$x = 0, a$	SS1–SS1		SS2–SS2	
$y = 0, b$	SS1–SS1	SS2–SS2	SS2–SS2	SS1–SS1
1	0.8333 $(m,n) = (0,1)$	0.8333 $(m,n) = (0,1)$	1.3017 $(m,n) = (1,1)$	1.0000 $(m,n) = (1,0)$
2	1.0000 $(m,n) = (1,0)$	1.3017 $(m,n) = (1,1)$	1.4086 $(m,i) = (0,1)$	1.3017 $(m,n) = (1,1)$
3	1.3017 $(m,n) = (1,1)$	1.6667 $(m,n) = (0,2)$	1.6903 $(m,i) = (1,0)$	1.4086 $(m,i) = (0,1)$
4	1.6667 $(m,n) = (0,2)$	1.6903 $(m,i) = (1,0)$	1.9437 $(m,n) = (1,2)$	1.9437 $(m,n) = (1,2)$
5	1.9437 $(m,n) = (1,2)$	1.9437 $(m,n) = (1,2)$	2.1667 $(m,n) = (2,1)$	2.0000 $(m,n) = (2,0)$
6	2.0000 $(m,n) = (2,0)$	2.1667 $(m,n) = (2,1)$	2.2003 $(m,i) = (1,1)$	2.1667 $(m,n) = (2,1)$
7	2.1667 $(m,n) = (2,1)$	2.2003 $(m,i) = (1,1)$	2.6034 $(m,n) = (2,2)$	2.2003 $(m,i) = (1,1)$
8	2.2003 $(m,i) = (1,1)$	2.5000 $(m,n) = (0,3)$	2.6926 $(m,n) = (1,3)$	2.6034 $(m,n) = (2,2)$
9	2.5000 $(m,n) = (0,3)$	2.6034 $(m,n) = (2,2)$	2.8172 $(m,i) = (0,2)$	2.6926 $(m,n) = (1,3)$
10	2.6034 $(m,n) = (2,2)$	2.6926 $(m,n) = (1,3)$	3.1136 $(m,n) = (3,1)$	2.8172 $(m,i) = (0,2)$
	$T = 0$, yes $Z = 0$, no $\Omega = 0$, $Tb = n\pi$, yes $\Omega = 0$, $Zb = i\pi$, no	$T = 0$, no $Z = 0$, yes $\Omega = 0$, $Tb = n\pi$, yes $\Omega = 0$, $Zb = i\pi$, no	$T = 0$, no $Z = 0$, yes $\Omega = 0$, $Tb = n\pi$, no $\Omega = 0$, $Zb = i\pi$, yes	$T = 0$, yes $Z = 0$, no $\Omega = 0$, $Tb = n\pi$, no $\Omega = 0$, $Zb = i\pi$, yes

4.1. The edges $y=0$ and b are simply supported

For the simply-supported opposite edges $y=0$ and b, all possible eigensolutions can be obtained readily by using Eqs. (28), (30) and (32) and boundary conditions, which are given in Table 4. It follows from Tables 2 and 4 that there are many different combinations of SS1 and SS2, which is a significant difference from the free out-of-plane vibrations of plate when the four edges are simply supported.

For the rectangular plate with four simply-supported edges, the frequencies are available only for Cases 1 and 2, and the Case 2 is involved in Case 1. It is noteworthy that the Case 3 is available only for the plate with at least one free edge. Using the obtained eigenvalues Ω, T and Z from the eigenvalue equations, the frequencies can be calculated from Eq. (27) as

$$\omega_{mn} = c\sqrt{T_n^2 + \Omega_m^2}$$
$$\omega_{mi} = c\sqrt{\frac{Z_i^2 + \Omega_m^2}{\upsilon_1}} \tag{33}$$

and $\omega_{mn} \neq \omega_{mi}$ for any positive integers n and i. There are no rigid body modes, and hence the eigenvalues T and Ω cannot be zero

