

**Steady State Bending Motion of a Prismatic Beam
Subject to a Time Periodic Point Force**
by
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Theory

We consider a uniform prismatic beam that is in steady state motion being forced by a time periodic point force. In standard notation where $y(x,t)$ is the vertical displacement, x is the distance along the neutral axis of the beam from $x=0$ at the left end to $x=L$ at the right end, and t =time, the equation of motion is

$$(1) \quad EI \frac{\partial^4 y}{\partial x^4} - \rho A \frac{\partial^2 y}{\partial t^2} = F \delta(x - x_0) e^{i\omega t} .$$

This equation is supplemented by two boundary conditions (BC's) at each of the ends $x=0$ and $x=L$. The following types of BC's can be applied:

- | | |
|------------------------|-------------------------------------------------------|
| (i) Pinned: | $y=0$ and $y''=0$ |
| (ii) Fixed: | $y=0$ and $y'=0$ |
| (iii) Free: | $y''=0$ and $y'''=0$ |
| (iv) Linear spring: | $y''=0$ and $EIy''' + ksy=0$ |
| (v) Torsion spring: | $y=0$ and $EIy'' - kty'=0$ |
| (vi) Point mass: | $y''=0$ and $EIy''' + m\ddot{y}=0$ |
| (vii) Linear dashpot: | $y''=0$ and $EIy''' + cs\dot{y}=0$ |
| (viii) Torsion damper: | $y=0$ and $EIy'' - ct\dot{y}'=0$ |
| (ix) Finite mass: | $EIy'' - Im\ddot{y}''=0$ and $EIy''' + m\dot{y}'''=0$ |

Here, as usual, the primes after the variables denote differentiation wrt x and the dot above the variable denotes differentiation wrt time. The coefficients are the usual ones, but note that Im is the moment of inertia of a mass rigidly attached to the end of the beam. The moment of inertia being about the point of attachment.

Note: The 2nd parts of BC's (iv), (vi) and (vii) can be combined to give a mass/spring/damper system at an end. Similarly the 2nd parts of (v), (viii) and the first part of (ix) can be combined to give a rotational mass/spring/damper system at an end.

We will only scale the lengths (y and x) by the beam length L and consider all lengths thusly scaled. Furthermore we separate variables so that

$$(2) \quad y(x, t) = Y(x) e^{i\omega t} .$$

In which case equation (1) and the boundary conditions can be written as

$$(3) \quad Y'''' + \beta^4 Y = \hat{F} \delta(x - x_0)$$

- | | |
|-------------|----------------------|
| (i) Pinned: | $Y=0$ and $Y''=0$ |
| (ii) Fixed: | $Y=0$ and $Y'=0$ |
| (iii) Free: | $Y''=0$ and $Y'''=0$ |

- (iv) Linear spring: $Y''=0$ and $Y'''+ks*Y=0$
(v) Torsion spring: $Y=0$ and $Y''-kt*Y'=0$
(vi) Point mass: $Y''=0$ and $Y'''+\omega^2m*Y=0$
(vii) Linear dashpot: $Y''=0$ and $Y'''+i\omega cs*Y=0$
(viii) Torsion damper: $Y=0$ and $Y''-i\omega ct*Y'=0$
(ix) Finite mass: $Y''+\omega^2Im*Y''=0$ and $Y'''+\omega^2m*Y=0$

where now Y , $x \in [0,1]$, and x_0 are dimensionless and the coefficients in all these equations are defined as

$$\begin{aligned} ks^* &= ksL^3/EI & kt^* &= ktL/EI & m^* &= mL^3/EI \\ cs^* &= cs/L^3/EI & ct^* &= ctL/EI & Im^* &= ImL/EI \\ \hat{F} &= FL^3/EI & \beta^4 &= (EI/\rho AL^2)\omega^2 \end{aligned}$$

We can write any of the BC's in a compact way (the way they are programmed).

$$(4) \quad [a, b, c, d] \begin{bmatrix} Y \\ Y' \\ Y'' \\ Y''' \end{bmatrix} = 0$$

where the column vector is evaluated at $x=0$ or $x=1$, whichever case is desired and the entries in the row vector, a, b, c, d are values corresponding to the boundary condition desired.

