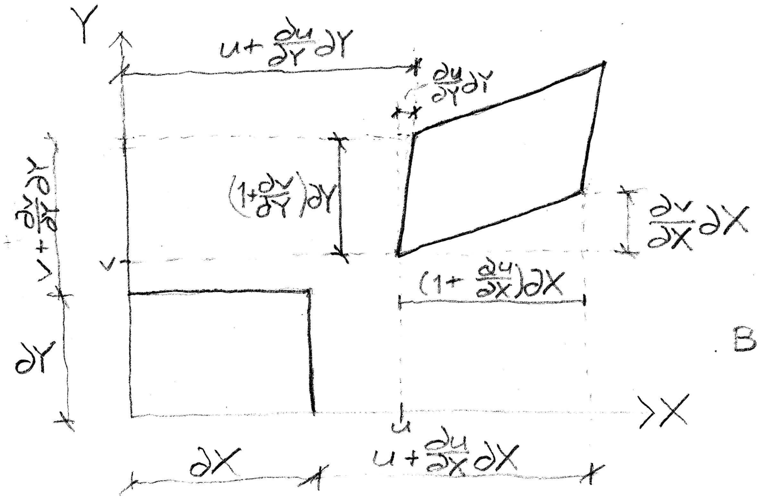

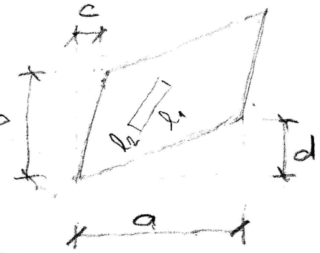

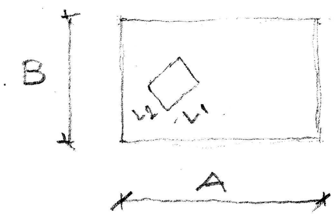

AREA $= AB$

area $= (a+c)(b+d) - (ad+bc+2cd)$

$= ab - cd$

$$\frac{l_1 l_2}{L_1 L_2} \doteq \frac{\text{area}}{\text{AREA}} \qquad \begin{pmatrix}\varepsilon_X & \gamma \\ \gamma & \varepsilon_Y\end{pmatrix} \sim \begin{pmatrix}\varepsilon_1 & 0 \\ 0 & \varepsilon_2\end{pmatrix}$$

$$\varepsilon_X + \varepsilon_Y = \varepsilon_1 + \varepsilon_2$$

$$\varepsilon_X = \frac{1}{2}\left(\frac{a^2+d^2}{A^2} - 1\right), \quad \varepsilon_Y = \frac{1}{2}\left(\frac{b^2+c^2}{B^2} - 1\right), \quad \varepsilon_X \varepsilon_Y - \gamma^2 = \frac{1}{2}\left(\frac{l_1^2}{L_1^2} - 1\right)\frac{1}{2}\left(\frac{l_2^2}{L_2^2} - 1\right)$$

$$= \frac{1}{4}\left(\frac{l_1^2 l_2^2}{L_1^2 L_2^2} - \frac{l_1^2}{L_1^2} - \frac{l_2^2}{L_2^2} + 1\right)$$

$$= \frac{1}{4}\left(\underbrace{\left(\frac{l_1 l_2}{L_1 L_2}\right)^2}_{\left(\frac{\text{area}}{\text{AREA}}\right)^2} - 1\right) - \frac{1}{2}\underbrace{\left[\frac{1}{2}\left(\frac{l_1^2}{L_1^2} - 1\right) + \frac{1}{2}\left(\frac{l_2^2}{L_2^2} - 1\right)\right]}_{\varepsilon_X + \varepsilon_Y}$$

$$\therefore \gamma^2 = \varepsilon_X \varepsilon_Y + \frac{1}{2}(\varepsilon_X + \varepsilon_Y) - \frac{1}{4}\left(\left(\frac{\text{area}}{\text{AREA}}\right)^2 - 1\right)$$

$$= \frac{1}{4}\left(\frac{a^2b^2 + a^2c^2 + d^2b^2 + d^2c^2}{A^2B^2} - \frac{a^2+d^2}{A^2} - \frac{b^2+c^2}{B^2} + 1 + \frac{a^2+d^2}{A^2} + \frac{b^2+c^2}{B^2} - 2 - \frac{(ab-cd)^2}{A^2B^2} + 1\right)$$

$$= \frac{1}{4}\left(\frac{a^2c^2 + d^2b^2 + 2abcd}{A^2B^2}\right) = \frac{(ac+bd)^2}{4A^2B^2} \Rightarrow \gamma = \pm\frac{ac+bd}{2AB}$$

$$\pm\frac{ac+bd}{2AB} = \pm\frac{1}{2}\left[\left(1+\frac{\partial u}{\partial X}\right)\frac{\partial u}{\partial Y} + \left(1+\frac{\partial v}{\partial Y}\right)\frac{\partial v}{\partial X}\right], \quad \gamma \ll 1 \Rightarrow \gamma \approx \oplus\frac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right)$$

$$\varepsilon_X = \frac{1}{2}\left[\left(1+\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 - 1\right] = \frac{\partial u}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right]$$

$$\varepsilon_Y = \frac{1}{2}\left[\left(1+\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial u}{\partial Y}\right)^2 - 1\right] = \frac{\partial v}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2\right]$$

$$\boxed{\begin{aligned} \varepsilon_X &= \frac{\partial u}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right] \\ \varepsilon_Y &= \frac{\partial v}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2\right] \\ \gamma &= \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right)}_{\text{linear part}} + \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Y}\right)}_{\text{quadratic part}} \end{aligned}}$$

GREEN-LAGRANGE STRAIN

Special Case: Pure Rotation 

$$1 + \frac{\partial u}{\partial X} = \cos\theta, \quad \frac{\partial v}{\partial X} = \sin\theta, \quad 1 + \frac{\partial v}{\partial Y} = \cos\theta, \quad \frac{\partial u}{\partial Y} = -\sin\theta$$

$$\varepsilon_X = \cos\theta - 1 + \frac{1}{2}\left[\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta\right]$$

$$\varepsilon_Y = \cos\theta - 1 + \frac{1}{2}\left[\sin^2\theta + \cos^2\theta - 2\cos\theta + 1\right]$$

$$\gamma = 0 + \frac{1}{2}\left[(\cos\theta - 1)(-\sin\theta) + (\sin\theta)(\cos\theta - 1)\right]$$

$$\varepsilon_X = \cos\theta - 1 + [1 - \cos\theta] = 0 \checkmark$$

$$\varepsilon_Y = \cos\theta - 1 + [1 - \cos\theta] = 0 \checkmark$$

$$\gamma = \underbrace{0}_{\text{linear part}} + [\ 0\ ] = 0 \checkmark$$

linear part $\Rightarrow$ fictitious ε (and σ)

D. Cooper
9.30.09