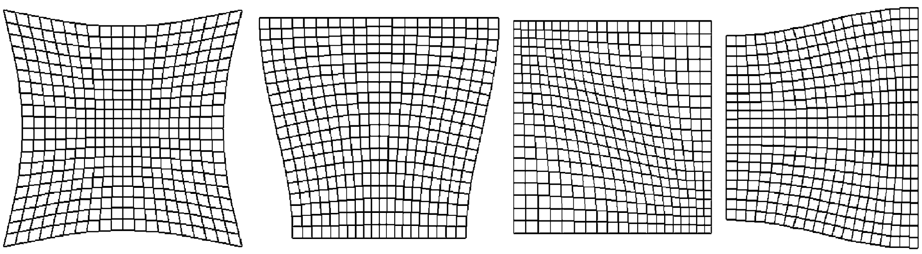

Fig. 2. Mode shapes of plate with four simply-supported edges, $\beta = 1.3017$. (a) SS1–SS1–SS1–SS1, (b) SS1–SS2–SS1–SS2, (c) SS2–SS2–SS2–SS2 and (d) SS2-SS1-SS2-SS1.

Table 7
The eigenvalue equations when the edges $y=0$ and b are arbitrary.

Case 1	
C–C $\dfrac{1-\cos Tb\cos Zb}{\sin Tb\sin Zb} = -\dfrac{1}{2}\left(\dfrac{k_3}{k_4}+\dfrac{k_4}{k_3}\right)$	F–F $\dfrac{1-\cos Tb\cos Zb}{\sin Tb\sin Zb} = \dfrac{1}{2}\left(\dfrac{\alpha_1}{\alpha_2}+\dfrac{\alpha_2}{\alpha_1}\right)$
SS2–C $k_3\tan Zb = -k_4\tan Tb$	SS1–C $k_3\tan Tb = -k_4\tan Zb$
SS2–F $\alpha_1\tan Tb = \alpha_2\tan Zb$	SS1–F $\alpha_2\tan Tb = \alpha_1\tan Zb$
C–F $\left(\alpha_1\dfrac{k_3}{k_4}-\alpha_2\right)\cos Tb\cos Zb + \left(\alpha_2\dfrac{k_3}{k_4}-\alpha_1\right)\sin Tb\sin Zb = \alpha_1\alpha_2 - \dfrac{k_3}{k_4}$	
Case 2	
C–C $\dfrac{1-\cos Tb\cosh Zb}{\sin Tb\sinh Zb} = \dfrac{k_3^2-k_4^2}{2k_3k_4}$	F–F $\dfrac{1-\cos Tb\cosh Zb}{\sin Tb\sinh Zb} = \dfrac{1}{2}\left(\dfrac{\alpha_3}{\alpha_1}-\dfrac{\alpha_1}{\alpha_3}\right)$
SS2–C $k_3\tanh Zb = k_4\tan Tb$	SS1–C $k_3\tan Tb = -k_4\tanh Zb$
SS2–F $\alpha_1\tan Tb = \alpha_3\tanh Zb$	SS1–F $\alpha_3\tan Tb = -\alpha_1\tanh Zb$
C–F $\left(\alpha_1\dfrac{k_3}{k_4}+\alpha_3\right)\cos Tb\cosh Zb + \left(\alpha_3\dfrac{k_3}{k_4}-\alpha_1\right)\sin Tb\sinh Zb = -\alpha_3\alpha_1 - \dfrac{k_3}{k_4}$	
Case 3	
F–F $\dfrac{1-\cosh Tb\cosh Zb}{\sinh Tb\sinh Zb} = -\dfrac{1}{2}\left(\dfrac{\alpha_3}{\alpha_4}+\dfrac{\alpha_4}{\alpha_3}\right)$	
SS2–F $\alpha_4\tanh Tb = \alpha_3\tanh Zb$	SS1–F $\alpha_3\tanh Tb = \alpha_4\tanh Zb$
C–F $\left(\alpha_4\dfrac{k_3}{k_4}+\alpha_3\right)\cosh Tb\cosh Zb - \left(\alpha_3\dfrac{k_3}{k_4}+\alpha_4\right)\sinh Tb\sinh Zb = -\alpha_3\alpha_4 - \dfrac{k_3}{k_4}$	