BC's (iii) $[0,0,1,0]$ and $[0,0,0,1]$

BC's (vi) $[0,0,1,0]$ and $[-\omega^2m^*,0,0,1]$

BC's (vi) combined with the 2nd of (iv) and 2nd of (vii) to give a mass/spring dashpot at the end

$$[0,0,1,0] \text{ and } [-\omega^2m^*+i\omega cs^*+ks,0,0,1].$$

Equation (3) is basically the equation for the Green's function. Since the solution to the homogeneous equation (3) is

$$Y(x) = A \sin(\beta x) + B \cos(\beta x) + C \sinh(\beta x) + D \cosh(\beta x)$$

the solution of equation (3) with the delta function forcing (the Green's function) is

$$(5) \quad Y(x) = \begin{cases} A_1 \sin(\beta x) + B_1 \cos(\beta x) + C_1 \sinh(\beta x) + D_1 \cosh(\beta x) & \text{for } x < x_0 \\ A_2 \sin(\beta x) + B_2 \cos(\beta x) + C_2 \sinh(\beta x) + D_2 \cosh(\beta x) & \text{for } x > x_0 \end{cases}.$$

The coefficients are chosen so that 4 boundary conditions are satisfied and the following matching conditions at $x=x_0$ are satisfied. Let $Y_1(x)$ be the solution for $x < x_0$ and $Y_2(x)$ be the solution for $x > x_0$. Then

$$\begin{aligned}
Y_2(x_0) - Y_1(x_0) &= 0 \\
Y_2'(x_0) - Y_1'(x_0) &= 0 \\
Y_2''(x_0) - Y_1''(x_0) &= 0 \\
Y_2'''(x_0) - Y_1'''(x_0) &= \hat{F}
\end{aligned}$$

These matching conditions and the four BC's are sufficient to determine the 8 constants in solution (5).

Solution method to be programmed:

Let $Q(x)$ be defined as

$$(6) \quad Q(x) = \begin{bmatrix} \sin(\beta x) & \cos(\beta x) & \sinh(\beta x) & \cosh(\beta x) \\ \beta \cos(\beta x) & -\beta \sin(\beta x) & \beta \cosh(\beta x) & \beta \sinh(\beta x) \\ -\beta^2 \sin(\beta x) & -\beta^2 \cos(\beta x) & \beta^2 \sinh(\beta x) & \beta^2 \cosh(\beta x) \\ -\beta^3 \cos(\beta x) & \beta^3 \sin(\beta x) & \beta^3 \cosh(\beta x) & \beta^3 \sinh(\beta x) \end{bmatrix}$$

Note that this is actually the Wronskian matrix of the differential equation.

Let the coefficients in the boundary condition row vectors $[a,b,c,d]$ be designated as follows:

At $x=0$, $\vec{p} = [a_1, b_1, c_1, d_1]$ and $\vec{q} = [a_2, b_2, c_2, d_2]$.

At $x=1$, $\vec{r} = [a_1, b_1, c_1, d_1]$ and $\vec{s} = [a_2, b_2, c_2, d_2]$.

Furthermore let

$$\vec{C} = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} \vec{C}_0 \\ \vec{C}_1 \end{bmatrix} \quad \text{and} \quad \vec{R} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hat{F} \\ 0 \\ 0 \end{bmatrix}$$

be the unknown coefficient vector and the right hand side vector (forcing vector). Then we obtain \vec{C} by solving the system of equations

$$(7) \quad \begin{bmatrix} \begin{bmatrix} \vec{p} \\ \vec{q} \end{bmatrix} Q(0) & Z_{2,4} \\ -Q(x_0) & Q(x_0) \\ Z_{2,4} & \begin{bmatrix} \vec{r} \\ \vec{s} \end{bmatrix} Q(1) \end{bmatrix} \vec{C} = \vec{R}$$

where $Z_{2,4}$ is a 2 row and 4 column matrix of 0's and both

$$\begin{bmatrix} \vec{p} \\ \vec{q} \end{bmatrix} Q(0) \text{ and } \begin{bmatrix} \vec{r} \\ \vec{s} \end{bmatrix} Q(1)$$

are each 2 row and 4 column matrices. Equation (7) is an 8 by 8 complex system of algebraic equations. The system is complex only if there are damping end conditions. Nevertheless, Matlab solves the system the same way whether it is real or complex. The final solution is

