Couple stress theory for polar solids

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Abstract

The existing couple stress theory for polar media suffers from an indeterminacy of the spherical part of the couple-stress tensor, which limits significantly its applicability in the study of micro and nanoscale mechanics. Here we rely on concepts from virtual work, along with some kinematical considerations, to establish a consistent polar theory for solids that resolves all of the indeterminacies by recognizing the character of the couple stress tensor. We then develop the corresponding theory of small deformations in elastic bodies, including the energy and constitutive relations, displacement formulations, the uniqueness theorem for the corresponding boundary value problem and the reciprocal theorem for linear elasticity theory. Next, we consider the more restrictive case of isotropic materials and present general solutions for two-dimensional problems based on stress functions and for problems of anti-plane deformation. Finally, we examine several additional elementary boundary value problems within this consistent theory of polar elasticity.

1. Introduction

Classical first gradient approaches in continuum mechanics do not address the size-dependency that is observed in smaller scales. Consequently, a number of theories that include higher gradients of deformation have been proposed to capture, at least partially, size-effects at the nano-scale. Additionally, consideration of the second gradient of deformation leads naturally to the introduction of the concept of couple-stresses. Thus, in the current form of these theories, the material continuum may respond to body and surface couples, as well as spin inertia for dynamical problems.

The existence of couple-stress in materials was originally postulated by Voigt (1887). However, Cosserat and Cosserat (1909) were the first to develop a mathematical model to analyze materials with couple stresses. The idea was revived and generalized much later by Toupin (1962), Mindlin and Tiersten (1962), Mindlin (1964), Koiter (1964), Nowacki (1986) and others. In these developments, the gradient of the rotation vector, as a curvature tensor, has been recognized as the effect of the second gradient of deformation in polar materials. Unfortunately, there are some difficulties with the present formulations. Perhaps the most disturbing troubles are the indeterminacy of the spherical part of the couple-stress tensor and the appearance of the body couple in the constitutive relation for the force-stress tensor (Mindlin and Tiersten, 1962).

Here we develop a consistent couple stress theory for polar media and organize the current paper in the following manner. In Section 2, we present stresses, couple stresses and the equilibrium equations per the usual definitions in the existing couple stress literature. Based on purely kinematical considerations as provided in Section 3, we first suggest the mean curvature tensor as the measure of deformation compatible with the couple stress tensor for the infinitesimal theory. Then, by using the virtual work formulation of Section 4, we demonstrate that in couple stress materials, body couples must be transformed to an equivalent body force system. More importantly, based on resolving properly the boundary conditions, we show that the couple-stress tensor is skew-symmetric and, thus, completely determinate. This also confirms the mean curvature tensor as the fundamental deformation measure, energetically conjugate to the couple stress tensor. Afterwards, in Section 5, the general theory of small deformation polar elasticity is developed. The constitutive and equilibrium equations for a linear elastic material also are derived under the assumption of infinitesimal deformations in Section 6, along with the uniqueness theorem for well-posed boundary value problems and the reciprocal theorem. Section 7 provides the general solution based on stress functions for two-dimensional infinitesimal linear polar elasticity, while the corresponding anti-plane deformation problem is examined in Section 8. Section 9 presents solutions for several elementary problems in polar elasticity. Finally, Section 10 contains a summary and some general conclusions.

2. Stresses and equilibrium

For a polar material, it is assumed that the transfer of the interaction in the current configuration occurs between two particles of the body through a surface element dS with unit normal vector n_i by means of a force vector $t_i^{(n)}dS$ and a moment vector $m_i^{(n)}dS$, where $t_i^{(n)}$ and $m_i^{(n)}$ are force and couple traction vectors. Surface forces and couples are then represented by generally non-symmetric force-stress σ_{ji} and couple-stress μ_{ji} tensors, where

$$t_i^{(n)} = \sigma_{ji} n_j \tag{1}$$

$$m_i^{(n)} = \mu_{ji} n_j \tag{2}$$

Consider an arbitrary part of the material continuum occupying a volume V enclosed by boundary surface S as the current configuration. Under quasistatic conditions, the linear and angular balance equations for this part of the body are

$$\int_S t_i^{(n)} dS + \int_V F_i \, dV = 0 \tag{3}$$

$$\int_S \left[\varepsilon_{ijk} x_j t_k^{(n)} + m_i^{(n)} \right] dS + \int_V \left[\varepsilon_{ijk} x_j F_k + C_i \right] dV = 0 \tag{4}$$

where F_i and C_i are the body force and the body couple per unit volume of the body, respectively. Here ε_{ijk} is the permutation tensor or Levi-Civita symbol.

By using the relations (1) and (2), along with the divergence theorem, and noticing the arbitrariness of volume V, we finally obtain the differential form of the equilibrium equations, for the usual couple stress theory, as

$$\sigma_{ji,j} + F_i = 0 \tag{5}$$

$$\mu_{ji,j} + \varepsilon_{ijk} \sigma_{jk} + C_i = 0 \tag{6}$$

where the comma denotes differentiation with respect to the spatial variables.

3. Kinematics

Here we consider the kinematics of a polar continuum under the assumptions of infinitesimal deformation. In Cartesian coordinates, we define u_i to represent the displacement field of the continuum material. Consider the neighboring points P and Q with position vectors x_i and $x_i + dx_i$ in the reference configuration. The relative displacement of point Q with respect to P is

$$du_i = u_{i,j} dx_j \tag{7}$$

where $u_{i,j}$ is the displacement gradient tensor at point P. As we know, although this tensor is important in analysis of deformation, it is not itself a suitable measure of deformation. This tensor can be decomposed into symmetric and skew-symmetric parts

$$u_{i,j} = e_{ij} + \omega_{ij} \tag{8}$$

where

$$e_{ij} = u_{(i,j)} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{9}$$

$$\omega_{ij} = u_{[i,j]} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{10}$$

Notice that here we have introduced parentheses surrounding a pair of indices to denote the symmetric part of a second order tensor, whereas square brackets are associated with the skew-symmetric part. Of course, in (9) and (10), the tensors e_{ij} and ω_{ij} are the small deformation strain and rotation tensor, respectively. The rotation vector ω_i dual to the rotation tensor ω_{ij} is defined by

$$\omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{kj} = \frac{1}{2}\varepsilon_{ijk}u_{k,j} \tag{11a}$$

which in vectorial form is written

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u} \tag{11b}$$

Alternatively, this rotation vector is related to the rotation tensor through

$$\omega_{ji} = \varepsilon_{ijk}\omega_k \tag{12}$$

which shows

$$\omega_1 = -\omega_{23}, \quad \omega_2 = \omega_{13}, \quad \omega_3 = -\omega_{12} \tag{13}$$

Therefore, the relative displacement is decomposed into

$$du_i = du_i^{(1)} + du_i^{(2)} \tag{14}$$

where

$$du_i^{(1)} = e_{ij} dx_j \tag{15}$$

$$du_i^{(2)} = \omega_{ij} dx_j \tag{16}$$

Then, ω_{ij} is seen to generate a rigid-like rotation of element dx_i about point *P*, where

$$du_i^{(2)} dx_i = \omega_{ij} dx_i dx_j = 0 \tag{17}$$

Since ω_{ij} does not contribute to the elongation or contraction of element dx_i, it cannot appear in a tensor measuring material stretches. Therefore, as we know, the symmetric strain tensor e_{ij} is the suitable measure of deformation in classical infinitesimal theories, such as Cauchy elasticity.