simultaneously, as also Z and Ω. There are four combinations, as shown in Table 5, for the eigenvalue equations in Tables 2 and 4. Here only the first combination (the first column of Table 5) is discussed for the analysis of frequency properties. For this combination, Eq. (33) can be written as

$$\omega_{mn} = c\sqrt{(n\pi/b)^2 + (m\pi/a)^2}$$
$$\omega_{mi} = c\sqrt{\frac{(i\pi/b)^2 + (m\pi/a)^2}{\upsilon_1}} \qquad (34)$$

It can be seen from Tables 2 and 4 that the same frequencies can be calculated from Eq. (34) for the four combinations SS2–SS2–SS2–SS2, SS2–SS1–SS2–SS1, SS1–SS2–SS1–SS2 and SS1–SS1–SS1–SS1. The first 10 frequencies are given in Table 6 from which it follows that, if Ω, T and Z are not equal to zero, there are four repeated frequencies; if one of Ω and T or one of Ω and Z equals zero, the boundary conditions should be checked, and there are two repeated frequencies. For the case SS1–SS1–SS1–SS1, T can be zero, but not Z; and if $\Omega = 0$, $Tb = n\pi$ is the root, but not $Zb = i\pi$. Similar analysis can be performed for the other three combinations.

The mode shapes are drawn in Fig. 2 for $\beta = 1.3017$, which corresponds to the four repeated frequency.

4.2. The edges $y = 0$ and b are arbitrary

According to the coefficient relations in Table 3 and Eqs. (28), (30) and (32), one can derive the eigenfunctions $\phi_2(y)$, $\psi_2(y)$ and the corresponding eigenvalue equations by means of the arbitrary boundary conditions of the edges $y = 0$ and b. But only for the seven cases C–C, F–F, SS1–C, SS2–C, SS1–F, SS2–F and C–F, the eigenvalue equations and the coefficient relations of eigenfunctions are derived here, which are given in Table 7 and Table A1, respectively.

The simply-supported and clamped boundary conditions are separable, but the free boundary conditions are not separable for FIV of plate, as is the case for the free transverse vibrations. For the sake of brevity, only the case F–F is considered below. The free boundary conditions are

$$\tau = 0 \Rightarrow \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \Rightarrow \phi_1\phi_2' + \psi_1'\psi_2 = 0$$
$$\sigma_y = 0 \Rightarrow \upsilon\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \upsilon\phi_1'\phi_2 + \psi_1\psi_2' = 0 \qquad (35)$$

Table 8
The frequency parameters β for the case $y = 0$ and b are clamped (C–C).

	1(1)	2(1)	3(2)	4(1)	5(1)	6(2)	7(1)	8(1)	9(1)	10(2)
$\Omega a/\pi$	0	0	1	0	1	2	1	0	1	2
Tb/π	1	1.6903	1.4063	2	1.7737	1.2350	2.6068	3	2.8455	2.3126
Zb/π	0.5916	1	0.4938	1.1832	0.4063	1.7917	1.2010	1.7748	1.3776	1.3863
β	0.8333	1.4086	1.5406	1.6667	1.7846	2.2493	2.3915	2.5000	2.5735	2.7774
β^*	0.8334	1.4087	1.5406	1.6668	1.7847	2.2496	2.3917	2.5007	2.5739	2.7779
$x = 0,a$	SS1–SS1	SS2–SS2	Both	SS1–SS1	Both	Both	Both	SS1–SS1	Both	Both

Table 9
The frequency parameters β for the case $y = 0$ and b are free (F–F).

