Finite Deformation: General Theory

The notes on finite deformation have been divided into two parts: special cases (http://imechanica.org/node/5065) and general theory. In class I start with special cases, and then sketch the general theory. But the two parts can be read in any order.

Subject to loads, a body deforms. We would like to develop a theory to evolve this deformation in time. In continuum mechanics, we model the body by a field of particles, and update the positions of the particles by using an equation of motion. We formulate the equation of motion by mixing the following ingredients:

- kinematics of deformation,
- conservation of mass, momentum, and energy,
- production of entropy,
- models of materials.

Each of these ingredients has alternative, but equivalent, mathematical representations. To focus on essential ideas, we will first adopt one set of representations, using nominal quantities. At the end of the notes, we will sketch some of the alternatives.

We will use the equation of motion to study several phenomena, including wave, vibration, and bifurcation.

Model a body by a field of particles. A body is made of atoms, each atom is made of electrons, protons and neutrons, and each proton or neutron is made of... This kind of description is too detailed. We will not go very far in helping the engineer by thinking of a bridge as a pile of atoms.

Instead, we will develop a continuum theory. This theory models the body by a field of material particles, or particles for brevity. Each particle consists of many atoms. As time progress, the clouds of electrons deform, and the protons jiggle, all at a maddeningly high frequency. The material particle, however, represents the collective behavior of many atoms, over a large size and a long time. A common method to devise a material model is to regard each material particle as a small specimen, undergoing homogenous deformation.

The body deforms in a three-dimensional space. The space consists of places, labeled by a system of coordinates. At a given time, each material particle in the body occupies a place in the space. As time progresses, the material particle moves from one place to another. The motion of the material particle is affected by the external loads, as well as the by the neighboring particles.

When a body deforms, does each material particle preserve its identity? We have tacitly assumed that, when a body deforms, each material particle preserves its identity. Whether this assumption is valid can be determined by experiments. For example, we can paint a grid on the body. After deformation, if the grid is distorted but remains intact, then we say that the deformation preserves the identity of each particle. If, however, after the deformation the grid disintegrates, we should not assume that the deformation preserves the identity of each particle.
Whether a deformed body preserves the identity of the particle is subjective, depending on the size and time scale over which we look at the body. A rubber, for example, consists of cross-linked long molecules. If our grid is over a size much larger than the individual molecular chains, then deformation will not disintegrate the grid. By contrast, a liquid consists of molecules that can change neighbors. A grid painted on a body of liquid, no matter how coarse the grid is, will disintegrate over a long enough time. Similar remarks may be made for metals undergoing plastic deformation. Also, in many situations, the body will grow over time. Examples include growth of cells in a tissue, and growth of thin films when atoms diffuse into the films. The combined growth and deformation clearly does not preserve the identity of each material particle.

In these notes, we will assume that the identity of each material particle is preserved as the body deforms.

Name a material particle by the coordinate of the place occupied by the material particle when the body is in a reference state. Each material particle can be named any way we want. For example, we often name a material particle by using an English letter, a Chinese character, or a color. More systematically, we name each material particle by the coordinate \( X \) of the place occupied by the material particle when the body is in a particular state, called the reference state.

Often we choose the reference state to be the state when the body is unstressed. However, even without external loading, a body may be under a field of residual stress. Thus, we may not be able to always set the reference state as the unstressed state. Rather, any state of the body may be used as a reference state.

Indeed, the reference state need not be an actual state of the body, and can be a hypothetical state of the body. For example, we can use a flat plate as a reference state for a shell, even if the shell is always curved and is never flat. To enable us to use differential calculus, all that matters is that material particles can be mapped from the reference state to any actual state by a 1-to-1 smooth function.

We will use the phrase “the particle \( X \)” as shorthand for “the material particle that occupies the place with coordinate \( X \) when the body is in the reference state”.

Field of deformation. Now we are given a body in a three-dimensional space. We have set up a system of coordinates in the space, and have chosen a reference state of the body to name material particles. When the body is in the reference state, a material particle occupies a place whose coordinate is \( X \). At time \( t \), the body deforms to a current state, and the material particle \( X \) moves to a place whose coordinate is \( x \). The time-dependent field \( x = x(X,t) \) describes the history of deformation of the body. A central aim of continuum mechanics is to evolve the field of deformation \( x(X,t) \) by developing an equation of motion.
Exercise. Give a pictorial interpretation of the following field of deformation:

\[
\begin{align*}
X_i &= X_1 + X_2 \tan \gamma(t) \\
X_2 &= X_2 \\
X_3 &= X_3
\end{align*}
\]

Compare the above field with another field of deformation:

\[
\begin{align*}
X_i &= X_1 + X_2 \sin \gamma(t) \\
X_2 &= X_2 \cos \gamma(t) \\
X_3 &= X_3
\end{align*}
\]

Displacement, velocity, and acceleration. At time \( t \), the particle \( X \) occupies the place \( x(X,t) \). At a slightly later time \( t + \delta t \), the particle \( X \) occupies the place \( x(X,t + \delta t) \). During the short time between \( t \) and \( t + \delta t \), the particle \( X \) moves by a small displacement:

\[
\Delta x = x(X,t + \delta t) - x(X,t).
\]

The velocity of the particle \( X \) at time \( t \) is

\[
\frac{\Delta x}{\delta t} = \frac{\partial x}{\partial t}.
\]

The acceleration of the particle \( X \) at time \( t \) is

\[
\frac{\partial^2 x}{\partial t^2}.
\]

The fields of velocity and acceleration are linear in the field of deformation \( x(X,t) \).

We have just interpreted the partial derivative of the function \( x(X,t) \) with respect to \( t \). We next interpret the partial derivative of the function \( x(X,t) \) with respect to \( X \).

Deformation gradient. For a bar undergoing homogenous deformation, we have defined the stretch by
We now extend this definition to a body undergoing inhomogeneous deformation in three dimensions.

Consider two nearby material particles in the body. When the body is in the reference state, first particle occupies the place with the coordinate \( \mathbf{X} \), and the second particle occupies the place with the coordinate \( \mathbf{X} + \mathbf{dX} \). When the body is in the current state at time \( t \), the first particle occupies the place with the coordinate \( \mathbf{x}(\mathbf{X},t) \), and the second particle occupies the place with the coordinate \( \mathbf{x}(\mathbf{X} + \mathbf{dX},t) \).

The stretch is generalized to

\[
\frac{\mathbf{x}_i(\mathbf{X} + \mathbf{dX},t) - \mathbf{x}_i(\mathbf{X},t)}{d\mathbf{X}_K}.
\]

This object represents 9 ratios, and is given a symbol,

\[
\mathbf{F}_{ik} = \frac{\partial \mathbf{x}_i(\mathbf{X},t)}{\partial \mathbf{X}_K},
\]

and is called the deformation gradient. That is, we have generalized the stretch to a time-dependent field of tensor, \( \mathbf{F}(\mathbf{X},t) \). The deformation gradient \( \mathbf{F}(\mathbf{X},t) \) is linear in the field of deformation \( \mathbf{x}(\mathbf{X},t) \).

As a body deforms, each material element of line rotates and stretches. Now focus on the vector connecting the two nearby material particles, \( \mathbf{X} \) and \( \mathbf{X} + \mathbf{dX} \). We call the vector a material element of line. When the body is in the reference state, the material element of line is represented by the vector \( \mathbf{dX} \). When the body is in the current state at time \( t \), this material element of line is represented by the vector

\[
d\mathbf{x} = \mathbf{x}(\mathbf{X} + \mathbf{dX},t) - \mathbf{x}(\mathbf{X},t).
\]

Thus,
\[
dx_i = \frac{\partial x_i(X,t)}{\partial X_k} dX_k .
\]

We have defined the partial derivative as the deformation gradient, so that
\[
dx_i = F_{ki}(X,t) dX_k ,
\]
or
\[
dx = F(X,t) dX .
\]

Thus, the deformation gradient \( F(X,t) \) is a linear operator that maps the vector between two nearby material particles from the reference state, \( dX \), to the vector between the same two material particles in the current state, \( dx \).