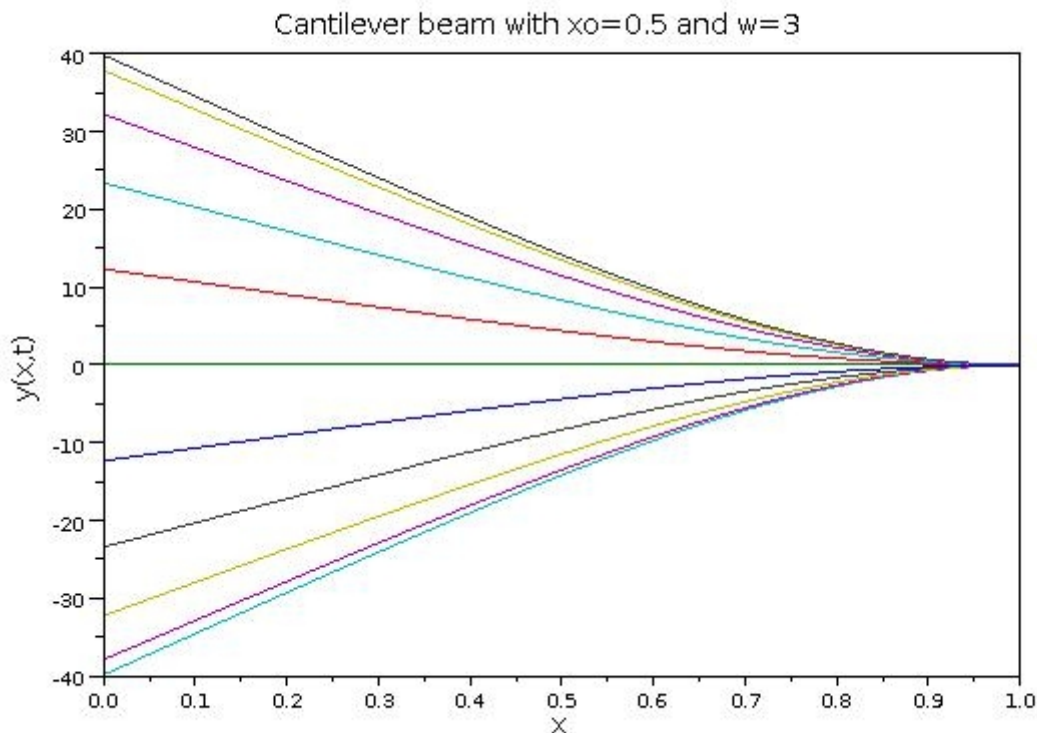
$$(8) \quad y(x,t) = \text{Real} \left(\begin{bmatrix} \sin(\beta x) & \cos(\beta x) & \sinh(\beta x) & \cosh(\beta x); & x < x_0 \\ \sin(\beta x) & \cos(\beta x) & \sinh(\beta x) & \cosh(\beta x); & x > x_0 \end{bmatrix} \begin{bmatrix} \vec{C}_0 \\ \vec{C}_1 \end{bmatrix} \exp(i\omega t) \right) .$$

Examples

These examples will be run at various locations x_0 and various frequencies w of the load. We assume the magnitude of the point load is $\hat{F}=100$ in all cases. In all the graphs, the various color curves are the beam positions at different instances of time over one cycle.