	1(3)	2(1)	3(2)	4(1)	5(1)	6(1)	7(1)	8(3)	9(3)	10(1)
$\Omega a/\pi$	1	0	1	0	0	1	1	2	2	1
Tb/π	0.7242	1	0.9493	1.6903	2	1.6917	1.7380	1.0946	0.6518	2.6453
Zb/π	1.0581	0.5916	0.7878	1	1.1832	0.2563	0.4482	2.0404	1.9730	1.2301
β	0.7974	0.8333	1.2751	1.4086	1.6667	1.7284	1.7600	1.7799	1.9248	2.4206
β^*	0.7974	0.8334	1.2752	1.4087	1.6669	1.7285	1.7602	1.7803	1.9252	2.4208
$x = 0,a$	Both	SS1–SS1	Both	SS2–SS2	SS1–SS1	Both	Both	Both	Both	Both

Table 10
The frequency parameters β for the case $y = 0$ is clamped, and $y = b$ is free (C–F).

	1(1)	2(1)	3(3)	4(1)	5(2)	6(3)	7(1)	8(1)	9(1)	10(1)
$\Omega a/\pi$	0	0	1	0	1	2	1	1	0	0
Tb/π	0.5	0.8452	0.0922	1.5	1.4834	0.9439	2.1463	2.1863	2.5	2.5355
Zb/π	0.2958	0.5	0.9690	0.8874	0.4073	2.0139	0.8224	0.8585	1.4790	1.5
β	0.4167	0.7043	0.9970	1.2500	1.5900	1.8388	2.0492	2.0783	2.0833	2.1129
β^*	0.4167	0.7043	0.9972	1.2502	1.5902	1.8395	2.0494	2.0786	2.0838	2.1130
$x = 0,a$	SS1–SS1	SS2–SS2	Both	SS1–SS1	Both	Both	Both	Both	SS1–SS1	SS2–SS2

Table 11
The frequency parameters β for the case SS1–C–SS2–F.

	1(2)	2(1)	3(3)	4(1)	5(1)	6(2)	7(3)	8(1)	9(1)	10(1)
$\Omega a/\pi$	0.5	0.5	1.5	0.5	0.5	1.5	2.5	0.5	1.5	1.5
Tb/π	0.6480	0.9022	0.6551	1.7291	2.2735	1.6587	1.1973	2.8317	2.4979	2.5942
Zb/π	0.2950	0.2256	1.5021	0.9014	1.2550	1.0691	2.5203	1.6039	0.2790	0.4994
β	0.7359	0.9029	1.3971	1.5252	1.9594	2.0398	2.2923	2.4121	2.5657	2.6312
β^*	0.7360	0.9029	1.3975	1.5254	1.9597	2.0401	2.2933	2.4124	2.5662	2.6315

There are two distinct types of eigenfunctions for the simply-supported edges $x=0$ and a as given in Table 2, which are

$$\begin{array}{l}\phi_1(x)=\sin\Omega x\\ \psi_1(x)=\cos\Omega x\end{array}\quad \text{and} \quad \begin{array}{l}\phi_1(x)=\cos\Omega x\\ \psi_1(x)=\sin\Omega x\end{array} \tag{36a,b}$$

for which the signs of the coefficients of ϕ_2 and ψ_2 are opposite. This means that the sign of the second column in Table 3 is opposite to that of the third column for three cases. Thus, the substitution of Eq. (36a) or Eq. (36b) into Eq. (35) leads to the same result, as

$$\begin{array}{l}\phi_2'+\Omega\psi_2=0\\ \upsilon\Omega\phi_2-\psi_2'=0\end{array} \tag{37}$$