When the body deforms, the material element of line both stretches and rotates. In the reference state, let \( dL \) be the length of the element and \( M \) be the unit vector in the direction of the element, namely,
\[
dx = MdL .
\]
In the current state, let \( dl \) be the length of the element and \( m \) be the unit vector in the direction of the element, namely,
\[
dx = mdl .
\]

Thus, the deformation of the body stretches the length of the material element of line from \( dL \) to \( dl \), and rotates the direction of the material element from \( M \) to \( m \). Recall that the stretch of the element is defined by
\[
\hat{\lambda} = \frac{dl}{dL} .
\]

Combining the above four equations, we obtain that
\[
\hat{\lambda} m = F(X,t) M .
\]

In the current state at time \( t \), once the field of deformation, \( x(X,t) \), is known, the deformation gradient \( F(X,t) \) can be calculated. Once the direction \( M \) of a material element of line is given when the body is in the reference state, the above expression gives in the current state the direction of the element, \( m(X,t,M) \), and
the stretch of the element, \( \lambda(X,t,M) \).

**Green deformation tensor.** We can also obtain an explicit expression for the stretch \( \lambda(X,t,M) \). Taking the inner product of the vector \( \lambda m = F(X,t)M \), we obtain that

\[
\lambda^2 = M'CM,
\]

where

\[
C = F'^TF.
\]

The time-dependent field, \( C(X,t) \), known as the *Green deformation tensor*, is symmetric and positive-definite.

In the current state at time \( t \), once the field of deformation, \( x(X,t) \), is known, the Green deformation tensor \( C(X,t) \) can be calculated, and so can the stretch of any material element of line, \( \lambda(X,t,M) \).

**Exercise.** Given a field of deformation:

\[
\begin{align*}
x_1 &= X_1 + X_3 \sin \gamma(t) \\
x_2 &= X_2 \cos \gamma(t) \\
x_3 &= X_3
\end{align*}
\]

Calculate the tensors \( F(X,t) \) and \( C(X,t) \). When the body is in the reference state, a material element of line is in the direction \( M = [0,1,0]' \). Calculate the stretch and the direction of the element when the body is in the current state at time \( t \).

**Exercise.** Any symmetric and positive-definite matrix has three orthogonal eigenvectors, along with three real and positive eigenvalues. Interpret the geometric significance of the eigenvectors and the eigenvalues of \( C(X,t) \).

**Exercise.** For the above field of deformation, determine the eigenvectors and eigenvalues of the tensor \( C(X,t) \). Interpret your result.

**Conservation of mass.** In our previous work, we have tacitly assumed that mass is conserved as a body deforms. We now make this assumption explicit.

When a body is in the reference state, a material particle occupies a place with coordinate \( X \), and a material element of volume around the particle is denoted by \( dV(X) \). When the body is in a current state at \( t \), the element deforms to some other volume. Let \( \rho \) be the nominal density of mass, namely,

\[
\rho = \frac{\text{mass in current state}}{\text{volume in reference state}}.
\]

During deformation, we assume that a material particle does not gain or lose mass, so that the nominal density of mass, \( \rho \), is time-independent. If the body in the reference state is inhomogeneous, the nominal density of mass in general varies from one material particle to another. Combining these two considerations, we write the nominal density of mass as a function of material particle:

\[
\rho = \rho(X).
\]
This function is often given as an input to our theory. Consider again a material element of volume. When the body is in the reference state, the volume of the material element is \( dV(X) \). When the body is in the current state at time \( t \), the material element may deform to some other volume. The mass of the material element of volume at all time is \( \rho(X)dV(X) \). Consequently, at all time the mass of any part of the body is

\[
\int \rho(X)dV.
\]

The integral extends over the volume of the part in the reference state. Thus, the domain of integration remains fixed, independent of time, even though the body deforms.

**Conservation of linear momentum.** Let \( dV(X) \) be a material element of volume. In the current state, the material element of volume has the mass \( \rho(X)dV(X) \), and the linear momentum

\[
\frac{\partial x(X,t)}{\partial t} \rho(X)dV(X).
\]

The linear momentum in any part of the body is

\[
\int \frac{\partial x(X,t)}{\partial t} \rho(X)dV(X).
\]

The integral extends over the volume of the part in the reference state. The rate of change of the linear momentum of the part is

\[
\frac{d}{dt} \left( \int \frac{\partial x(X,t)}{\partial t} \rho(X)dV(X) \right) = \int \frac{\partial^2 x(X,t)}{\partial t^2} \rho(X)dV(X).
\]

The integrals extend over the volume of the part in the reference state.

Conservation of linear momentum requires that the rate of change of the linear momentum in any part of a body should equal the force acting on the part. We next express this principle in useful terms.

**Body force.** Consider a material element of volume around material particle \( X \). When the body in the reference state, the volume of the element is \( dV(X) \). When the body is in the current state at time \( t \), the force acting on the element is denoted by \( B(X,t)dV(X) \), namely,

\[
B(X,t) = \frac{\text{force in current state}}{\text{volume in reference state}}.
\]

The force \( B(X,t)dV(X) \) is called the body force, and the vector \( B(X,t) \) the nominal density of the body force. The body force is applied by an agent external to the body, and is often taken as an input to our theory.

**Traction.** Consider a material element of area at material particle \( X \). When the body is in the reference state, the area of the element is \( dA(X) \), and the unit vector normal to the element is \( N(X) \). When the body is in the current state at time \( t \), the force acting on the element of area is denoted as \( T(X,t)dA(X) \),
namely,

\[ T(X, t) = \frac{\text{force in current state}}{\text{area in reference state}}. \]

The force \( T(X, t) \, dA(X) \) is called the surface force, and the vector \( T(X, t) \) the nominal traction. For a material element of area on the external surface, the traction may be prescribed by an agent external to the body, and is taken as an input to our theory. For a material element of area inside the body, the traction represents the force exerted by one material particle on its neighboring material particle. In this case, the traction is internal to the body, and need be determined by the theory.

Conservation of linear momentum stipulates that, for any part of a body and at any time, the force acting on the part equals the rate of change in the linear momentum, namely,

\[
\int T(X, t) \, dA + \int B(X, t) \, dV = \frac{d}{dt} \int \rho(X) \frac{\partial x(X, t)}{\partial t} \, dV. 
\]

The integrals extend over the surface and the volume of any part of the body.

Inertial force. From the above equation, it is evident that we can also regard

\[- \rho(X) \frac{\partial^2 x_i(X, t)}{\partial t^2}\]

as a special type of the body force, called the inertial force. Using this interpretation, conservation of linear momentum is viewed as a balance of forces on any part of the body, including the surface force, body force, and inertial force, namely,

\[
\int T(X, t) \, dA + \int \left[ B(X, t) - \rho(X) \frac{\partial^2 x_i(X, t)}{\partial t^2} \right] \, dV = 0
\]

We next derive from the conservation of linear momentum two consequences.

Stress. For a bar pulled by a force, the nominal stress is defined as the force in the current state divided by the cross-sectional area of the bar in the reference state. We now generalize this definition to three dimensions.

Consider a material tetrahedron around a material particle \( X \). When the body is in the reference state, the material tetrahedron has three faces on the coordinate planes, and the fourth face is inclined to the axes. When the body is in the current state at time \( t \), the faces deform. Consider one face of the tetrahedron. When the body is in the reference state, the face is a flat triangle normal to the axis \( X_k \). When the body is in the current state at time \( t \), the face deforms into a curved triangle. Denote by \( s_{ik}(X, t) \) the component \( i \) of the force acting on this face in the current state divided by the area of the face in the reference state.

We adopt the following sign convention. When the outward normal vector of the area points in the positive direction of axis \( X_k \), we take \( s_{ik} \) to be positive if the component \( i \) of the force points in the positive direction of axis \( x_i \). When the outward normal vector of the area points in the negative direction of the axis \( X_k \),
we take $s_{ik}$ to be positive if the component $i$ of the force points in the negative direction of axis $x_j$.