In couple stress theory, we expect to have an additional tensor measuring the curvature of the arbitrary fiber element dx_i. To find this tensor, we consider the field of rotation vector ω_i. The relative rotation of two neighboring points *P* and *Q* is given by

$$d\omega_i = \omega_{i,j} dx_j \tag{18}$$

where the tensor $\omega_{i,j}$ is the gradient of the rotation vector at point *P*. It is seen that the components $\omega_{1,1}$, $\omega_{2,2}$ and $\omega_{3,3}$ represent the torsion of the fibers along corresponding coordinate directions x_1, x_2 and x_3, respectively, at point *P*. The off-diagonal components represent the curvature of these fibers in planes parallel to coordinate planes. For example, $\omega_{1,2}$ is the curvature of a fiber element in the x_2 direction in a plane parallel to the x_2x_3 plane, while $\omega_{2,1}$ is the curvature of a fiber element in the x_1 direction in a plane parallel to the x_1x_3 plane.

The suitable measure of curvature must be a tensor measuring pure curvature of an arbitrary element dx_i. Therefore, in this tensor, the components $\omega_{1,1}$, $\omega_{2,2}$ and $\omega_{3,3}$ cannot appear. However, simply deleting these components from the tensor $\omega_{i,j}$ does not produce a tensor. Consequently, we expect that the required tensor is the skew-symmetric part of $\omega_{i,j}$. By decomposing the tensor $\omega_{i,j}$ into symmetric and skew-symmetric parts, we obtain

$$\omega_{i,j} = \chi_{ij} + \kappa_{ij} \tag{19}$$

where

$$\chi_{ij} = \omega_{(i,j)} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right) \tag{20}$$

$$\kappa_{ij} = \omega_{[i,j]} = \frac{1}{2}\left(\omega_{i,j} - \omega_{j,i}\right) \tag{21}$$

The symmetric tensor χ_{ij} results from applying the strain operator to the rotation vector, while the tensor κ_{ij} is the rotation of the rotation vector at point *P*. From (20),

$$\chi_{11} = \omega_{1,1}, \quad \chi_{22} = \omega_{2,2}, \quad \chi_{33} = \omega_{3,3} \tag{22}$$

and

$$\chi_{12} = \chi_{21} = \frac{1}{2}\left(\omega_{1,2} + \omega_{2,1}\right) \tag{23a}$$

$$\chi_{23} = \chi_{32} = \frac{1}{2}\left(\omega_{2,3} + \omega_{3,2}\right) \tag{23b}$$

$$\chi_{13} = \chi_{31} = \frac{1}{2}\left(\omega_{1,3} + \omega_{3,1}\right) \tag{23c}$$

The diagonal elements χ_{11}, χ_{22} and χ_{33} defined in (22) represent pure torsion of fibers along the x_1, x_2 and x_3 directions, respectively, as mentioned above. On the other hand, from careful examination of (23), we find that χ_{12}, χ_{23} and χ_{13} measure the deviation from sphericity (Hamilton, 1866) of deforming planes parallel to x_1x_2, x_2x_3 and x_1x_3, respectively. Furthermore, we may recognize that this symmetric χ_{ij} tensor must have real principal values, representing the pure twists along the principal directions. Thus, we refer to χ_{ij} as the torsion tensor and we expect that this tensor will not contribute as a fundamental measure of deformation in a polar material. Instead, we anticipate that the fundamental curvature tensor is the skew-symmetric rotation of rotation tensor κ_{ij}. This will be confirmed in the next section through consideration of couple stresses and virtual work.

We also may arrive at this outcome by noticing that only the part of $d\omega_i$ that is normal to element dx_i produces pure curvature. Therefore, by decomposing $d\omega_i$ into

$$d\omega_i = d\omega_i^{(1)} + d\omega_i^{(2)} \tag{24}$$

where

$$d\omega_i^{(1)} = \chi_{ij} dx_j \tag{25}$$

$$d\omega_i^{(2)} = \kappa_{ij} dx_j \tag{26}$$

we notice

$$d\omega_i^{(2)} dx_i = \kappa_{ij} dx_i dx_j = 0 \tag{27}$$

This shows that $d\omega_i^{(2)}$ is the component of $d\omega_i$ normal to dx_i. Therefore, the tensor κ_{ij} seems to be the suitable curvature tensor, which is represented by

$$\left[\kappa_{ij}\right] = \begin{bmatrix} 0 & \kappa_{12} & \kappa_{13} \\ -\kappa_{12} & 0 & \kappa_{23} \\ -\kappa_{13} & -\kappa_{23} & 0 \end{bmatrix} \tag{28}$$

where the non-zero components of this tensor are

$$\kappa_{12} = -\kappa_{21} = \frac{1}{2}\left(\omega_{1,2} - \omega_{2,1}\right) \tag{29a}$$

$$\kappa_{23} = -\kappa_{32} = \frac{1}{2}\left(\omega_{2,3} - \omega_{3,2}\right) \tag{29b}$$

$$\kappa_{13} = -\kappa_{31} = \frac{1}{2}\left(\omega_{1,3} - \omega_{3,1}\right) \tag{29c}$$

Now we may recognize that κ_{12}, κ_{23}, and κ_{13} are the mean curvatures of planes parallel to the x_1x_2, x_2x_3, x_3x_1 planes, respectively, at point *P* after deformation. Therefore, the skew-symmetric tensor κ_{ij} will be referred to as the mean curvature tensor or simply the curvature tensor. The curvature vector κ_i dual to this tensor is defined by

$$\kappa_i = \frac{1}{2}\varepsilon_{ijk}\omega_{k,j} = \frac{1}{2}\varepsilon_{ijk}\kappa_{kj} \tag{30}$$

Thus, this axial vector is related to the mean curvature tensor through

$$\kappa_{ji} = \varepsilon_{ijk}\kappa_k \tag{31}$$

which shows

$$\kappa_1 = -\kappa_{23},\ \kappa_2 = \kappa_{13},\ \kappa_3 = -\kappa_{12} \tag{32}$$

It is seen that the mean curvature vector can be expressed as

$$\boldsymbol{\kappa} = \frac{1}{2}\nabla \times \boldsymbol{\omega} \tag{33}$$

This shows that $\boldsymbol{\kappa}$ is the rotation of the rotation vector, which can also be expressed as

$$\boldsymbol{\kappa} = \frac{1}{4}\nabla \times \left(\nabla \times \mathbf{u}\right) = \frac{1}{4}\nabla\left(\nabla \bullet \mathbf{u}\right) - \frac{1}{4}\nabla^2\mathbf{u} \tag{34a}$$

$$\kappa_i = \frac{1}{4}u_{k,ki} - \frac{1}{4}u_{i,kk} = \frac{1}{4}u_{k,ki} - \frac{1}{4}\nabla^2 u_i \tag{34b}$$

What we have presented here is applicable to small deformation polar theory, which requires the components of the strain tensor and mean curvature vector to be infinitesimal. These conditions can be written as

$$\left|e_{ij}\right| << 1 \tag{35}$$

$$|\kappa_i| << \frac{1}{l_S} \quad (36)$$

where, l_S is the smallest characteristic length in the body.

While analogous measures of strain and curvature can be obtained for finite deformation polar theory, this would take us beyond the scope of the present work, which is directed toward the infinitesimal linear couple stress theory.

4. Virtual work formulation and its consequences for polar media

Consider now a polar material continuum occupying a volume V bounded by a surface S as the current configuration. The standard form of the equilibrium equations for this medium were given in (5) and (6).