Then one has

$$\begin{array}{l}\phi_2'(0)+\Omega\psi_2(0)=0\\ \upsilon\Omega\phi_2(0)-\psi_2'(0)=0\end{array},\quad \begin{array}{l}\phi_2'(b)+\Omega\psi_2(b)=0\\ \upsilon\Omega\phi_2(b)-\psi_2'(b)=0\end{array} \tag{38a,b}$$

By solving Eqs. (38), one can obtain the eigenvalue equations as shown in Table 7, and the eigenfunction coefficients as shown in Table A1. The parameters $\alpha_i (i=1,2,3,4)$ in these two tables are given by

$$\begin{array}{l}\alpha_1=\dfrac{k_3T-\Omega}{k_4Z+\Omega}=\dfrac{T^2-\Omega^2}{2\Omega^2},\quad \alpha_2=\dfrac{k_3\Omega\upsilon-T}{k_4\Omega\upsilon+Z}=-\dfrac{ZT(1-\upsilon)}{\Omega^2\upsilon+Z^2}\\ \alpha_3=\dfrac{k_3\Omega\upsilon-T}{Z-k_4\Omega\upsilon}=\dfrac{ZT(\upsilon-1)}{Z^2-\Omega^2\upsilon},\quad \alpha_4=-\dfrac{k_3T+\Omega}{k_4Z+\Omega}=-\dfrac{T^2+\Omega^2}{2\Omega^2}\end{array} \tag{39}$$

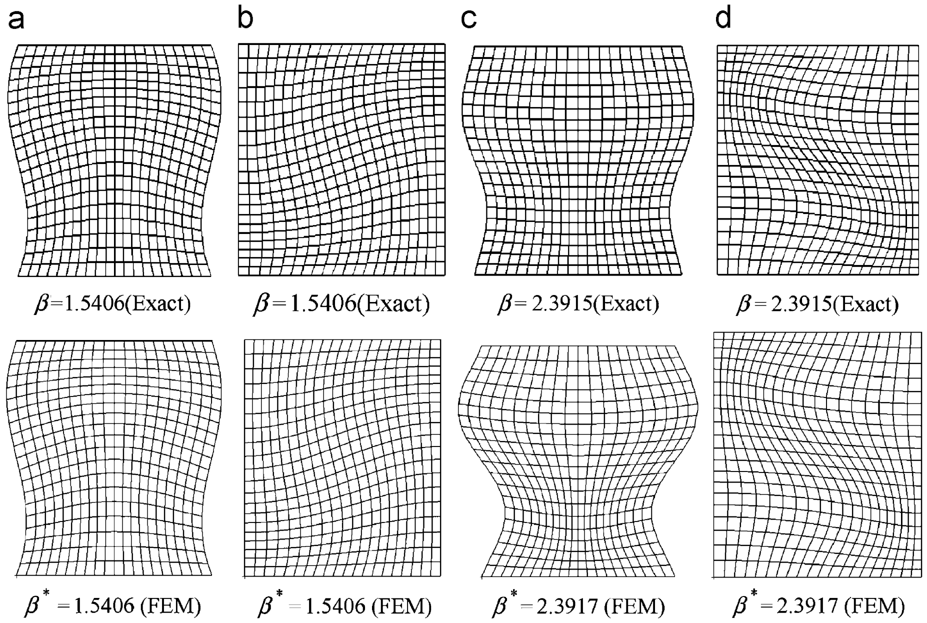

Fig. 3. Mode shapes of plate with both clamped edges $y=0$ and b (C–C). (a) SS1–C–SS1–C, (b) SS2–C–SS2–C, (c) SS1–C–SS1–C and (d) SS2–C–SS2–C.