**Stress-traction relation.** Consider the material tetrahedron again. When the body is in the reference state, the four faces of the tetrahedron are triangles: the three triangles on the coordinate planes, and one triangle on the plane normal to the unit vector $\mathbf{N}$. Let the areas of the three triangles on the coordinate planes be $A_K$, and the area of the triangle normal to $\mathbf{N}$ be $A$. The geometry dictates that

$$A_K = N_K A .$$

When the body is in the current state, the tetrahedron deforms to a shape of four curved faces. Regard this deformed tetrahedron in the current state as a free-body diagram. Now apply conservation of momentum to this deformed tetrahedron. Acting on each of the four faces is a surface force. As the volume of the tetrahedron decreases, the ratio of area over volume becomes large, so that the surface forces prevail over the body force and the change in the linear momentum. Consequently, the surface forces on the four faces of the tetrahedron must balance, giving

$$s_{ik} A_K = T_i A .$$

A combination of the above two equations give

$$s_{ik} N_K = T_i .$$

Consequently, the nominal stress is a linear operator that maps the unit vector normal to a material plane in the reference state to the traction acting on the material plane in the current state.

**Conservation of linear momentum in differential form.** Consider any part of the body. The force due to traction is

$$\int T_i dA = \int s_{ik} N_K dA = \int \frac{\partial S_k}{\partial \dot{X}_K} dV .$$

The first equality comes from the stress-traction relation, and the second equality invokes the divergence theorem. Consequently, conservation of linear momentum requires that
This equation holds for arbitrary part of the body, so that the integrands must equal:

\[
\frac{\partial s_{ik}(x,t)}{\partial x_k} + B_i(x,t) = \rho(x) \frac{\partial^2 x_i(x,t)}{\partial t^2}.
\]

This equation expresses conservation of linear momentum in a differential form. The equation is linear in the field of stress and the field of deformation.

An alternative derivation of the above equation goes as follows. Consider a block of material \(ABC\). When the body is in the reference state, the block is rectangular. When the body is in the current state, the block deforms to some other shape. Conserving linear momentum of this free-body diagram, we obtain that

\[
BCs_{ik}(x_i + A, x_2, x_3, t) - BCS_{ik}(x_i, x_2, x_3, t) + CAS_{ik}(x_i, x_2 + B, x_3, t) - CAS_{ik}(x_i, x_2, x_3, t) + ABS_{ik}(x_i, x_2, x_3 + C, t) - ABS_{ik}(x_i, x_2, x_3, t) + ABCB_i(x, t) = ABC\rho(x) \frac{\partial^2 x_i(x,t)}{\partial t^2},
\]

Divide this equation be the volume \(ABC\), and we obtain that

\[
\frac{\partial s_{ik}(x,t)}{\partial x_k} + B_i(x,t) = \rho(x) \frac{\partial^2 x_i(x,t)}{\partial t^2}.
\]

**Conservation of angular momentum.** This principle stipulates that, for any part of a body and at any time, the moment acting on the part equals the rate of change in the angular momentum, namely,
\[ \int \mathbf{x}(X, t) \times \mathbf{T}(X, t) \, dA + \int \mathbf{x}(X, t) \times \mathbf{B}(X, t) \, dV = \frac{d}{dt} \int \mathbf{x}(X, t) \cdot \frac{\partial \mathbf{x}(X, t)}{\partial t} \rho(X) \, dV. \]

The first term can be converted into a volume integral as follows:

\[ \int \varepsilon_{ijp} s_{jk} dA = \int \varepsilon_{ijp} s_{jk} N_k \, dA = \int \varepsilon_{ijp} \frac{\partial (x_j s_{pk})}{\partial X_k} \, dV\]

\[ = \int \left[ \varepsilon_{ijp} s_{jk} \frac{\partial x_j}{\partial X_k} + \varepsilon_{ijp} \frac{\partial s_{jk}}{\partial X_k} \right] \, dV\]

where \( \varepsilon_{ijk} \) is the symbol for permutation.

Combining the above two equations, and using the equation for conservation of linear momentum, we obtain that

\[ \int \varepsilon_{ijp} s_{jk} \frac{\partial x_j}{\partial X_k} \, dV = 0.\]

This equation holds for arbitrary part of the body, so that the integrand must vanish:

\[ \varepsilon_{ijp} s_{jk} \frac{\partial x_j}{\partial X_k} = 0.\]

This equation is equivalent to

\[ \mathbf{sF}^T = \mathbf{Fs}^T.\]

That is, conservation of angular momentum requires that the product \( \mathbf{sF}^T \) be a symmetric tensor. In general, neither the deformation gradient \( \mathbf{F} \), nor the nominal stress \( \mathbf{s} \), is a symmetric tensor.

An alternative derivation of the above equation goes as follows. Consider a small block in a body. In the reference state, the block is rectangular, of lengths \( A, B, C \). In the current state, the block deforms to some other shape. Consider the free-body diagram of the block in the current state: a pair of forces \( s_i, BC \) act on the pair of the material elements of area \( BC \), a pair of forces \( s_{i2}, CA \) act on the pair of the material elements of area \( CA \), and a pair of forces \( s_{i3}, AB \) act on the pair of the material elements of area \( AB \).

In the current state, in the coordinate plane \((i,j)\), we balance the moments of the forces acting on the block:

\[
\begin{align*}
&BCs_{i1} \left[ x_j (X_1 + A, X_2, X_3, t) - x_j (X_1, X_2, X_3, t) \right] \\
&+ CAS_{i2} \left[ x_j (X_1, X_2 + B, X_3, t) - x_j (X_1, X_2, X_3, t) \right] \\
&+ ABS_{i3} \left[ x_j (X_1, X_2, X_3 + C, t) - x_j (X_1, X_2, X_3, t) \right] \\
&= BCs_{j1} \left[ x_i (X_1 + A, X_2, X_3, t) - x_i (X_1, X_2, X_3, t) \right] \\
&+ CAS_{j2} \left[ x_i (X_1, X_2 + B, X_3, t) - x_i (X_1, X_2, X_3, t) \right] \\
&+ ABS_{j3} \left[ x_i (X_1, X_2, X_3 + C, t) - x_i (X_1, X_2, X_3, t) \right]
\end{align*}
\]

Divide the equation by the volume \( ABC \), and we obtain that
\[ s_{ik} F_{jk} = s_{jk} F_{ik}. \]

**Conservation of energy.** We now view any material part of the body as a system. In the current state, the part is subject to the body force, surface force, and inertial force. The forces do work to the part when the body deforms. The part is also in thermal contact with reservoirs of energy and with neighboring parts of the body, so that the part receives heat from the rest of the world. Conservation of energy requires that the change in the energy of the part should equal the sum of the work done to the part and heat received by the part. We next express this principle in useful terms.

**Work.** Between a time \( t \) and a slightly later time \( t + \delta t \), a material particle \( X \) moves by a small displacement:

\[ \delta x = x(X, t + \delta t) - x(X, t). \]

Associated with this field of small displacement, the surface force, the body force, and the inertial force together do work to a part of the body:

\[ \text{work} = \int T_i \delta x_i dA + \int \left( B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \delta x_i dV. \]

The integrals extend over the surface and the volume of the part of the body.

We can express the work in terms of the stress. Associated with the small displacement \( \delta x_i \), the deformation gradient of a material particle \( X \) changes by

\[ \delta F_{ik} = F_{ik}(X, t + \delta t) - F_{ik}(X, t) = \frac{\partial \delta x_i}{\partial X_k}. \]

Recall that conservation of linear momentum results in two equations:

\[ T_i = s_{ik} N_k, \]

\[ \frac{\partial s_{ik}}{\partial X_k} + B_i = \rho \frac{\partial^2 x_i}{\partial t^2}. \]

Insert the two equations into the expression for work, and apply the divergence theorem

\[ \int s_{ik} N_k \delta x_i dA = \int \left( \frac{\partial (s_{ik} \delta x_i)}{\partial X_k} \right) dV. \]

We reduce the work done by all the forces to the following expression:

\[ \text{work} = \int s_{ik} \delta F_{ik} dV. \]

This expression holds for any part of the body.