Let us multiply equation (5) by a virtual displacement δu_i and integrate over the volume and also multiply equation (6) by the corresponding virtual rotation $\delta\omega_i$, where

$$\delta\omega_i = \frac{1}{2}\varepsilon_{ijk}\delta u_{k,j} \quad (37)$$

and integrate this over the volume as well. Therefore, we have

$$\int_V (\sigma_{ji,j} + F_i)\delta u_i dV = 0 \quad (38)$$

$$\int_V (\mu_{ji,j} + \varepsilon_{ijk}\sigma_{jk} + C_i)\delta\omega_i dV = 0 \quad (39)$$

By noticing the relation

$$\sigma_{ji,j}\delta u_i = (\sigma_{ji}\delta u_i)_{,j} - \sigma_{ji}\delta u_{i,j} \quad (40)$$

and using the divergence theorem, the relation (38) becomes

$$\int_V \sigma_{ji}\delta u_{i,j} dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i \delta u_i dV \quad (41)$$

Similarly, by using the relation

$$\mu_{ji,j}\delta\omega_i + \varepsilon_{ijk}\sigma_{jk}\delta\omega_i = (\mu_{ji}\delta\omega_i)_{,j} - \mu_{ji}\delta\omega_{i,j} - \sigma_{jk}\delta\omega_{jk} \quad (42)$$

equation (39) becomes

$$\int_V \mu_{ji}\delta\omega_{i,j}dV - \int_V \sigma_{ji}\delta\omega_{ij}dV = \int_S m_i^{(n)}\delta\omega_i dS + \int_V C_i\delta\omega_i dV \tag{43}$$

Then, by adding (41) and (43), we obtain

$$\begin{gathered}\int_V \mu_{ji}\delta\omega_{i,j}dV + \int_V \sigma_{ji}\left(\delta u_{i,j} - \delta\omega_{ij}\right)dV = \\ \int_S t_i^{(n)}\delta u_i dS + \int_V F_i\delta u_i dV + \int_S m_i^{(n)}\delta\omega_i dS + \int_V C_i\delta\omega_i dV\end{gathered} \tag{44}$$

However, by noticing the relation

$$\delta e_{ij} = \delta u_{i,j} - \delta\omega_{ij} \tag{45}$$

for compatible virtual displacement, we obtain the virtual work theorem as

$$\int_V \sigma_{ji}\delta e_{ij}dV + \int_V \mu_{ji}\delta\omega_{i,j}dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i\delta u_i dV + \int_S m_i^{(n)}\delta\omega_i dS + \int_V C_i\delta\omega_i dV \tag{46}$$

Now, by using this virtual work formulation, we investigate the fundamental character of the body couple and couple stress in a material continuum.

It is seen that the term

$$\int_V C_i\delta\omega_i dV \tag{47}$$

in (46) is the only term in the volume that involves $\delta\omega_i$. However, $\delta\omega_i$ is not independent of δu_i in the volume, because we have the relation

$$\delta\omega_i = \frac{1}{2}\varepsilon_{ijk}\delta u_{k,j} \tag{48}$$

Therefore, by using (48), we find

$$C_i\delta\omega_i = \frac{1}{2}C_i\varepsilon_{ijk}\delta u_{k,j} = \frac{1}{2}\left(\varepsilon_{ijk}C_i\delta u_k\right)_{,j} - \frac{1}{2}\varepsilon_{ijk}C_{i,j}\delta u_k \tag{49}$$

and, after applying the divergence theorem, the body couple virtual work in (47) becomes

$$\int_V C_i \delta\omega_i dV = \int_V \frac{1}{2}\varepsilon_{ijk} C_{k,j} \delta u_i dV + \int_S \frac{1}{2}\varepsilon_{ijk} C_j n_k \delta u_i dS \tag{50}$$

which means that the body couple C_i transforms into an equivalent body force $\frac{1}{2}\varepsilon_{ijk} C_{k,j}$ in the volume and a force traction vector $\frac{1}{2}\varepsilon_{ijk} C_j n_k$ on the bounding surface. This shows that in polar materials, the body couple is not distinguishable from the body force. Therefore, in the couple stress theory for polar media, we must only consider body forces. This is analogous to the impossibility of distinguishing a distributed moment load in Euler–Bernoulli beam theory, in which the moment load must be replaced by the equivalent distributed force load and end concentrated loads. Therefore, for a proper couple stress theory, the equilibrium equations become

$$\sigma_{ji,j} + F_i = 0 \tag{51}$$

$$\mu_{ji,j} + \varepsilon_{ijk}\sigma_{jk} = 0 \tag{52}$$

where

$$\mathbf{F} + \frac{1}{2}\nabla \times \mathbf{C} \to \mathbf{F} \qquad \text{in} \quad V \tag{53a}$$

and

$$\mathbf{t}^{(n)} + \frac{1}{2}\mathbf{C} \times \mathbf{n} \to \mathbf{t}^{(n)} \qquad \text{on} \ \ S \tag{53b}$$

and the virtual work theorem reduces to

$$\int_V \sigma_{ji} \delta e_{ij} dV + \int_V \mu_{ji} \delta\omega_{i,j} dV = \int_S t_i^{(n)} \delta u_i dS + \int_V F_i \delta u_i dV + \int_S m_i^{(n)} \delta\omega_i dS \tag{54}$$

Next, we investigate the fundamental character of the couple stress tensor based on boundary conditions.

The prescribed boundary conditions on the surface of the body can be either vectors u_i and ω_i, or $t_i^{(n)}$ and $m_i^{(n)}$, which makes a total number of six boundary values for either case. However, this is in contrast to the number of geometric boundary conditions that can be imposed (Koiter, 1964). In particular, if components of u_i are specified on the boundary surface, then the normal component of the rotation ω_i corresponding to twisting

$$\omega_i^{(n)} = \omega^{(nn)} n_i = \omega_k n_k n_i \tag{55}$$

where

$$\omega^{(nn)} = \omega_k n_k \tag{56}$$

cannot be prescribed independently. However, the tangential component of rotation ω_i corresponding to bending, that is,

$$\omega_i^{(ns)} = \omega_i - \omega^{(nn)} n_i = \omega_i - \omega_k n_k n_i \tag{57}$$

may be specified in addition, and the number of geometric or essential boundary conditions that can be specified is therefore five.

Next, we let $m^{(nn)}$ and $m_i^{(ns)}$ represent the normal and tangential components of the surface couple vector $m_i^{(n)}$, respectively, where

$$m^{(nn)} = m_k^{(n)} n_k = \mu_{ji} n_i n_j \tag{58}$$

causes twisting, while

$$m_i^{(ns)} = m_i^{(n)} - m^{(nn)} n_i \tag{59}$$

is responsible for bending.

From kinematics, since $\omega^{(nn)}$ is not an independent generalized degree of freedom, its apparent corresponding generalized force must be zero. Thus, for the normal component of the surface couple vector $m_i^{(n)}$, we must enforce the condition

$$m^{(nn)} = m_k^{(n)} n_k = \mu_{ji} n_i n_j = 0 \quad \text{on } S \tag{60}$$

Furthermore, the boundary couple surface virtual work in (54) becomes

$$\int_S m_i^{(n)} \delta\omega_i dS = \int_S m_i^{(ns)} \delta\omega_i dS = \int_S m_i^{(ns)} \delta\omega_i^{(ns)} dS \tag{61}$$

This shows that a polar material in couple stress theory does not support independent distributions of normal surface couple $m^{(nn)}$, and the number of mechanical boundary conditions also is five. In practice, it might seem that a given $m^{(nn)}$ has to be replaced by an equivalent shear stress and force system. Koiter (1964) gives the detail analogous to the Kirchhoff bending theory of plates. However, we should realize that there is a difference between couple stress theory and the Kirchhoff bending theory of plates. Plate theory is an approximation for elasticity, which is a continuum mechanics theory. However, couple stress theory is a continuum mechanics theory itself without any approximation.

From the above discussion, we should realize that on the surface of the body, a normal couple $m^{(nn)}$ cannot be applied. By continuing this line of reasoning, we may reveal the subtle character of the couple stress-tensor. First, we notice that the virtual work theorem can be written for every arbitrary volume with arbitrary surface within the body. Therefore, for any point on any arbitrary surface with unit normal n_i, we must have

$$m^{(nn)} = \mu_{ji} n_i n_j = 0 \quad \text{in} \quad V \tag{62}$$

Since $n_i n_j$ is symmetric and arbitrary in (62), μ_{ji} must be skew-symmetric. Thus,

$$\mu_{ji} = -\mu_{ij} \quad \text{in} \quad V \tag{63}$$

This is the fundamental property of the couple-stress tensor in polar continuum mechanics, which has not been recognized previously. Here we can see the crucial role of the virtual work theorem in this result.