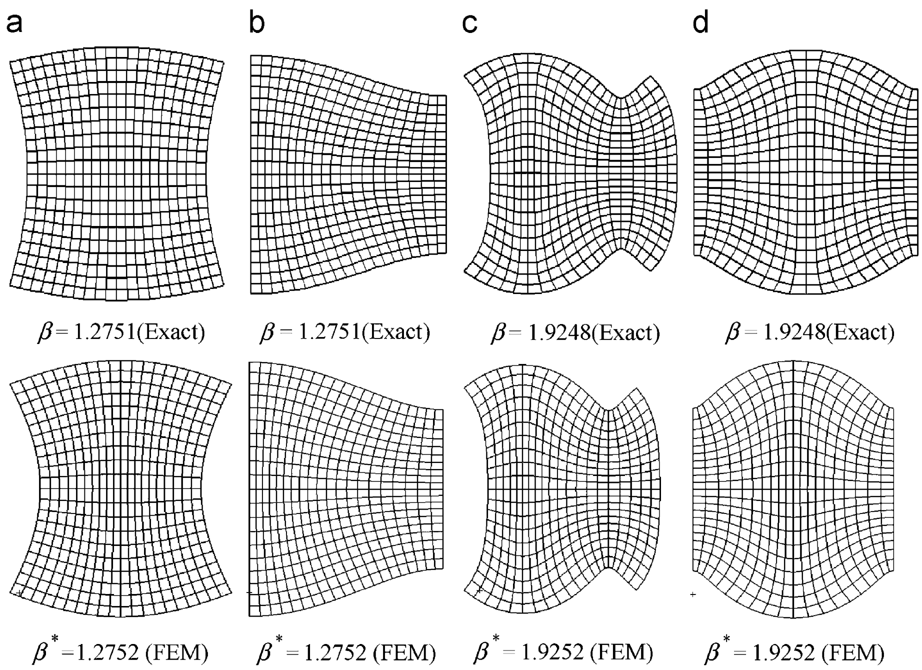

Fig. 4. Mode shapes of plate with both free edges $y=0$ and b (F–F). (a) SS1–F–SS1–F, (b) SS2–F–SS2–F, (c) SS1–F–SS1–F and (d) SS2–F–SS2–F.

The numerical results and comparisons are presented in Tables 8–11, wherein the numbers in parentheses (*) of the first rows indicate that the frequencies are computed from the eigenvalue equation of Case*, see Table 7. The 'both' in Tables 8–10 means that the two cases SS1–SS1 and SS2–SS2 (for $x = 0$ and a) have the same frequencies. The boundary conditions for the edges $x = 0$ and a are SS1–SS2 in Table 11. The mode shapes are shown and compared with those by FEM in Figs. 3–6.

There are several points pertaining to the numerical results as follows.

(1) The computed frequencies and mode shapes agree with those by FEM, see Tables 8–11 and Figs. 3–6.
(2) For the cases when the edges $y = 0$ and b are C–C, F–F and C–F, the eigenvalues T and Z cannot be zero, but Ω can be zero, see Tables 8–10. If $\Omega = 0$, $Tb = n\pi/2$ $(n = 1,2,3,\ldots)$ correspond to SS1–SS1 (for $x = 0$ and a), but $Zb = i\pi/2$ $(i = 1,2,3,\ldots)$ correspond to SS2–SS2 (for $x = 0$ and a).
(3) If the edges $x = 0$ and a are SS2–SS1 or SS1–SS2 corresponding to the eigenvalue equation $\cos \Omega a = 0$, Ω, T and Z cannot be zeros, and there are no repeated frequencies, see Table 11.