Consequently, associated with a field of small displacement \( \delta x_i \), all the forces together do this amount work per unit volume:

\[ \frac{\text{work in the current state}}{\text{volume in the reference state}} = s_{ik} \delta F_{ik}. \]

That is, the nominal stress is work-conjugate to the deformation gradient.

The above result generalizes the one dimensional result for a bar pulled by a force:
\[
\frac{\text{work in the current state}}{\text{volume in the reference state}} = s\delta\lambda.
\]

To appreciate the general result, it might be instructive for you to review how this one-dimensional result arises from elementary considerations (http://imechanica.org/node/5065).

**Heat.** The body is in thermal contact with a field of reservoirs of energy. Consider a material element of volume around the particle \(X\). When the body is in the reference state, the volume of the element is \(dV(X)\). When the body is in the current state at time \(t\), the element deforms to some other volume. Between a reference time and the current time \(t\), the material element receives from the reservoirs an amount of energy \(Q(X,t)dV(X)\), namely,

\[
Q(X,t) = \frac{\text{energy received up to current state}}{\text{volume in reference state}}. 
\]

We adopt the sign convention that \(Q > 0\) when the material element of volume receives energy from the reservoirs.

Consider a material element of area at the particle \(X\). When the body is in the reference state, the area of the element is \(dA(X)\), and the unit vector normal to the element is \(N(X)\). When the body is in the current state at time \(t\), the element deforms to some other shape. Between a reference time and the current time, the energy across the material element of area is denoted by \(q(X,t)dA(X)\), namely,

\[
q(X,t) = \frac{\text{energy across up to current state}}{\text{area in reference state}}. 
\]

We adopt the sign convention that \(q > 0\) when energy transfers in the direction of \(N(X)\).

Consider a material tetrahedron around a material particle \(X\). When the body is in the reference state, the tetrahedron has three faces on the coordinate planes, and the fourth face is inclined to the coordinate axes. When the body is in a current state at time \(t\), the faces deform. Consider one face of the tetrahedron. When the body is in the reference state, the fact is normal to the axis \(X_k\). Between a reference time and the current time, the energy per area across the face is denoted by \(I_k(X,t)\). We adopt the sign convention that \(I_k > 0\) if the energy goes out the tetrahedron.

**Conservation of energy.** In the current state at time \(t\), denote the nominal density of internal energy by \(u(X,t)\), namely,

\[
u(X,t) = \frac{\text{internal energy in current state}}{\text{volume in reference state}}. 
\]

Thus, when the body is in the current state at time \(t\), the internal energy of any part of the body is

\[
\int u(X,t)dV.
\]
The integral extends over the volume of the part. Between time \( t \) and \( t + \delta t \), the nominal density of internal energy changes by

\[
\delta u(X,t) = u(X,t + \delta t) - u(X,t).
\]

Similarly define various amounts of heat transfer during this short time:

\[
\delta Q = Q(X,t + \delta t) - Q(X,t),
\]

\[
\delta q = q(X,t + \delta t) - q(X,t),
\]

\[
\delta I_k = I_k(X,t + \delta t) - I_k(X,t).
\]

Consider any part of the body. Conservation of energy requires that the work done by the forces upon the part and the heat transferred into the part equal the change in the internal energy of the part. Thus,

\[
\int \delta udV = \int s_{ik} \delta F_{ik} dV + \int \delta QdV - \int \delta qdA.
\]

The integrals extend over the volume and the surface of the part. We next reduce conservation of energy into a differential form.

\[ q - I_k \] relation. Consider a material element of tetrahedron. When the body is in the reference state, the four faces of the tetrahedron are triangles: the three triangles on the coordinate planes, and one triangle on the plane normal to the unit vector \( N \). Let the areas of the three triangles on the coordinate planes be \( A_k \), and the area of the triangle normal to \( N \) be \( A \). The geometry dictates that

\[
A_k = N_k A.
\]

When the body is in the current state, the tetrahedron deforms to a shape of four curved faces. Now apply conservation of energy to this deformed tetrahedron. As the volume of the tetrahedron decreases, the ratio of area over volume becomes large, so that conduction across the surface prevails over terms due to volumetric changes. Consequently, the conduction through all four faces must balance:

\[
I_k A_k = qA
\]

A combination of the above two equations give
\[ I_k N_k = q. \]

Consequently, \( I_k \) is a linear operator that maps the unit vector normal to an area to the energy across per unit area \( q \).

Applying the divergence theorem, we obtain that
\[
\int q dA = \int I_k N_k dA = \int \frac{\partial I_k}{\partial X_k} dV.
\]
Inserting this expression into the expression for conservation of energy, we obtain that
\[
\int \delta u dV = \int\left( s_{ik} \partial F_{l,k} + \partial Q - \frac{\partial}{\partial X_k} \partial I_k \right) dV.
\]
This equation holds for any part of the body, so that the integrands must equal:
\[
\delta u = s_{ik} \partial F_{l,k} + \partial Q - \frac{\partial}{\partial X_k} \partial I_k.
\]
This equation expresses conservation of energy in a differential form.

**Production of entropy.** We regard a material particle as a thermodynamic system. When the body is in the current state at time \( t \), denote the nominal density of entropy by \( \eta(X,t) \), namely,
\[
\eta(X,t) = \frac{\text{entropy in current state}}{\text{volume in reference state}}.
\]
Between time \( t \) and time \( t + \delta t \), the nominal density of entropy changes by
\[
\delta \eta = \eta(X,t + \delta t) - \eta(X,t),
\]
and the entropy of any part of the body changes by
\[
\int \delta \eta dV.
\]
The integral extends over the volume of the part.

Each material particle \( X \) is in thermal contact with a reservoir of energy held at temperature \( \theta_k(X,t) \). Upon losing energy \( \partial Q \) to the material particle, the reservoir changes its entropy by \( -\partial Q / \theta_k \). Consequently, the change in the entropy of the reservoirs in thermal contact with the part of the body is
\[
-\int \frac{\partial Q}{\theta_k} dV.
\]
When the body is in the current state at time \( t \), the field of temperature in the body is denoted by \( \theta(X,t) \). The part of the body is also in thermal contact with neighboring parts of the body. We may as well think the material particles on the surface of the part as reservoirs of energy. When energy flows from the part to the reservoirs, the entropy of these hypothetical reservoirs changes by
\[
\int \frac{\partial Q}{\theta} dA.
\]
The integral extends over the surface of the part. Recall that conservation of
energy results in two equations:

\[ I_k N_k = q , \]
\[ \partial u = s_{ik} \partial F_{ik} + \partial Q - \frac{\partial \partial A_k}{\partial X_k} \]

Using these two equations, and applying the divergence theorem,

\[ \int \frac{N_k \partial A_k}{\theta} dA = \int \left( \frac{\partial \partial A_k}{\partial X_k} \frac{\partial}{\partial \theta} \right) dV , \]

we obtain that

\[ \int \left[ \frac{\partial u}{\theta} + s_{ik} \frac{\partial F_{ik}}{\theta} + \frac{\partial (1/\theta)}{\partial X_k} \partial A_k + \frac{1}{\theta} \frac{1}{\theta_k} \partial Q \right] dV . \]

We form a composite from the following things:

- a part of the body,
- the reservoirs of energy in thermal contact with the part
- the particles on the surface of the body represented as hypothetical reservoirs of energy
- all the mechanical forces.

Between time \( t \) and time \( t + \delta t \), the entropy of the composite changes by

\[ \text{change in entropy} = \int \left[ \frac{\partial \eta}{\theta} - \frac{\partial u}{\theta} + s_{ik} \frac{\partial F_{ik}}{\theta} + \frac{\partial (1/\theta)}{\partial X_k} \partial A_k + \left( \frac{1}{\theta} - \frac{1}{\theta_k} \right) \partial Q \right] dV . \]

The mechanical forces do not contribute to the entropy.