In terms of components, the couple stress tensor now can be written as

$$[\mu_{ij}] = \begin{bmatrix} 0 & \mu_{12} & \mu_{13} \\ -\mu_{12} & 0 & \mu_{23} \\ -\mu_{13} & -\mu_{23} & 0 \end{bmatrix} \tag{64}$$

and one can realize that the couple stress actually can be considered as an axial vector. This couple stress vector μ_i dual to the tensor μ_{ij} can be defined by

$$\mu_i = \frac{1}{2} \varepsilon_{ijk} \mu_{kj} \tag{65}$$

where we also have

$$\varepsilon_{ijk}\mu_k = \mu_{ji} \tag{66}$$

These relations simply show

$$\mu_1 = -\mu_{23}, \quad \mu_2 = \mu_{13}, \quad \mu_3 = -\mu_{12} \tag{67}$$

It is seen that the surface couple vector can be expressed as

$$m_i^{(n)} = \mu_{ji}n_j = \varepsilon_{ijk}n_j\mu_k \tag{68}$$

which can be written in vectorial form

$$\mathbf{m}^{(n)} = \mathbf{n} \times \boldsymbol{\mu} \tag{69}$$

This obviously shows that the surface couple vector $\mathbf{m}^{(n)}$ is tangent to the surface.

Interestingly, the angular equilibrium equation (52) can be expressed as

$$\varepsilon_{ijk}\left(\mu_{k,j} + \sigma_{jk}\right) = 0 \tag{70}$$

which indicates that $\mu_{k,j} + \sigma_{jk}$ is symmetric. Therefore, its skew-symmetric part vanishes and

$$\sigma_{[ji]} = -\mu_{[i,j]} \tag{71}$$

which produces the skew-symmetric part of the force stress tensor in terms of the couple stress vector. This result could have been expected on the grounds that the skew-symmetric stress tensor $\sigma_{[ji]}$ is actually an axial vector and should depend on the axial couple stress vector μ_i. Therefore, it is seen that the sole duty of the angular equilibrium equation (52) is to produce the skew-symmetric part of the force stress tensor. This relation can be elaborated if we consider the axial vector s_i dual to the skew-symmetric part of the force-stress tensor $\sigma_{[ij]}$, where

$$s_i = \frac{1}{2}\varepsilon_{ijk}\sigma_{[kj]} \tag{72a}$$

which also satisfies

$$\varepsilon_{ijk}s_k = \sigma_{[ji]} \tag{72b}$$

or simply

$$s_1 = -\sigma_{[23]}, \quad s_2 = \sigma_{[13]}, \quad s_3 = -\sigma_{[12]} \tag{73}$$

By using (71) and (72a), we obtain

$$s_i = -\frac{1}{2}\varepsilon_{ijk}\mu_{[j,k]} = \frac{1}{2}\varepsilon_{ijk}\mu_{k,j} \tag{74a}$$

which can be written in vectorial form

$$\mathbf{s} = \frac{1}{2}\nabla \times \boldsymbol{\mu} \tag{74b}$$

This simply shows that half of the curl of the couple stress vector $\boldsymbol{\mu}$ produces the skew-symmetric part of the force-stress tensor through **s**. Interestingly, it is seen that

$$\nabla \bullet \mathbf{s} = 0 \tag{75}$$

Returning to the virtual work theorem, we notice since μ_{ji} is skew-symmetric

$$\mu_{ji}\delta\omega_{i,j} = \mu_{ji}\delta\kappa_{ij} \tag{76}$$

which shows that the skew-symmetric mean curvature tensor κ_{ij} is energetically conjugate to the skew-symmetric couple-stress tensor μ_{ji}. This confirms our speculation of κ_{ij} as a suitable curvature tensor in Section 2. Furthermore, the virtual work theorem (54) becomes

$$\int_V \sigma_{ji}\delta e_{ij} dV + \int_V \mu_{ji}\delta\kappa_{ij} dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i \delta u_i dV + \int_S m_i^{(ns)}\delta\omega_i^{(ns)} dS \tag{77}$$

Interestingly, by using the dual vectors of these tensors, we have

$$\mu_{ji}\delta\kappa_{ij} = \varepsilon_{ijp}\mu_p\varepsilon_{jiq}\delta\kappa_q = -\varepsilon_{ijp}\varepsilon_{ijq}\mu_p\delta\kappa_q = -2\delta_{pq}\mu_p\delta\kappa_q = -2\mu_i\delta\kappa_i \tag{78}$$

which shows the conjugate relation between twice the mean curvature vector $-2\kappa_i$ and the couple-stress vector μ_i.

Since δe_{ij} is symmetric, we also have

$$\sigma_{ji}\delta e_{ij} = \sigma_{(ji)}\delta e_{ij} \tag{79}$$

where

$$\sigma_{(ji)} = \frac{1}{2}\left(\sigma_{ji} + \sigma_{ij}\right) \tag{80}$$

is the symmetric part of the force-stress tensor. Thus, the principle of virtual work can be written

$$\int_V \sigma_{(ji)}\delta e_{ij} dV + \int_V \mu_{ji}\delta\kappa_{ij} dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i \delta u_i dV + \int_S m_i^{(n)}\delta\omega_i dS \tag{81}$$

Therefore, it is seen that the symmetric small deformation strain tensor e_{ij} is the kinematical tensor energetically conjugate to the symmetric part of the force-stress tensor $\sigma_{(ji)}$. Finally, the virtual work theorem (81) can be rewritten as

$$\int_V \left(\sigma_{ji}\delta e_{ij} - 2\mu_i \delta\kappa_i\right) dV = \int_S t_i^{(n)}\delta u_i dS + \int_V F_i \delta u_i dV + \int_S m_i^{(n)}\delta\omega_i dS \tag{82}$$

What we have presented so far is a continuum mechanical theory of couple stress polar materials, independent of the material properties. In the following section, we specialize the theory for elastic materials.

5. Infinitesimal polar elasticity

Now, we develop the theory of small deformation for elastic polar materials. In a polar elastic material, there is a strain energy function W, where for arbitrary virtual deformations about the equilibrium position, we have

$$\delta W = \sigma_{ji}\delta e_{ij} + \mu_{ji}\delta\kappa_{ij} = \sigma_{(ji)}\delta e_{ij} - 2\mu_i \delta\kappa_i \tag{83}$$

Therefore, W is a positive definite function of the symmetric strain tensor e_{ij} and the mean curvature vector κ_i. Thus,

$$W = W(\mathbf{e}, \boldsymbol{\kappa}) = W\left(e_{ij}, \kappa_i\right) \tag{84}$$

However, for a variational analysis the relation (83) should be written as

$$\delta W = \sigma_{(ji)}\delta u_{i,j} - 2\mu_i \delta\kappa_i \tag{85}$$

where the all components of $\delta u_{i,j}$ and $\delta\kappa_i$ can be taken independent of each other. From the relations (84) and (85), we obtain

$$\sigma_{(ji)} = \frac{\partial W}{\partial u_{i,j}} \tag{86}$$

$$2\mu_i = -\frac{\partial W}{\partial \kappa_i} \tag{87}$$

However, it is seen that

$$\frac{\partial W}{\partial u_{i,j}} = \frac{\partial W}{\partial e_{kl}} \frac{\partial e_{kl}}{\partial u_{i,j}} \tag{88}$$

By noticing

$$e_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \tag{89}$$

we obtain

$$\frac{\partial e_{kl}}{\partial u_{i,j}} = \frac{1}{2}\left(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}\right) \tag{90}$$