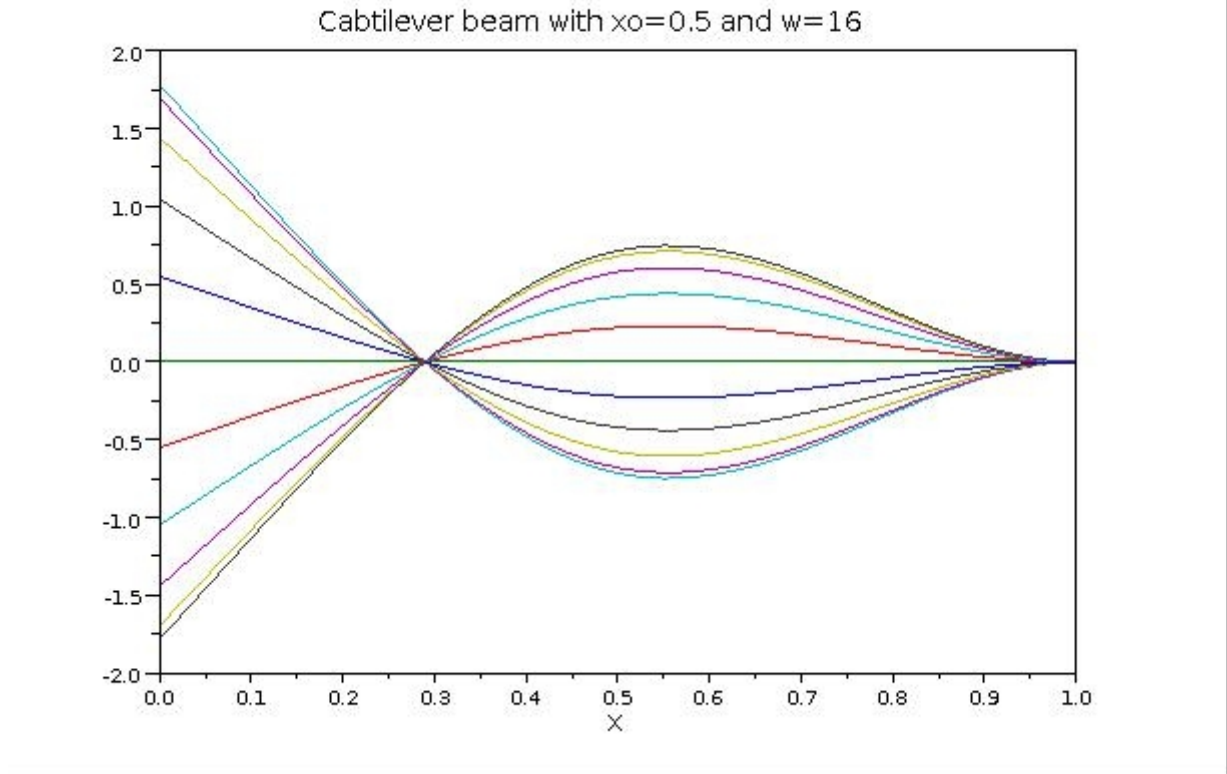
(i) A cantilevered beam from the right end:

$$\vec{p}=[0,0,1,0] \quad \vec{q}=[0,0,0,1] \quad \vec{r}=[1,0,0,0] \quad \vec{s}=[0,1,0,0]$$



The first mode free vibration of a cantilever beam has a frequency of $w=3.516$. The second mode free vibration frequency is $w= 22.034$ and the third mode frequency is $w= 61.7$. Thus the first case above excites the first mode. The second case below excites the second mode since the frequency is nearer the second mode of free vibrations. The different modes of vibration are excited by varying