By construction, the composite is an isolated system. Consequently, all the changes must proceed in the direction to increase the entropy of the composite. This principle applies to any part of the body, so that all the changes must proceed in the direction

\[ \frac{\partial \eta}{\theta} - \frac{\partial u}{\theta} + s_{ik} \frac{\partial F_{ik}}{\theta} + \frac{\partial (1/\theta)}{\partial X_k} \partial A_k + \left( \frac{1}{\theta} - \frac{1}{\theta_k} \right) \partial Q \geq 0 . \]

This isolated system has a large number of internal variables, including \( \eta, u, F, I, Q \).

**Thermodynamic model of a material particle.** We regard each material particle as a thermodynamic system characterized by the entropy as a function of the internal energy and deformation gradient:

\[ \eta = \eta(u, F) . \]

Associated with the variation \( \delta u \) and \( \delta F \), the entropy varies by

\[ \delta \eta = \frac{\partial \eta(u, F)}{\partial u} \delta u + \frac{\partial \eta(u, F)}{\partial F_{ik}} \delta F_{ik} . \]

Assume that the particle is in local equilibrium, so that

\[ \frac{\partial \eta(u, F)}{\partial u} = \frac{1}{\theta} . \]
The change in entropy of the isolated system reduces to
\[ \frac{\partial (1/\theta)}{\partial X_k} \partial I_k + \left( \frac{1}{\theta} \frac{1}{\theta_R} \right) \partial Q \geq 0. \]

This inequality can be satisfied by invoking Fourier’s law, as discussed in the notes on heat conduction (http://imechanica.org/node/4942).

**Isothermal deformation.** In particular, we will be mostly interested in situations where the body is in thermal equilibrium, so that the temperature is uniform in the body, and is an input to the theory. Let \( W \) be the nominal density of the Helmholtz free-energy state
\[ W = \text{free energy in current state} \]
\[ \text{volume in reference state} \]
so that
\[ W = u - \theta \eta \]

As a material model, we assume that the free-energy density is a function of the deformation gradient,
\[ W = W(F). \]
The temperature is uniform in the body, and we will not indicate temperature explicitly.

A combination of the above equations gives that
\[ s_{ik} = \frac{\partial W(F)}{\partial F_{ik}}. \]

This equation relates the stress to the deformation gradient once the function \( W(F) \) is prescribed. We may as well regard this equation as the defining equation for a material model, known as the hyperelastic model.

**The free-energy density is unchanged if the body undergoes a rigid-body rotation.** The free energy is invariant if the particle undergoes a rigid body rotation. Thus, \( W \) depends on \( F \) only through the product
\[ C_{KL} = F_{ik} F_{kl}, \]
or
\[ C = F^T F. \]
We write
\[ W(F) = f(C). \]

Recall that the stress is work-conjugate to the deformation gradient. Using the chain rule in differential calculus, we obtain that
\[ s_{ik} = \frac{\partial W(F)}{\partial F_{ik}} = \frac{\partial f(C)}{\partial C_{MN}} \frac{\partial C_{MN}}{\partial F_{ik}}, \]
or
\[ s_{ik} = 2F_{IL} \frac{\partial f(C)}{\partial C_{ik}}. \]

Note that this equation reproduces the equation resulting from conservation of angular momentum:

\[ s_{ik} F_{jk} = s_{ik} F_{ik}. \]

**Tensor of tangent modulus for a hyperelastic material.** The tensor of tangent modulus is defined by

\[ K_{ikjl} = \frac{\partial s_{ik}(F)}{\partial F_{jl}}, \]

or

\[ \delta s_{ik} = K_{ikjl} \delta F_{jl}. \]

For a hyperelastic material, the tensor of tangent modulus relates to the free-energy density as

\[ K_{ikjl} = \frac{\partial W(F)}{\partial F_{ik} \partial F_{jl}}. \]

Consequently, for a hyperelastic material, the tensor of tangent modulus possesses the major symmetry:

\[ K_{ijkl} = K_{jilk}. \]

**Isotropic material.** To specify a hyperelastic material model, we need to specify the free energy density as a function of the Green deformation tensor, \( C \). That is, we need to specify the function

\[ W = f(C). \]

The tensor \( C \) is positive-definite and symmetric. In three dimensions, this tensor has 6 independent components. Thus, to specify a hyperelastic material model, we need to specify the free energy density as a function of the 6 variables. For a given material, such a function is specified by a combination of experimental measurements and theoretical considerations. Trade off is made between the amount of effort and the need for accuracy.

For an isotropic material, the free energy density is invariant when the coordinates rotate in the reference state. That is, \( W \) depends on \( C \) through 3 scalars:

\[ \alpha = C_{kk}, \]
\[ \beta = C_{kl}C_{lk}, \]
\[ \gamma = C_{kl}C_{lk}C_{jk}. \]

These scalars are known as the invariants of the tensor \( C \). That is, to specify an isotropic hyperelastic material model, we need to prescribe a function of three variables:

\[ W = f(\alpha, \beta, \gamma). \]

Once this function is specified, we can derive the stress by using the chain rule:
Algebra of this kind is recorded in many textbooks on continuum mechanics.

**Incompressible material.** Rubbers and metals can undergo large change in shape, but very small change in volume. A commonly used idealization is to assume that such materials are incompressible, namely, 
\[ \det \mathbf{F} = 1. \]

In arriving at \( s_{ik} = \partial W(\mathbf{F}) / \partial F_{ik} \), we have tacitly assumed that each component of \( F_{ik} \) can vary independently. The condition of incompressibility places a constraint among the components. To enforce this constraint, we replace the free-energy function \( W(\mathbf{F}) \) by a function
\[ W(\mathbf{F}) - \Pi (\det \mathbf{F} - 1), \]
with \( \Pi \) as a Lagrange multiplier. We then allow each component of \( F_{ik} \) to vary independently, and obtain the stress from
\[ s_{ik} = \frac{\partial}{\partial F_{ik}} [W(\mathbf{F}) - \Pi (\det \mathbf{F} - 1)]. \]

Recall an identity in the calculus of matrix:
\[ \frac{\partial \det \mathbf{F}}{\partial F_{ik}} = H_{ik} \det \mathbf{F}, \]
where \( \mathbf{H} \) is defined by \( H_{ik} \delta_{kl} = \delta_{ik} \) and \( H_{ik} F_{jk} = \delta_{ij} \).

Thus, for an incompressible material, the stress relates to the deformation gradient as
\[ s_{ik} = \frac{\partial W(\mathbf{F})}{\partial F_{ik}} - \Pi H_{ik} \det \mathbf{F}. \]

The Lagrange multiplier \( \Pi \) is not a material parameter. For a given boundary-value problem, \( \Pi \) is to be determined as part of the solution.

**Neo-Hookean material.** For a neo-Hookean material, the free-energy function takes the form
\[ W(\mathbf{F}) = \frac{\mu}{2} (F_{ik} F_{ik} - 3). \]

The material is also taken to be incompressible, namely, 
\[ \det \mathbf{F} = 1. \]

The stress relates to the deformation gradient as
\[ s_{ik} = \mu F_{ik} - H_{ik} \Pi. \]

**Equation of motion.** We now state an initial and boundary value problem to evolve the field of deformation, \( \mathbf{x}(\mathbf{X}, t) \). Given a body, we choose a reference state to name a field of material particles. The body is prescribed with a
field of mass density, \( \rho(X) \). The material is modeled by an energy function \( W(F) \), which depends on \( F \) through the product \( F^T F \). The body is subject to a field of body force \( B(X,t) \). The field equations are

\[
F_{ik} = \frac{\partial x_i(X,t)}{\partial X_k},
\]

\[
s_{ik} = \frac{\partial W(F)}{\partial F_{ik}},
\]

\[
\frac{\partial s_{ik}(X,t)}{\partial X_k} + B_i(X,t) = \rho(X) \frac{\partial^2 x_i(X,t)}{\partial t^2}.
\]

A combination of the above three equations gives the equation of motion. This equation evolves the field of deformation \( x(X,t) \) in time, subject to the following initial and boundary conditions.

Initial conditions are given by prescribing at time \( t_o \) the places of all the particles, \( x(X,t_o) \), and the velocities of all the particles, \( V(X,t_o) \).