Therefore

$$\frac{\partial W}{\partial u_{i,j}} = \frac{1}{2}\frac{\partial W}{\partial e_{kl}}\left(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}\right) \tag{91}$$

which shows

$$\frac{\partial W}{\partial u_{i,j}} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right) \tag{92}$$

Then

$$\sigma_{(ji)} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right) \tag{93}$$

$$\mu_i = -\frac{1}{2}\frac{\partial W}{\partial \kappa_i} \tag{94}$$

It is also seen that

$$\mu_{[i,j]} = \frac{1}{2}\left(\mu_{i,j} - \mu_{j,i}\right) = -\frac{1}{4}\left[\left(\frac{\partial W}{\partial \kappa_i}\right)_{,j} - \left(\frac{\partial W}{\partial \kappa_j}\right)_{,i}\right] \tag{95}$$

Therefore, for the skew-symmetric part of the force-stress tensor, we have

$$\sigma_{[ji]} = -\mu_{[i,j]} = \frac{1}{4}\left[\left(\frac{\partial W}{\partial \kappa_i}\right)_{,j} - \left(\frac{\partial W}{\partial \kappa_j}\right)_{,i}\right] \tag{96}$$

Finally, we obtain the constitutive relations as

$$\sigma_{ji} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right) + \frac{1}{4}\left[\left(\frac{\partial W}{\partial \kappa_i}\right)_{,j} - \left(\frac{\partial W}{\partial \kappa_j}\right)_{,i}\right] \tag{97}$$

$$\mu_i = -\frac{1}{2}\frac{\partial W}{\partial \kappa_i} \tag{98}$$

The total potential energy functional for polar elastic body is defined as

$$\Pi\{\mathbf{u}\} = \int_V W dV - \int_V F_i u_i dV - \int_S t_i^{(n)} u_i dS - \int_S m_i^{(n)} \omega_i dS \tag{99}$$

It can be easily shown that this functional attains its absolute minimum when the displacement field corresponds to the elastic solution that satisfies the equilibrium equations. The kinematics of deformation and variation of (99) reveal an important character of the strain energy function W. We know there are two sets of equilibrium equations (51) and (52) corresponding to linear and angular equilibrium of an infinitesimal element of material. Therefore, the geometrical boundary conditions are the displacement u_i and rotation ω_i as we discussed previously. As we showed in Section 4, polar continuum mechanics supports the geometrical boundary conditions u_i and $\omega_i^{(ns)}$, and their corresponding energy conjugate mechanical boundary conditions $t_i^{(n)}$ and $m_i^{(ns)}$. Consequently, there is no other possible type of boundary condition in polar continuum mechanics. Therefore, in the variation of the total potential energy Π in (99), the strain energy function W at most can be in the form (84). This means at most the strain energy function W is a function of the second derivative of deformation in the form of the mean curvature vector κ_i. In other words, the continuum mechanics strain energy function W cannot depend on third and higher order derivative of deformations.

6. Infinitesimal linear polar elasticity

Strain energy and constitutive relations

For a linear elastic material, based on our development, the quadratic positive definite strain energy must be in the form

$$W(\mathbf{e},\boldsymbol{\kappa})=\frac{1}{2}A_{ijkl}e_{ij}e_{kl}+\frac{1}{2}B_{ij}\kappa_i\kappa_j \tag{100}$$

The tensors A_{ijkl} and B_{ij} contain the elastic constitutive coefficients. It is seen that the tensor A_{ijkl} is actually equivalent to its corresponding tensor in Cauchy elasticity. The symmetry relations

$$A_{ijkl}=A_{klij}=A_{jikl} \tag{101}$$

$$B_{ij}=B_{ji} \tag{102}$$

are trivial. These symmetry relations show that for the most general case the number of distinct components for A_{ijkl} and B_{ij} are 21 and 6, respectively. It is seen that the couple stress vector and symmetric part of stress tensor can be found as

$$\mu_i=-\frac{1}{2}B_{ij}\kappa_j \tag{103}$$

$$\sigma_{(ji)}=A_{ijkl}e_{kl} \tag{104}$$

Additionally, we find that

$$\mu_{i,j}=-\frac{1}{2}B_{ik}\kappa_{k,j} \tag{105}$$

The skew-symmetric part of this tensor is

$$\mu_{[i,j]}=-\sigma_{[ji]}=-\frac{1}{4}B_{ik}\kappa_{k,j}+\frac{1}{4}B_{jk}\kappa_{k,i} \tag{106}$$

Therefore, for the force stresses, we find

$$\sigma_{ji}=A_{ijkl}e_{kl}+\frac{1}{4}B_{ik}\kappa_{k,j}-\frac{1}{4}B_{jk}\kappa_{k,i} \tag{107}$$

For an isotropic polar material, the symmetry relations require

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \tag{108}$$

$$B_{ij} = 16\eta\delta_{ij} \tag{109}$$

The moduli λ and μ have the same meaning as the Lamé constants for an isotropic material in Cauchy elasticity. It is seen that only one extra material constant η accounts for couple-stress effects in an isotropic polar material and the strain energy becomes

$$W(\mathbf{e},\boldsymbol{\kappa}) = \frac{1}{2}\lambda\left(e_{kk}\right)^2 + \mu e_{ij}e_{ij} + 8\eta\kappa_i\kappa_i \tag{110}$$

with the following restrictions on elastic constants for positive definite strain energy

$$3\lambda + 2\mu > 0, \qquad \mu > 0, \qquad \eta > 0 \tag{111}$$

Then, the constitutive relations can be written

$$\mu_i = -8\eta\kappa_i \tag{112}$$

$$\sigma_{(ji)} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{113}$$

Interestingly, it is seen that for an isotropic material

$$\nabla \bullet \boldsymbol{\mu} = \mu_{i,i} = 0 \tag{114}$$

By using the relation

$$\kappa_i = \frac{1}{4}u_{k,ki} - \frac{1}{4}u_{i,kk} \tag{115}$$

we obtain

$$\mu_i = 2\eta\left(\nabla^2 u_i - u_{k,ki}\right) \tag{116a}$$

or in vectorial form

$$\boldsymbol{\mu} = 2\eta\left[\nabla^2\mathbf{u} - \nabla\left(\nabla \bullet \mathbf{u}\right)\right] \tag{116b}$$

Additionally,

$$\mu_{i,j} = 2\eta\left(\nabla^2 u_{i,j} - u_{k,kij}\right) \tag{117}$$

Therefore,

$$\mu_{[i,j]} = \eta\nabla^2\left(u_{i,j} - u_{j,i}\right) \tag{118}$$

or

$$\mu_{[i,j]} = 2\eta\nabla^2\omega_{ij} \tag{119}$$

and we obtain

$$\sigma_{[ji]} = -\mu_{[i,j]} = 2\eta\nabla^2\omega_{ji} \tag{120}$$

or by exchanging indices

$$\sigma_{[ij]} = 2\eta\nabla^2\omega_{ij} \tag{121}$$

Recall that the axial vector s_i is dual to $\sigma_{[ij]}$, as shown in (72). Then, from (74a) and (112), s_i can be written in terms of the curvature vector as

$$s_i = -4\eta\varepsilon_{ijk}\kappa_{k,j} \tag{122}$$

Therefore, the constitutive relation for vector $\mathbf{s}$ is

$$\mathbf{s} = -4\eta\nabla\times\boldsymbol{\kappa} \tag{123a}$$

which can be written alternatively as

$$\mathbf{s} = -2\eta\nabla\times\nabla\times\boldsymbol{\omega} = 2\eta\nabla^2\boldsymbol{\omega} \tag{123b}$$

or

$$\mathbf{s} = -\eta\nabla\times\nabla\times\nabla\times\mathbf{u} \tag{123c}$$

This remarkable result shows that in an isotropic polar material the vector $\mathbf{s}$, corresponding to skew-symmetric part of stress tensor, is proportional to the curl of curl of curl of the displacement vector $\mathbf{u}$.