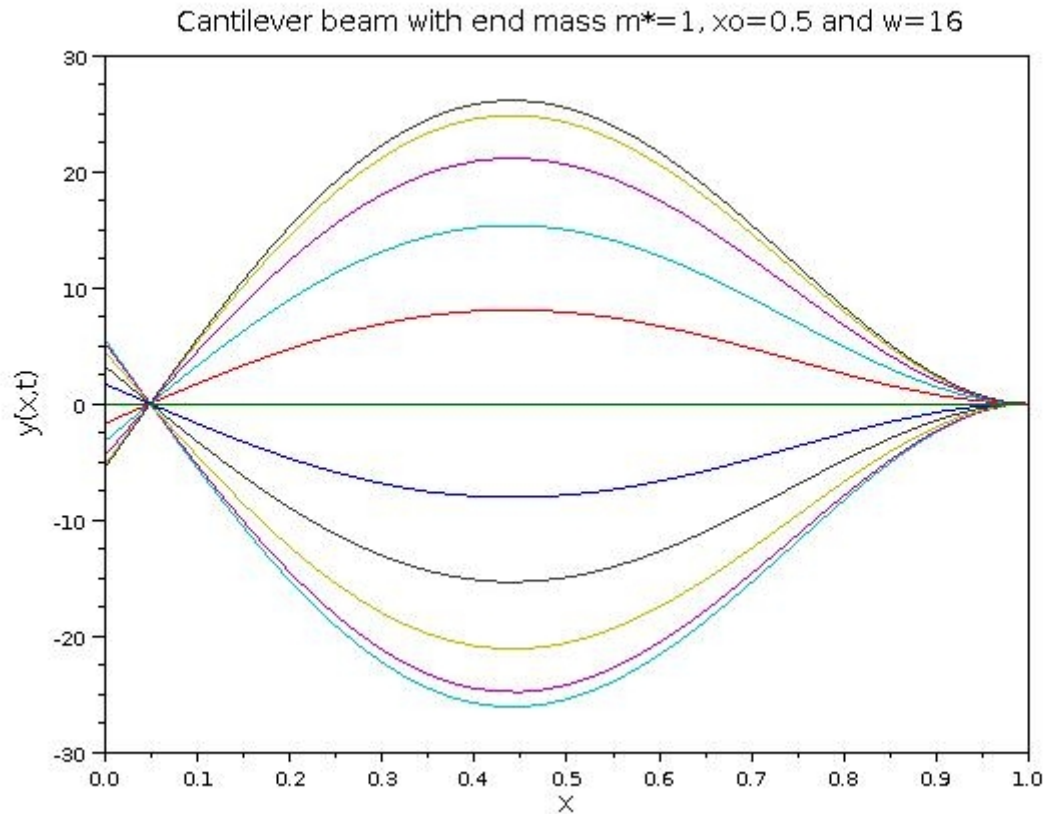
the location as well as frequency of forcing. We leave such illustrations to the numerical experiments of the reader using the program provided below.



(ii) The same cantilevered beam with various values of mass m^* at the left free end:

$$\vec{p}=[0,0,1,0] \quad \vec{q}=[-w^2 m^*,0,0,1] \quad \vec{r}=[1,0,0,0] \quad \vec{s}=[0,1,0,0]$$

Here we see the severe motion constraining effect of the mass at the free end of the beam. It almost has the effect of a pin constraint at the left end of the beam. The first mode of vibration of a beam pinned at one end and cantilevered at the other is 9.87. The second mode frequency is 39.48. Thus the excitation frequency of 16 does excite a mode close to the first mode of the pin-cantilever beam.