For every material particle on the surface of the body, we prescribe either one of the following two boundary conditions. On part of the surface of the body, \( S_i \), the traction is prescribed, so that

\[
s_{ik}(X,t)N_k(X) = \text{prescribed}, \quad \text{for } X \in S_i,
\]

On the other part of the surface of the body, \( S_u \), the position is prescribed, so that

\[
x(X,t) = \text{prescribed}, \quad \text{for } X \in S_u.
\]

**Perturb an equilibrium state of finite deformation.** A body is made of a material characterized by a relation between the stress and the deformation gradient:

\[
s_{ik} = g_{ik}(F).
\]

The body is subject to a dead load \( B(X) \) in the volume. One part of the surface of the body is subject to dead load \( T(X) \), and the other part of the surface is held at prescribed place.

Let \( x^o(X) \) be a state of equilibrium compatible with the above prescriptions. Associated with this state is the deformation gradient

\[
F_{ik}^o = \frac{\partial x_i^o(X)}{\partial X_k},
\]

and the stress

\[
s_{ik}^o = g_{ik}(F^o).
\]

The stress field balances the forces

\[
\frac{\partial s_{ik}^o(X)}{\partial X_k} + B_i(X) = 0
\]

in the volume of the body, and
on the part of surface where the traction is prescribed. On the part of the surface where the place is prescribed, \( x^o(X) \) is held at the prescribed place.

When the state of equilibrium, \( x^o(X) \), is now perturbed by an infinitesimal deformation, \( u(X,t) \), the field of deformation becomes

\[
x(X,t) = x^o(X) + u(X,t).
\]

Associated with this field of deformation is the deformation gradient

\[
F_{ik}(X,t) = F_{ik}^o(X) + \frac{\partial u_i(X,t)}{\partial X_k},
\]

and the stress

\[
s_{ik}(F) = s_{ik}(F^o) + K_{ijkl} \frac{\partial u_j(X,t)}{\partial X_l}.
\]

Here we have expanded the stress into the Taylor series up to terms linear in the perturbation. Recall the equation of force balance

\[
\frac{\partial}{\partial X_k} \left[ K_{ijkl} \frac{\partial u_j(X,t)}{\partial X_l} \right] = \rho \frac{\partial^2 X_i}{\partial t^2}.
\]

Inserting the stress-strain relation into the equation of force balance, we obtain a homogeneous partial differential equation:

\[
\frac{\partial}{\partial X_k} \left[ K_{ijkl} \frac{\partial u_j(X,t)}{\partial X_l} \right] = \rho(X) \frac{\partial^2 u_i(X,t)}{\partial t^2}.
\]

The boundary conditions also homogeneous:

\[
K_{ijkl} \frac{\partial u_j(X,t)}{\partial X_l} N_k = 0
\]

on the part of the surface where the dead force \( T(X) \) is prescribed, and

\( u(X,t) = 0 \)

on the part of the surface where the place is prescribed. These equations are solved subject to an initial condition.

The equation of motion for the perturbation \( u(X,t) \) looks similar to that in linear elasticity. Thus, solutions in linear elasticity may be reinterpreted for phenomena of this type. We give examples below.

**Linear vibration around a state of equilibrium.** We have already analyzed the vibration of beam subject to an axial force. In general, we look for solution of the form

\[
u(X,t) = a(X)\sin \omega t,
\]

where \( \omega \) is the frequency, and \( a(X) \) is the field of amplitude. The equation of motion becomes that
This equation, in conjunction with the homogeneous boundary conditions, forms an eigenvalue problem. The solution to this eigenvalue problem determines \( \omega \) and \( a(X) \) for a set of normal modes.

**Plane waves superimposed on a homogeneous deformation in an infinite body.** When a bar is subject to uniaxial tension, homogenous deformation satisfies all governing equations. However, the state of homogeneous solution may not be the only solution. Necking is an alternative solution, a bifurcation from the homogenous deformation. In the previous analysis, we have just considered the homogeneous deformation, and have identified the onset of necking with the axial force in the force-strain curve peaks. Now we wish to push the analysis a step further: we wish to consider the possible small perturbation from a homogenous deformation.

Consider an infinite body. Assume a plane wave of small amplitude:

\[
\mathbf{u}(X,t) = a f(\mathbf{N} \cdot \mathbf{X} - ct),
\]

where \( a \) is the unit vector in the direction of displacement, \( \mathbf{N} \) the unit vector in the direction of propagation, \( c \) the wave speed, and \( f \) the wave form.

The equation of motion becomes

\[
\frac{\partial}{\partial X_k} \left[ K_{ijkl}(F^o) \frac{\partial a_i(X)}{\partial X_k} \right] = -\rho(\mathbf{X}) \omega^2 a_i(\mathbf{X}).
\]

This is an eigenvalue problem.

When the modulus of tangent modulus possesses the major symmetry, \( K_{ijkl} = K_{ijlk} \), the acoustic tensor \( K_{ijkl} \mathbf{N}_k \mathbf{N}_l \) is also symmetric. According to a theorem in linear algebra, this symmetric acoustic tensor will have three real eigenvalues, and the three eigenvectors are orthogonal to one another. Furthermore, when the acoustic tensor is positive-definite, the three eigenvalues are positive, leading to three distinct plane waves.

**Exercise.** An infinite body of an incompressible material cannot support longitudinal wave, but can support shear wave. Assume that the body is in a homogeneous state of uniaxial stress. A small disturbance propagates along the axis of stress as a shear wave. Determine the speed of this shear wave.

**Bifurcation.** The tensor of tangent modulus depends on the state of homogeneous deformation, \( K_{ijkl}(F^o) \), and may no longer be positive-definite. When the tangent modulus reaches the critical condition

\[
\det[K_{ijkl}(F^o) \mathbf{N}_k \mathbf{N}_l] = 0
\]

in some direction \( \mathbf{N} \), we can find a nonvanishing displacement \( a \) at \( c = 0 \). This equation may be regarded as a 3D generalization of the Considère condition.

**Reading.** J. R. Rice, "The Localization of Plastic Deformation", in

http://esag.harvard.edu/rice/062_Rice_LocalPlasDef_IUTAM76.pdf

**Surface instability of rubber in compression**, M. A. Biot, *Appl. Sci. Res. A* 12, 168 (1963).  A semi-infinite block of a neo-Hookean material is in a homogeneous state of deformation, with $\lambda_1$ and $\lambda_2$ being the stretches in the directions parallel to the surface of the block, and $\lambda_3$ being the stretch in the direction normal to the surface. The material is taken to be incompressible, so that $\lambda_1\lambda_2\lambda_3 = 1$. The compression in direction 1 is taken to be more severe than that in direction 2, so that when the surface wrinkles in one orientation, leaving $\lambda_2$ unchanged. That is, the wrinkled block is in a state of generalized plane strain. Biot found the critical condition for this instability:

$$\lambda_3 / \lambda_1 = 3.4$$

The analysis of this problem is analogous to that of the Rayleigh wave.

**Weak statement of conservation of momentum.** We next consider a number of refinements of the theory. Recall that conservation of linear momentum results in two equations:

$$T_i = s_{ik} N_k,$$

$$\frac{\partial s_{ik}}{\partial x_k} + B_i = \rho \frac{\partial^2 x_i}{\partial t^2}.$$

This pair of equations may be called the **strong statement** of conservation of momentum.

We now express conservation of linear momentum in a different form. Denote an arbitrary field by $\Delta_i = \Delta_i(X)$

In the literature on the finite element method, $\Delta_i(X)$ is also called a **test function**. Consider any part of the body. Multiplying the two equations in the strong statement by $\Delta_i(X)$, integrating over the area of the part and the volume of the part, respectively, and then adding the two, we obtain that

$$\int s_{ik} \frac{\partial \Delta_i}{\partial x_k} dV = \int T_i \Delta_i dA + \int \left( B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \Delta_i dV.$$

The integrals are taken over the volume and the surface of the part. In reaching the above equation, we have used the divergence theorem. Consequently, conservation of linear momentum implies that the above equation holds for any test function $\Delta_i(X)$. This statement is known as the **weak statement** of conservation of linear momentum.