By using the relations (113) and (120), the total force-stress tensor can be written as

$$\sigma_{ji} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + 2\eta\nabla^2\omega_{ji} \tag{124}$$

We also notice that

$$\begin{aligned} \mu_{ji} &= -8\eta\kappa_{ji} \\ &= 4\eta\left(\omega_{i,j} - \omega_{j,i}\right) \end{aligned} \tag{125}$$

which is more useful than μ_i in practice.

It is seen that these relations are similar to those in the indeterminate couple stress theory (Mindlin and Tiersten, 1962), when $\eta' = -\eta$. Here we have derived the couple stress theory for polar materials in which all former troubles with indeterminacy disappear. There is no spherical indeterminacy and the second couple stress coefficient η' depends on η, such that the couple stress tensor becomes skew-symmetric.

Interestingly, the ratio

$$\frac{\eta}{\mu} = l^2 \tag{126}$$

specifies a characteristic material length l, which is absent in Cauchy elasticity, but is fundamental to small deformation couple-stress polar elasticity. We realize that this is the characteristic length in an elastic material and that $l_S \to l$ in (36). Thus, the requirements for small deformation polar elasticity are

$$|e_{ij}| << 1 \tag{127a}$$

$$|\kappa_i| << \frac{1}{l} \tag{127b}$$

Displacement formulations

When the force-stress tensor (107) is written in terms of displacements, as follows

$$\begin{aligned}\sigma_{ji} &= A_{ijkl}e_{kl} + \frac{1}{4}B_{ik}\kappa_{k,j} - \frac{1}{4}B_{jk}\kappa_{k,i} \\ &= A_{ijkl}u_{k,l} + \frac{1}{16}B_{ik}\left(u_{m,mkj} - \nabla^2 u_{k,j}\right) - \frac{1}{16}B_{jk}\left(u_{m,mki} - \nabla^2 u_{k,i}\right)\end{aligned} \tag{128}$$

and is carried into the linear equilibrium equation, we obtain

$$A_{ijkl}u_{k,lj} + \frac{1}{16}B_{ik}\left(\nabla^2 u_{m,mk} - \nabla^2\nabla^2 u_k\right) - \frac{1}{16}B_{jk}\left(u_{m,mkij} - \nabla^2 u_{k,ij}\right) = 0 \tag{129}$$

For an isotropic material, the force-stress tensor becomes

$$\begin{aligned}\sigma_{ji} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - 2\eta\nabla^2\omega_{ij} \\ &= \lambda u_{k,k}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right) - \eta\nabla^2\left(u_{i,j} - u_{j,i}\right)\end{aligned} \tag{130}$$

and for the linear equilibrium equation, we have

$$\left(\lambda+\mu+\eta\nabla^2\right)u_{k,ki}+(\mu-\eta\nabla^2)\nabla^2 u_i+F_i=0 \tag{131a}$$

which can be written in the vectorial form

$$\left(\lambda+\mu+\eta\nabla^2\right)\nabla\left(\nabla\bullet\mathbf{u}\right)+(\mu-\eta\nabla^2)\nabla^2\mathbf{u}+\mathbf{F}=\mathbf{0} \tag{131b}$$

This relation can also be written as

$$\left(\lambda+2\mu\right)\nabla\left(\nabla\bullet\mathbf{u}\right)-(\mu-\eta\nabla^2)\nabla\times\nabla\times\mathbf{u}+\mathbf{F}=\mathbf{0} \tag{132}$$

which was derived previously by Mindlin and Tiersten (1962) within the context of the indeterminate couple stress theory. However, recall that the Mindlin-Tiersten formulation involved two couple stress parameters η and η'. In hindsight, the fact that η' does not appear in (132) should have been an indication that this coefficient is not independent of η. We now know that $\eta'=-\eta$.

The general solution for the displacement in isotropic polar elasticity also has been derived by Mindlin and Tiersten (1962) as

$$\mathbf{u}=\mathbf{G}-l^2\nabla\nabla\bullet\mathbf{G}-\frac{1}{4(1-\nu)}\nabla\left[\mathbf{r}\bullet\left(1-l^2\nabla^2\right)\mathbf{G}+G_0\right] \tag{133}$$

where the vector function $\mathbf{G}$ and scalar function G_0 satisfy the relations

$$\mu\left(1-l^2\nabla^2\right)\nabla^2\mathbf{G}=-\mathbf{F} \tag{134a}$$

$$\mu\nabla^2 G_0=\mathbf{r}\bullet\mathbf{F} \tag{134b}$$

These functions reduce to the Papkovich functions in the classical theory, when $l=0$. It is easily seen that

$$\frac{\lambda+2\mu}{\mu}\nabla\bullet\mathbf{u}=\left(1-l^2\nabla^2\right)\nabla\bullet\mathbf{G} \tag{135a}$$

$$\nabla\times\mathbf{u}=\nabla\times\mathbf{G} \tag{135b}$$

Uniqueness theorem for boundary value problems

Now we investigate the uniqueness of the linear polar elasticity boundary value problem. The proof follows from the concept of strain energy, similar to the approach for Cauchy elasticity. By replacing the virtual deformation with the actual deformation in the virtual work theorem (82) and accounting for the symmetry of e_{ij}, we obtain

$$\int_V \left(\sigma_{(ji)}e_{ij} - 2\mu_i\kappa_i\right) dV = \int_S t_i^{(n)} u_i dS + \int_V F_i u_i dV + \int_S m_i^{(n)} \omega_i dS \tag{136}$$

Using the constitutive relations (103) and (104), we have

$$\sigma_{(ji)}e_{ij} - 2\mu_i\kappa_i = A_{ijkl}e_{ij}e_{kl} + B_{ij}\kappa_i\kappa_j = 2W(\mathbf{e},\boldsymbol{\kappa}) \tag{137}$$

Therefore, (136) can be written as

$$\int_V \left(A_{ijkl}e_{ij}e_{kl} + B_{ij}\kappa_i\kappa_j\right) dV = \int_S t_i^{(n)} u_i dS + \int_V F_i u_i dV + \int_S m_i^{(n)} \omega_i dS \tag{138}$$

This relation gives twice of total strain energy in terms of the work of external body forces and surface tractions.

Now, we consider the general boundary value problem. The prescribed boundary conditions on the surface of the body can be any well-posed combination of vectors u_i and ω_i, $t_i^{(n)}$ and $m_i^{(n)}$ as discussed on Section 4. Assume that there exist two different solutions $\left\{u_i^{(1)}, e_{ij}^{(1)}, \kappa_i^{(1)}, \sigma_{ji}^{(1)}, \mu_i^{(1)}\right\}$ and $\left\{u_i^{(2)}, e_{ij}^{(2)}, \kappa_i^{(2)}, \sigma_{ji}^{(2)}, \mu_i^{(2)}\right\}$ to the same problem with identical body forces and boundary conditions. Thus, we have the equilibrium equations

$$\sigma_{ji,j}^{(\alpha)} + F_i = 0 \tag{139}$$

$$\sigma_{[ji]}^{(\alpha)} = -\mu_{[i,j]}^{(\alpha)} \tag{140}$$

where

$$\mu_i^{(\alpha)} = -\frac{1}{2} B_{ij}\kappa_j^{(\alpha)} \tag{141a}$$

$$\sigma_{(ji)}^{(\alpha)} = A_{ijkl}e_{kl}^{(\alpha)} \tag{141b}$$

and the superscript $^{(\alpha)}$ references the solutions $^{(1)}$ and $^{(2)}$.