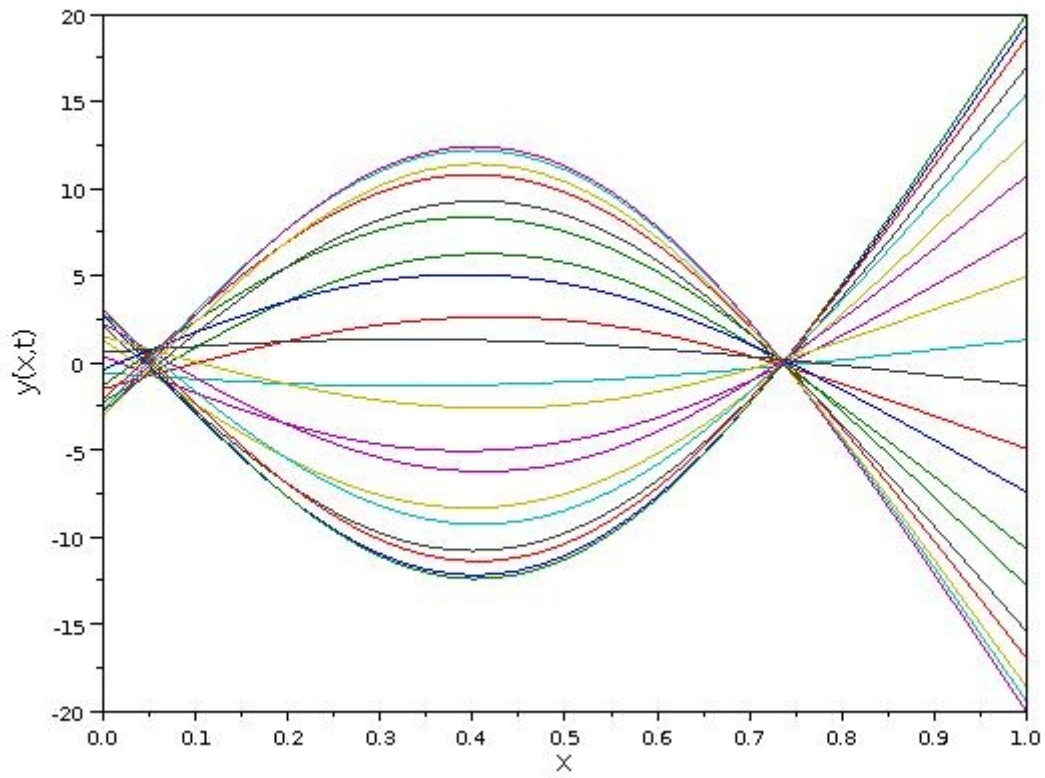


(iii) A free-free beam with a mass/spring/damper at the left free end:

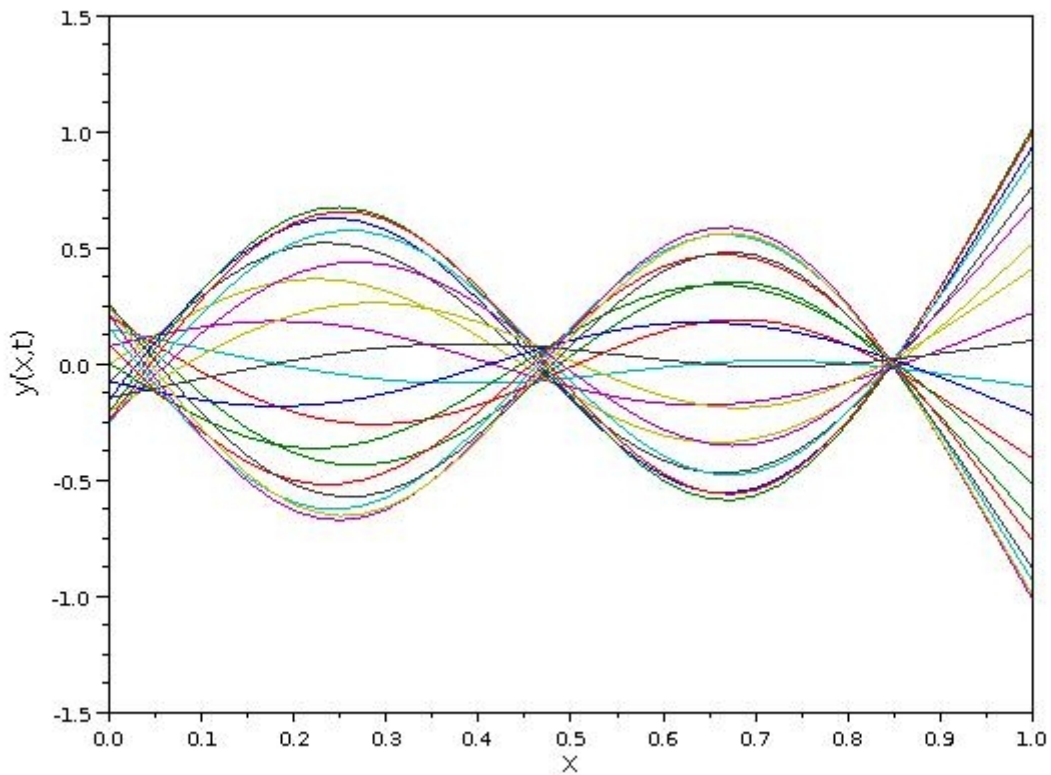
$$\vec{p}=[0,0,1,0] \quad \vec{q}=[-w^2 m^* + iwcs^* + ks^*, 0, 0, 1] \quad \vec{r}=[0,0,1,0] \quad \vec{s}=[0,0,0,1]$$