Conversely, we can start with the weak statement, and show that the weak statement implies the strong statement.
An alternative formulation of isothermal deformation of a hyperelastic material. A hyperelastic material is modeled by a free-energy density as a function of the deformation gradient, $W(F)$. To ensure that the free energy is invariant when the body undergoes a rigid-body rotation, we require that $W$ depend on $F$ only through the Green deformation tensor $C = F^T F$.

As an alternative starting point of the theory, we stipulate that the virtual work done by all the forces equals the virtual change in the energy, namely,

$$
\int \frac{\partial W(F)}{\partial F_{ik}} \Delta_{ij} dV = \int T_i \Delta_i dA + \int (B_i - \rho \frac{\partial^2 x_i}{\partial t^2}) \Delta_i dV.
$$

This equation holds for arbitrary test function $\Delta_i(x)$. 

Exercise. The above equation is the basis for the finite element method (http://imechanica.org/node/324). In that context, the above volume integrals extend over the entire body. The test function $\Delta_i(x)$ is set to vanish on part of the surface where displacement is prescribed, and on the remaining part of the surface, the surface force $T_i$ is prescribed. Sketch the finite element method by using the above equation.

Exercise. This formulation is so swift that it does not even mention the stress. Recover all the previous equations by identifying the stress by the relation

$$s_{ik} = \frac{\partial W(F)}{\partial F_{ik}}.$$

Exercise. You can go back to verify the following statements.

- This alternative formulation conserves mass, linear momentum, angular momentum, and energy.
- Consider the composite of a part of the body in thermal contact with various reservoirs. Under the isothermal condition, the entropy of the composite does not change.

Polar decomposition. By definition, the deformation gradient is a linear operator that maps one vector to another vector. For any linear operator there is a theorem in linear algebra called polar decomposition. Any linear operator $F$ can be written as a product:

$$F = RU,$$

where $R$ is an orthogonal operator, satisfying $R^T R = I$, and $U$ is calculated from $F^T F = U^2$. Thus, $U$ is symmetric and positive-definite.

Exercise. For a given $F$, show that the polar decomposition yields unique $U$ and $R$.

Exercise. Give pictorial interpretation of the polar decomposition.

Lagrange strain. Define the Lagrange strain $E_{KL}$ by
\[ dl^2 - dL^2 = E_{KL} dX_K dX_L. \]

Recall
\[ dL^2 = dX_K dX_K, \]
and
\[ dl^2 = F_{ik} F_{il} dX_k dX_l. \]

We obtain that
\[ E_{KL} = \frac{1}{2} \left( F_{ik} F_{il} - \delta_{KL} \right), \]
where \( \delta_{KL} = 1 \) when \( K = L \), and \( \delta_{KL} = 0 \) when \( K \neq L \). The \( E_{KL} \) tensor is symmetric.

The three tensors are related as
\[ \mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{I}), \quad \mathbf{C} = \mathbf{U}^* \]

**Exercise.** The Lagrange strain forms a link between finite deformation and the infinitesimal deformation approximation. Let the displacement field be
\[ \mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}, \]
Show that the Lagrange strain is
\[ E_{kl} = \frac{1}{2} \left( \frac{\partial U_k}{\partial X_L} + \frac{\partial U_L}{\partial X_k} + \frac{\partial U_k}{\partial X_L} + \frac{\partial U_L}{\partial X_k} \right). \]

Thus, the Lagrange strain coincides with the strain obtained in the infinitesimal strain formulation if
\[ \frac{\partial U_l}{\partial X_K} \ll 1. \]

This in turn requires that all components of linear strain and rotation be small. However, even when all strains and rotations are small, we still need to balance force in the deformed state, as remarked before.

**Exercise.** We can relate the general definition of the Lagrange strain to that introduced in the beginning of the notes for a tensile bar. Consider a material element of line in a three dimensional body. In the reference state, the element is in the direction specified by a unit vector \( \mathbf{M} \). In the current state, the stretch of the element is \( \lambda \). We have defined the Lagrange strain in one dimension by
\[ \eta = \frac{\lambda^2 - 1}{2}. \]

Show that the Lagrange strain of the element in direction \( \mathbf{M} \) is given by
\[ \eta = E_{KL} M_K M_L. \]

**Exercise.** The tensor of Lagrange strain can also be used to calculate the engineering shear strain. Consider two material elements of line. In the reference state, the two elements are in two orthogonal directions \( \mathbf{M} \) and \( \mathbf{N} \). In the current state, the stretches of the two elements are \( \lambda_M \) and \( \lambda_N \), and the angle
between the two elements become $\frac{\pi}{2} - \gamma$. We have defined $\gamma$ as the engineering shear strain. Show that
\[
\sin \gamma = \frac{2E_{KL} M_{K} N_{L}}{\lambda_{M} \lambda_{N}}.
\]

Deformation of a material element of volume. Let $dV = dX_{1} dX_{2} dX_{3}$ be an element of volume in the reference state. After deformation, the line element $dX_{1}$ becomes $dx^{1}$, with components given by
\[
dx_{i}^{1} = F_{i} dX_{1}.
\]
We can write similar relations for $dx^{2}$ and $dx^{3}$. In the current state, the volume of the element is
\[
dv = (dx^{1} \times dx^{2}) \cdot dx^{3}.
\]
We can confirm that this expression is the same as
\[
dv = \det(F) dV.
\]

Deformation of a material element of area. Let an element of area be $NdA$ in the reference state, where $N$ is the unit vector normal to the element and $dA$ is the area of the element. After deformation, the element of area becomes $nda$ in the current state, where $n$ is the unit vector normal to the element and $da$ is the area of the element. An element of length $dX$ in the reference state becomes $dx$ in the current state. The relation between the volume in the reference state and the volume in the current state gives
\[
dx \cdot nda = \det(F) dX \cdot NdA,
\]
or
\[
dx_{k} F_{ik} n_{i} da = \det(F) dX_{k} N_{k} dA.
\]
This relation holds for arbitrary $dX$, so that
\[
F_{ik} n_{i} da = \det(F) N_{k} dA,
\]
or, in vector form,
\[
F^{T} da = \det(F) dA.
\]

The second Piola-Kirchhoff stress. Define the tensor of the second Piola-Kirchhoff stress, $S_{KL}$, by
\[
\frac{\text{work in the current state}}{\text{volume in the reference state}} = S_{KL} \delta E_{KL}.
\]
This expression defines a new measure of stress, $S_{KL}$. Because $E_{KL}$ is a symmetric tensor, $S_{KL}$ should also be symmetric.

Recall that we have also expressed the same work by $s_{ik} \delta F_{ik}$. Equating the two expressions for work,
Recall definition of the Lagrange strain,
\[ E_{KL} = \frac{1}{2} \left( F_{ik} F_{ik} - \delta_{KL} \right). \]
We obtain that
\[ \partial E_{KL} = \frac{1}{2} \left( F_{il} \partial F_{ik} + F_{ik} \partial F_{il} \right) \]
and
\[ s_{ik} = S_{KL} F_{il}. \]
For a hypoelastic material, the free energy is a function of the Lagrange strain, \( \Omega(E) \). The second Piola-Kirchhoff stress is work-conjugate to the Lagrange strain:
\[ S_{KL} = \frac{\partial \Omega(E)}{\partial E_{KL}}. \]

**Lagrangian vs. Eulerian formulations.** The above formulation uses the material coordinate \( X \) as the independent variable. The function \( x(X,t) \) gives the place occupied by the material particle \( X \) at time \( t \). This formulation is known as Lagrangian. The inverse function, \( X(x,t) \), tells us which material particle is at place \( x \) and time \( t \). The formulation using the spatial coordinate \( x \) is known as Eulerian. In the above, we have used the Eulerian formulation to study cavitation because we know the current configuration. In general, the current configuration is unknown, and we have used the Lagrangian formulation to state the general problem. In the following, we list a few results related to the Eulerian formulation.