Let us now define the *difference solution*

$$u_i' = u_i^{(2)} - u_i^{(1)} \tag{142a}$$

$$e'_{ij} = e_{ij}^{(2)} - e_{ij}^{(1)} \tag{142b}$$

$$\kappa'_i = \kappa_i^{(2)} - \kappa_i^{(1)} \tag{142c}$$

$$\sigma'_{ji} = \sigma_{ji}^{(2)} - \sigma_{ji}^{(1)} \tag{142d}$$

$$\mu'_i = \mu_i^{(2)} - \mu_i^{(1)} \tag{142e}$$

Since the solutions $\left\{u_i^{(1)}, e_{ij}^{(1)}, \kappa_i^{(1)}, \sigma_{ji}^{(1)}, \mu_i^{(1)}\right\}$ and $\left\{u_i^{(2)}, e_{ij}^{(2)}, \kappa_i^{(2)}, \sigma_{ji}^{(2)}, \mu_i^{(2)}\right\}$ correspond to the same body forces and boundary conditions, the *difference solution* must satisfy the equilibrium equations

$$\sigma'_{ji,j} = 0 \tag{143}$$

$$\sigma'_{[ji]} = -\mu'_{[i,j]} \tag{144}$$

with zero corresponding boundary conditions. Consequently, twice the total strain energy (137) for the *difference solution* is

$$\int_V \left(A_{ijkl} e'_{ij} e'_{kl} + B_{ij} \kappa'_i \kappa'_j \right) dV = \int_V 2W' \, dV = 0 \tag{145}$$

Since the strain energy density W' is non-negative, this relation requires

$$2W' = A_{ijkl} e'_{ij} e'_{kl} + B_{ij} \kappa'_i \kappa'_j = 0 \quad \text{in } V \tag{146}$$

However, the tensors A_{ijkl} and B_{ij} are positive definite. Therefore the strain, curvature and associted stresses for *difference solution* must vanish

$$e'_{ij} = 0, \quad \kappa'_i = 0, \quad \sigma'_{ij} = 0, \quad \mu'_i = 0 \tag{147a-d}$$

These require that the *difference displacement* u'_i can be at most a rigid body motion. However, if displacement is specified on part of the boundary such that rigid body motion is prevented, then the *difference displacement* vanishes everywhere and we have

$$u_i^{(1)} = u_i^{(2)} \tag{148a}$$

$$e_{ij}^{(1)} = e_{ij}^{(2)} \tag{148b}$$

$$\kappa_i^{(1)} = \kappa_i^{(2)} \tag{148c}$$

$$\sigma_{ji}^{(1)} = \sigma_{ji}^{(2)} \tag{148d}$$

$$\mu_i^{(1)} = \mu_i^{(2)} \tag{148e}$$

Therefore, the solution to the boundary value problem is unique. On the other hand, if only force and couple tractions are specified over the entire boundary, then the displacement is not unique and is determined only up to an arbitrary rigid body motion.

Reciprocal theorem

We derive now the general reciprocal theorem for the equilibrium states of a linear polar elastic material under different applied loads. Consider two sets of equilibrium states of compatible elastic solutions $\left\{u_i^{(1)}, \omega_i^{(1)}, t_i^{(n)(1)}, m_i^{(n)(1)}, F_i^{(1)}\right\}$ and $\left\{u_i^{(2)}, \omega_i^{(2)}, t_i^{(n)(2)}, m_i^{(n)(2)}, F_i^{(2)}\right\}$. Let us apply the virtual work theorem (82) in the forms

$$\int_V \left(\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)}\right) dV = \int_S t_i^{(n)(1)} u_i^{(2)} dS + \int_V F_i^{(1)} u_i^{(2)} dV + \int_S m_i^{(n)(1)} \omega_i^{(2)} dS \tag{149}$$

$$\int_V \left(\sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)}\right) dV = \int_S t_i^{(n)(2)} u_i^{(1)} dS + \int_V F_i^{(2)} u_i^{(1)} dV + \int_S m_i^{(n)(2)} \omega_i^{(1)} dS \tag{150}$$

By using the general constitutive relations

$$\sigma_{ji}^{(1)} = A_{ijkl} e_{kl}^{(1)} + \frac{1}{4} B_{ik} \kappa_{k,j}^{(1)} - \frac{1}{4} B_{jk} \kappa_{k,i}^{(1)} \tag{151}$$

$$\mu_i^{(1)} = -\frac{1}{2} B_{ij} \kappa_j^{(1)} \tag{152}$$

$$\sigma_{ji}^{(2)} = A_{ijkl} e_{kl}^{(2)} + \frac{1}{4} B_{ik} \kappa_{k,j}^{(2)} - \frac{1}{4} B_{jk} \kappa_{k,i}^{(2)} \tag{153}$$

$$\mu_i^{(2)} = -\frac{1}{2} B_{ij} \kappa_j^{(2)} \tag{154}$$

it is seen that

$$\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)} = A_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} + \frac{1}{4} B_{ik} \kappa_{k,j}^{(1)} e_{ij}^{(2)} - \frac{1}{4} B_{jk} \kappa_{k,i}^{(1)} e_{ij}^{(2)} + B_{ij} \kappa_j^{(1)} \kappa_i^{(2)} \tag{155}$$

However, the symmetry relation (102) shows

$$\frac{1}{4} B_{ik} \kappa_{k,j}^{(1)} e_{ij}^{(2)} - \frac{1}{4} B_{jk} \kappa_{k,i}^{(1)} e_{ij}^{(2)} = 0 \tag{156}$$

Therefore

$$\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)} = A_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} + B_{ij} \kappa_j^{(1)} \kappa_i^{(2)} \tag{157}$$

Similarly,

$$\sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)} = A_{ijkl} e_{kl}^{(2)} e_{ij}^{(1)} + B_{ij} \kappa_j^{(2)} \kappa_i^{(1)} \tag{158}$$

Now we see that the symmetry relations (101) and (102) require

$$\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)} = \sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)} \tag{159}$$

which shows

$$\int_V \left(\sigma_{ji}^{(1)} e_{ij}^{(2)} - 2\mu_i^{(1)} \kappa_i^{(2)}\right) dV = \int_V \left(\sigma_{ji}^{(2)} e_{ij}^{(1)} - 2\mu_i^{(2)} \kappa_i^{(1)}\right) dV \tag{160}$$

Therefore, the general reciprocal theorem for these two elastic solutions is

$$\begin{aligned} &\int_S t_i^{(n)(1)} u_i^{(2)} dS + \int_V F_i^{(1)} u_i^{(2)} dV + \int_S m_i^{(n)(1)} \omega_i^{(2)} dS \\ &= \int_S t_i^{(n)(2)} u_i^{(1)} dS + \int_V F_i^{(2)} u_i^{(1)} dV + \int_S m_i^{(n)(2)} \omega_i^{(1)} dS \end{aligned} \tag{161}$$

7. Two-dimensional infinitesimal linear isotropic polar elasticity theory

In this section, we reconsider the two-dimensional infinitesimal linear isotropic polar elasticity developed by Mindlin (1963). We start this development by assuming that the displacement components are two-dimensional, where

$$u_1 = u(x,y),\ u_2 = v(x,y),\ u_3 = 0 \tag{162a-c}$$

This is exactly the conditions for plane strain theory in Cauchy elasticity. The non-zero components of strains are

$$e_x = \frac{\partial u}{\partial x},\quad e_y = \frac{\partial v}{\partial x},\quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{163a-c}$$

and the only non-zero rotation component is

$$\omega_z = \omega_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{164}$$

Therefore, the components of the mean curvature vector are

$$\kappa_x = -\kappa_{yz} = \frac{1}{2}\frac{\partial \omega_z}{\partial y}, \qquad \kappa_y = \kappa_{xz} = -\frac{1}{2}\frac{\partial \omega_z}{\partial x} \tag{165a,b}$$