The next two cases show the forced vibration of a free-free ended beam but including a mass/spring/damper system at the left end. For reference we note that the frequency of vibration of a free-free beam with nothing at the ends is 22.37 for the 1st mode (two nodes) and 61.67 for the second mode (3 nodes). It is interesting to note the effects of the mass/spring/damper system at the left end is again to restrict the motion at the left end, but also the damper causes phase shifts of the modes of motion as indicated by the changes in the position of the nodes near the left end.

Free-Free beam with $m^*=1$, $cs^*=5$, $ks=1$, $x_0=0.5$ $w=16$



Free-Free beam with $m^*=0.25$, $cs^*=10$, $ks^*=0.25$, $x_0=0.75$, $w=50$



Computer Program

Below is a Scilab program implementation of this.

```
clear
// Prismatic beam forced motion by harmonic point load
//
// 1. Input
//
function Q=bmatrix(b,x)
sbx=sin(b*x); cbx=cos(b*x); shbx=sinh(b*x); chbx=cosh(b*x);
Q=[sbx,cbx,shbx,cbx;...
   cbx,-sbx,shbx,shbx;...
   -sbx,-cbx,shbx,shbx;...
   -cbx,sbx,shbx,shbx];
Q(2,:)=b*Q(2,:); Q(3,:)=Q(3,:)*b^2; Q(4,:)=Q(4,:)*b^3;
endfunction
//
i=sqrt(-1);
F=100;           // Force magnitude
xo=0.75;        // Location of the force
w=40;          // Forcing frequency
N=41;          // # of beam grid points
M=21;          // # of time steps
// Beam data
// See notes for setting up these parameters and the BC coefficients
//
a=1;           // The beam parameter a in the notes ( $a^2=EI/\rho AL^2$ )
// See notes for the boundary coefficients below:
// left end BC's
al1=1; bl1=0; cl1=0; dl1=0;
al2=0; bl2=1; cl2=0; dl2=0;
// right end BC's
ar1=0; br1=0; cr1=1; dr1=0;
ar2=i*w*10-w^2; br2=0; cr2=0; dr2=1;
//
////////////////////////////////////
//
// 2. Data setup
go1=[al1,bl1,cl1,dl1]; go2=[al2,bl2,cl2,dl2];
g11=[ar1,br1,cr1,dr1]; g12=[ar2,br2,cr2,dr2];
b=sqrt(w/a);
x=linspace(0,1,N); t=linspace(0,2*%pi/w,M);
// Obtain the basic Q(x) matrix for each boundaryGeneral
Q0=bmatrix(b,0); Q1=bmatrix(b,1);
Qxo=bmatrix(b,xo);
// Construct the left and write boundary matrices.
LB=[go1;go2]*Q0; RB=[g11;g12]*Q1;
// Construction of the system matrix
S=[LB,zeros(2,4);-Qxo,Qxo;zeros(2,4),RB];
// Construction of the right side vector
```



```

R=[0,0,0,0,0,F,0,0]';
//
//      3. Solve for the solution constants
//
X=S\R;
//
//      4. Post processing
//
// Obtain the mode shape
A=X(1:4); B=X(5:8);
for j=1:N
    if x(j)<=xo
        y(j)=[sin(b*x(j)),cos(b*x(j)),sinh(b*x(j)),cosh(b*x(j))]*A;
    else
        y(j)=[sin(b*x(j)),cos(b*x(j)),sinh(b*x(j)),cosh(b*x(j))]*B;
    end
end
// Obtain the time variation
for k=1:M
    z(k,:)=real(y'*exp(i*w*t(k)));
end
//      5. Output
plot(x,z)

```