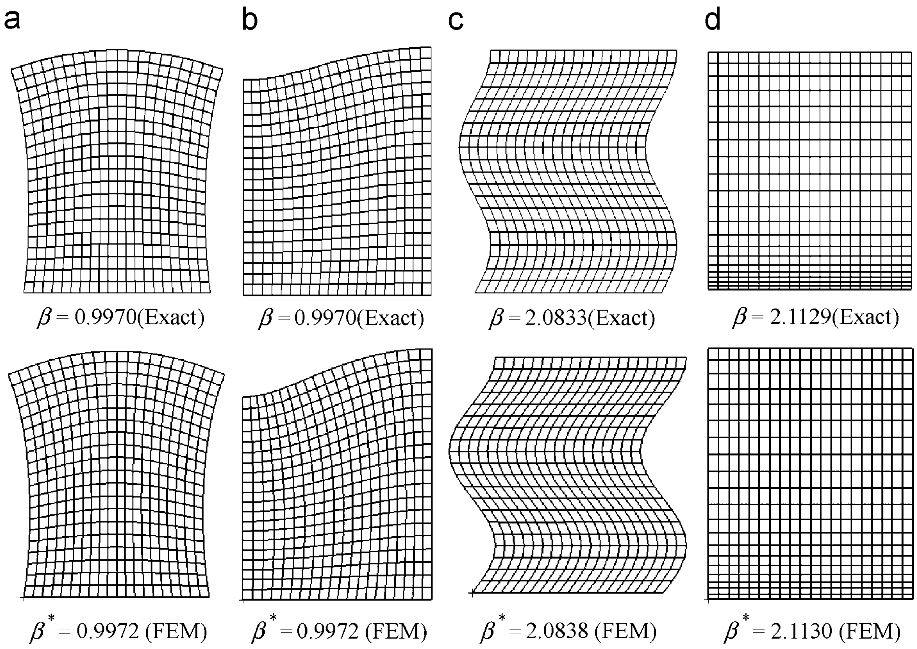

Fig. 5. Mode shapes of plate with clamped edge ($y = 0$) and free edge ($y = b$) (C–F). (a) SS1–C–SS1–F, (b) SS2–C–SS2–F, (c) SS1–C–SS1–F and (d) SS2–C–SS2–F.

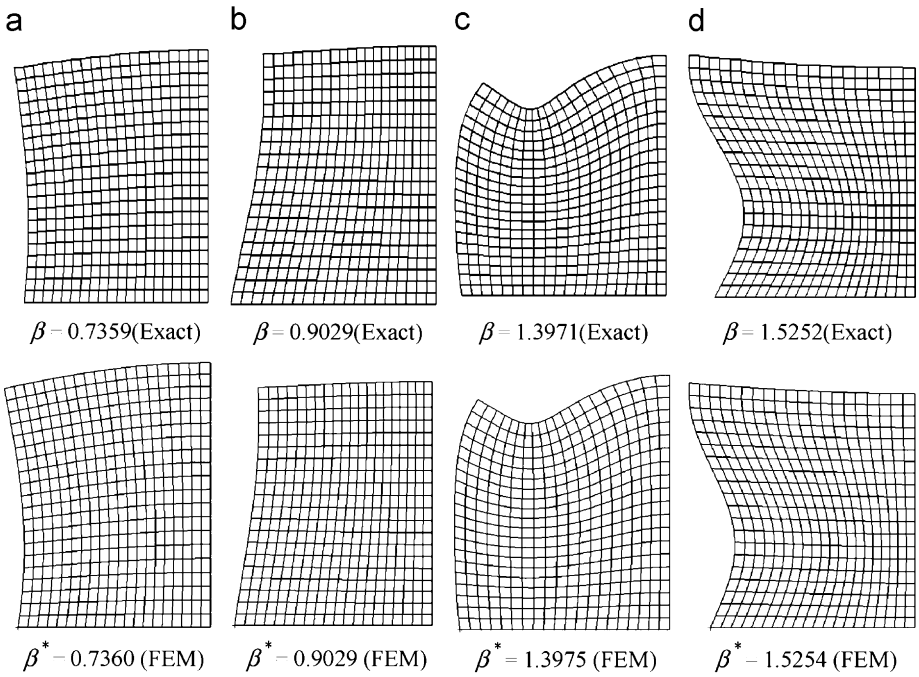

Fig. 6. Mode shapes of plate with SS1–C–SS2–F.