**Time derivative of a function of material particle.** At time \( t \), a material particle \( X \) moves to position \( x(X,t) \). The velocity of the material particle is
\[ V(X,t) = \frac{\partial x(X,t)}{\partial t}. \]
If we would like to use \( x \) as the independent variable, we change the variable from \( X \) to \( x \) by using the function \( X(x,t) \), and then write
\[ v(x,t) = V(X,t). \]
Let \( G(X,t) \) be a function of material particle and time. For example, \( G \) can be the temperature of material particle \( X \) at time \( t \). The rate of change in temperature of the material particle is
\[ \frac{\partial G(X,t)}{\partial t}. \]
This rate is known as the material time derivative. We can calculate the material time derivative by an alternative approach. Change the variable from \( X \) to \( x \) by
using the function \( X(x,t) \), and write

\[
g(x,t) = G(X,t)
\]

Using chain rule, we obtain that

\[
\frac{\partial G(X,t)}{\partial t} = \frac{\partial g(x,t)}{\partial t} + \frac{\partial g(x,t)}{\partial x_i} \frac{\partial x_i}{\partial t}.
\]

Thus, we can calculate the substantial time rate from

\[
\frac{\partial G(X,t)}{\partial t} = \frac{\partial g(x,t)}{\partial t} + \frac{\partial g(x,t)}{\partial x_i} v_i(x,t).
\]

Let us apply the idea to the acceleration of a material particle. In the Lagrangian formulation, the acceleration is

\[
A(X,t) = \frac{\partial V(X,t)}{\partial t} = \frac{\partial^2 x(X,t)}{\partial t^2}.
\]

In the Eulerian formulation, the acceleration of a material particle is

\[
a_i(x,t) = \frac{\partial v_i(x,t)}{\partial t} + \frac{\partial v_i(x,t)}{\partial x_j} v_j(x,t).
\]

**Conservation of mass.** In the Lagrange formulation, the nominal mass density is defined by

\[
\rho_k = \frac{\text{mass in current state}}{\text{volume in reference state}}.
\]

That is, \( \rho_k dV \) is the mass of a material element of volume. A subscript is added here to remind us that the volume is in the reference state.

In the Eularian formulation, the true mass density is defined by

\[
\rho = \frac{\text{mass in current state}}{\text{volume in current state}}.
\]

That is, \( \rho dv \) is the mass of a spatial element of volume.

The two definitions of density are related as

\[
\rho_k dV = \rho dv,
\]

or

\[
\rho_k = \rho \det F.
\]

Conservation of mass requires that the mass of the material element of volume be time-independent. Thus, the nominal density can only vary with material particle, \( \rho_k(X) \), and is time-independent. By contrast, the true density is a function of both place and time, \( \rho(x,t) \). Conservation of mass requires that

\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x_i} \left[ \rho(x,t) v_i(x,t) \right] = 0.
\]

When the material is incompressible, \( \det F = 1 \), we obtain that

\[
\rho_k(X) = \rho(x,t).
\]
**True strain.** Recall the definition of the true strain in one dimension. When the length of a bar changes by a small amount from \( l(t) \) to \( l + \delta l \), the increment in the true strain is defined as

\[
\delta \varepsilon = \frac{\delta l}{l}.
\]

Now consider a body undergoing a deformation \( x(X, t) \). Imagine a small displacement \( \delta u(x) \) in the current state. Consider two nearby material particles in the body. When the body is in the reference state, the coordinates of the two particles are \( X \) and \( X + dX \). The vector \( dX \) is a material element of line. When the body is in the current state at time \( t \), the two material particles occupy places \( x(X, t) \) and \( x(X + dX, t) \), so that the vector between the two particles is

\[
dx = x(X + dX, t) - x(X, t)\]

Consider a material element of line. In the reference state, the element is \( dX = M dL \), where \( dL \) is the length of the element, and \( M \) is the unit vector in the direction of the element. In the current state, the element becomes \( dx = m d l \), where \( dl \) is the length of the element, and \( m \) is the unit vector in the direction of the element.

The length of a material element of line at time \( t \) is

\[
dl = dx_i dx_j
\]

Subject to the small displacement, the above equation becomes

\[
\delta l \delta (dl) = d(\delta u_i) dx_i.
\]

Note that

\[
d(\delta u_i) = \frac{\partial (\delta u_i)}{\partial x_j} dx_j
\]

and

\[
\delta \varepsilon = \frac{\delta (dl)}{dl}.
\]

Thus,

\[
\delta \varepsilon = \frac{\partial (\delta u_i)}{\partial x_j} m m_j
\]

This is the increment of the strain of the material element of line. Only the symmetric part

\[
\frac{1}{2} \left[ \frac{\partial (\delta u_i)}{\partial x_j} + \frac{\partial (\delta u_j)}{\partial x_i} \right]
\]

will affect the increment of the strain of the material element of line. This tensor generalizes the notion of the true strain, and is given the symbol:

\[
\delta e_{ij}(x, t) = \frac{1}{2} \left[ \frac{\partial (\delta u_i)}{\partial x_j} + \frac{\partial (\delta u_j)}{\partial x_i} \right].
\]
Rate of deformation. Let \( v(x,t) \) be the field of true velocity in the current state. The same line of reasoning shows that the tensor

\[
 d_{ij}(x,t) = \frac{1}{2} \left[ \frac{\partial v_i(x,t)}{\partial x_j} + \frac{\partial v_j(x,t)}{\partial x_i} \right]
\]

allows us to calculate the rate of change the length of material element of line, namely,

\[
 \frac{1}{dl(X,t)} \frac{\partial [dl(X,t)]}{\partial t} = d_{ij} m_i m_j.
\]

The tensor \( d_{ij} \) is known as the rate of deformation.

Force balance. The true stress obeys that

\[
 \frac{\partial \sigma_{ij}(x,t)}{\partial x_j} + b_i(x,t) = \rho(x,t) \left[ \frac{\partial v_i(x,t)}{\partial t} + \frac{\partial v_j(x,t)}{\partial x_j} v_j(x,t) \right],
\]

in the volume of the body, and

\[
 \sigma_{ij} n_j = t_i
\]
on the surface of the body. These are familiar equations used in fluid mechanics.

Relation between true stress and nominal stress. Recall the definition of the nominal stress

\[
 \frac{\text{work in the current state}}{\text{volume in the reference state}} = s_{ik} \partial F_{ik},
\]

and the definition of the true stress

\[
 \frac{\text{work in the current state}}{\text{volume in the current state}} = \sigma_{ij} \delta_{ij}.
\]

Equating the work on an element of material, we obtain that

\[
 \sigma_{ij} \delta_{ij} dV = s_{ik} \partial F_{ik} dV.
\]

Because the true stress is a symmetric tensor, we obtain that

\[
 \sigma_{ij} \delta_{ij} = \sigma_{ij} \frac{\partial \delta u_j(x)}{\partial x_j}.
\]

Also note that

\[
 \partial F_{ik} = \frac{\partial \delta U_i(X)}{\partial X_k} = \frac{\partial \delta u_i(x)}{\partial x_j} \frac{\partial x_j}{\partial X_k} = \frac{\partial \delta u_i(x)}{\partial x_j} F_{jk}(X,t),
\]

and that

\[
 dv / dV = \det(F).
\]

Insisting that the two expressions for the work be the same for all motion, we obtain that
\[
\sigma_{ij} = \frac{s_{jk} F_{jk}}{\det(F)}.
\]

**Newtonian fluids.** The components of the stress relate to the components of the rate of deformation as

\[
\sigma_{ij} = \eta \left[ \frac{\partial v_i(x,t)}{\partial x_j} + \frac{\partial v_j(x,t)}{\partial x_i} \right] - p \delta_{ij},
\]

where \( \eta \) is the viscosity. The material is assumed to be compressible:

\[
\frac{\partial v_i(x,t)}{\partial x_i} = 0.
\]

**References**


匡震邦, 非线性连续介质力学, 上海交通大学出版社, 2002. Written by the man who taught me the subject in 1985 in Xian Jiaotong University.