5. Conclusions

All possible exact solutions for the free in-plane vibrations of rectangular plate have been derived by using the direct separation of variables for the first time. The exact solutions for the cases in which two opposite edges are simply supported and the other two opposite edges are asymmetrical such as SS1–C–SS1–F, etc. were not available before.

One can see from present work that the method of direct separation of variables is powerful. The present work provides further insight into the overall subject of the free in-plane vibrations of rectangular plate.

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Appendix A

See Table A1 for details.

Table A1
The coefficients of the eigenfunctions $\phi_2(y)$ and $\psi_2(y)$.

Case 1	
C–C $D_3=-D_1, D_4=\frac{k_3}{k_4}D_2, D_2=-f_1D_1$ $f_1=\frac{k_4(\cos Tb-\cos Zb)}{k_4\sin Tb+k_3\sin Zb}$	F–F $D_3=D_1\alpha_1, D_4=D_2\alpha_2, D_2=-D_1f_3$ $f_3=\frac{\alpha_1(\cos Tb-\cos Zb)}{\alpha_1\sin Tb-\alpha_2\sin Zb}$
SS2–C $D_1=D_3=0, D_4=-D_2\frac{\sin Tb}{\sin Zb}$	SS1–C $D_2=D_4=0, D_3=-D_1\frac{\cos Tb}{\cos Zb}$
SS2–F $D_1=D_3=0, D_4=D_2\alpha_1\frac{\sin Tb}{\sin Zb}$	SS1–F $D_2=D_4=0, D_3=D_1\alpha_1\frac{\cos Tb}{\cos Zb}$
C–F $D_3=-D_1, D_4=\frac{k_3}{k_4}D_2, D_2=D_1f_6,\ f_6=\frac{\alpha_1\cos Tb+\cos Zb}{-\alpha_1\sin Tb+\frac{k_3}{k_4}\sin Zb}$	
Case 2	
C–C $D_3=-D_1, D_4=-\frac{k_3}{k_4}D_2, D_2=-f_2D_1$ $f_2=\frac{k_4(\cos Tb-\cosh Zb)}{k_4\sin Tb-k_3\sinh Zb}$	F–F $D_3=D_1\alpha_1, D_4=D_2\alpha_3, D_2=-D_1f_4$ $f_4=\frac{\alpha_1(\cos Tb-\cosh Zb)}{\alpha_1\sin Tb-\alpha_3\sinh Zb}$
SS2–C $D_1=D_3=0, D_4=-D_2\frac{\sin Tb}{\sinh Zb}$	SS1–C $D_2=D_4=0, D_3=-D_1\frac{\cos Tb}{\cosh Zb}$
SS2–F $D_1=D_3=0, D_4=D_2\alpha_1\frac{\sin Tb}{\sinh Zb}$	SS1–F $D_2=D_4=0, D_3=D_1\alpha_1\frac{\cos Tb}{\cosh Zb}$
C–F $D_3=-D_1, D_4=-\frac{k_3}{k_4}D_2, D_2=-D_1f_7,\ f_7=\frac{\alpha_1\cos Tb+\cosh Zb}{\alpha_1\sin Tb+(k_3/k_4)\sinh Zb}$	
Case3	
F–F $D_3=D_1\alpha_4, D_4=D_2\alpha_3, D_2=-f_5D_1$ $f_5=\frac{\alpha_4(\cosh Tb-\cosh Zb)}{\alpha_4\sinh Tb-\alpha_3\sinh Zb}$	C–F $D_3=-D_1, D_4=-\frac{k_3}{k_4}D_2, D_2=-D_1f_8$ $f_8=\frac{\alpha_4\cosh Tb+\cosh Zb}{\alpha_4\sinh Tb+\frac{k_3}{k_4}\sinh Zb}$
SS2–F $D_1=D_3=0, D_4=D_2\alpha_4\frac{\sinh Tb}{\sinh Zb}$	SS1–F $D_2=D_4=0, D_3=D_1\alpha_4\frac{\cosh Tb}{\cosh Zb}$

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