It is seen that the compatibility equations for this case are

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{166a}$$

$$\frac{\partial \kappa_{yz}}{\partial x} = \frac{\partial \kappa_{xz}}{\partial y} \tag{166b}$$

$$\frac{\partial \omega_z}{\partial x} = \frac{\partial e_{xy}}{\partial x} - \frac{\partial e_x}{\partial y} \tag{166c}$$

$$\frac{\partial \omega_z}{\partial y} = \frac{\partial e_y}{\partial x} - \frac{\partial e_{xy}}{\partial y} \tag{166d}$$

Then, the corresponding couple stress and force stress components can be written

$$\mu_x = -4\eta\frac{\partial \omega_z}{\partial y}, \quad \mu_y = 4\eta\frac{\partial \omega_z}{\partial x} \tag{167a,b}$$

$$\sigma_{[xy]} = 2\eta\nabla^2\omega_{xy} = -2\eta\nabla^2\omega_z \tag{168a}$$

$$\sigma_{[yx]} = 2\eta\nabla^2\omega_{yx} = 2\eta\nabla^2\omega_z \tag{168b}$$

$$\sigma_{(xx)} = (\lambda + 2\mu)e_x + \lambda e_y \tag{169a}$$

$$\sigma_{(yy)} = \lambda e_x + (\lambda + 2\mu)e_y \tag{169b}$$

$$\sigma_{(xy)} = 2\mu e_{xy} \tag{169c}$$

It is also found that

$$\mu_{xz} = 4\eta\frac{\partial \omega_z}{\partial x} \tag{170a}$$

$$\mu_{yz} = 4\eta\frac{\partial \omega_z}{\partial y} \tag{170b}$$

Finally, it is seen that

$$\sigma_x = (\lambda + 2\mu)e_x + \lambda e_y \tag{171a}$$

$$\sigma_y = \lambda e_x + (\lambda + 2\mu)e_y \tag{171b}$$

$$\sigma_{xy} = 2\mu e_{xy} - 2\eta\nabla^2\omega_z \tag{171c}$$

$$\sigma_{yx} = 2\mu e_{xy} + 2\eta\nabla^2\omega_z \tag{171d}$$

where

$$\sigma_{xy} + \sigma_{yx} = 4\mu e_{xy} \tag{171e}$$

It is also seen that

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{171f}$$

similarly to plane strain Cauchy elasticity, while for the couple stresses

$$\mu_{zx} = -\mu_{xz} = -4\eta\frac{\partial\omega_z}{\partial x} \tag{171g}$$

$$\mu_{zy} = -\mu_{yz} = -4\eta\frac{\partial\omega_z}{\partial y} \tag{171h}$$

When there is no body force, these stresses satisfy the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} = 0 \tag{172a}$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \tag{172b}$$

$$\frac{\partial\mu_{xz}}{\partial x} + \frac{\partial\mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \tag{172c}$$

To solve for stresses, we need to derive compatibility equations in terms of stresses as follows. It is seen that

$$e_x = \frac{1}{2\mu}\left[(1-\nu)\sigma_x - \nu\sigma_y\right] \tag{173a}$$

$$e_y = \frac{1}{2\mu}\left[(1-\nu)\sigma_y - \nu\sigma_x\right] \tag{173b}$$

$$2e_{xy} = \frac{1}{2\mu}(\sigma_{xy} + \sigma_{yx}) \tag{173c}$$

$$\nabla^2 \omega_z = \frac{1}{4\eta}\left(\sigma_{yx} - \sigma_{xy}\right) \tag{173d}$$

By inserting these in (166), we obtain the compatibility equations in terms of the force and couple stress tensors. Thus,

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu \nabla^2 \left(\sigma_x + \sigma_y\right) = \frac{\partial^2}{\partial x \partial y}\left(\sigma_{yx} + \sigma_{xy}\right) \tag{174a}$$

$$\frac{\partial \mu_{xz}}{\partial y} = \frac{\partial \mu_{yz}}{\partial x} \tag{174b}$$

$$\mu_{xz} = l^2 \frac{\partial}{\partial x}\left(\sigma_{yx} + \sigma_{xy}\right) - 2l^2 \frac{\partial}{\partial y}\left[\sigma_x - \nu\left(\sigma_x + \sigma_y\right)\right] \tag{174c}$$

$$\mu_{yz} = 2l^2 \frac{\partial}{\partial x}\left[\sigma_x - \nu\left(\sigma_x + \sigma_y\right)\right] - l^2 \frac{\partial}{\partial y}\left(\sigma_{yx} + \sigma_{xy}\right) \tag{174d}$$

By combining these with the equilibrium equations, we obtain the set of equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \tag{175a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{175b}$$

$$\frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \tag{175c}$$

$$\nabla^2 \left(\sigma_x + \sigma_y\right) = 0 \tag{175d}$$

$$\frac{\partial \mu_{xz}}{\partial y} = \frac{\partial \mu_{yz}}{\partial x} \tag{175e}$$

By introducing the stress functions $\Phi = \Phi(x, y)$ and $\Psi = \Psi(x, y)$, we may write the force stresses and couple stresses as follows:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x \partial y} \tag{176a}$$

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x \partial y} \tag{176b}$$

$$\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y^2} \tag{176c}$$

$$\sigma_{yx} = -\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial x^2} \tag{176d}$$

$$\mu_{xz} = \frac{\partial \Psi}{\partial x} \tag{177a}$$

$$\mu_{yz} = \frac{\partial \Psi}{\partial y} \tag{177b}$$

Equilibrium equations satisfy and compatibility equations give

$$\nabla^2 \nabla^2 \Phi = 0 \tag{178}$$

$$\frac{\partial}{\partial x}\left(\Psi - l^2 \nabla^2 \Psi\right) = -2\left(1-\nu^2\right) l^2 \frac{\partial}{\partial y}\left(\nabla^2 \Phi\right) \tag{179a}$$

$$\frac{\partial}{\partial y}\left(\Psi - l^2 \nabla^2 \Psi\right) = 2\left(1-\nu^2\right) l^2 \frac{\partial}{\partial x}\left(\nabla^2 \Phi\right) \tag{179b}$$

Combining (179a) and (179b) by eliminating Φ gives

$$\nabla^2 \Psi - l^2 \nabla^4 \Psi = 0 \tag{180}$$

All these relations are exactly the equations derived by Mindlin (1963). This shows that the solutions for two-dimensional cases based on Mindlin's development, such as stress concentration relations for a plate with a circular hole, still can be used. However, we should notice that the couple stresses μ_{zx} and μ_{zy} are

$$\mu_{zx} = \frac{\eta'}{\eta} \mu_{xz} = 4\eta' \frac{\partial \omega_z}{\partial x} \tag{181a}$$

$$\mu_{zy} = \frac{\eta'}{\eta} \mu_{yz} = 4\eta' \frac{\partial \omega_z}{\partial y} \tag{181b}$$

in Mindlin's development. These relations become identical to those in the present theory, when we take $\eta' = -\eta$. Thus, we may solve the boundary value problem in an identical manner to Mindlin (1963), but then evaluate the couple stresses through a postprocessing operation.

More specifically, by comparing the relations (170) and (177), we can see

$$\mu_{xz} = 4\eta \frac{\partial \omega_z}{\partial x} = \frac{\partial \Psi}{\partial x} \tag{182a}$$

$$\mu_{yz} = 4\eta \frac{\partial \omega_z}{\partial y} = \frac{\partial \Psi}{\partial y} \tag{182b}$$

Therefore, we can take

$$4\eta\omega_z = \Psi \tag{183}$$

If Ψ is zero, there are no couple stress tensor components, and the relations for the force-stress tensor reduce to the relations in classical elasticity, where Φ is the Airy stress function.

For force and couple traction vectors, we have

$$t_x^{(n)} = \sigma_{xx} n_x + \sigma_{yx} n_y \tag{184a}$$

$$t_y^{(n)} = \sigma_{xy} n_x + \sigma_{yy} n_y \tag{184b}$$

$$m = m_z^{(n)} = \mu_{xz} n_x + \mu_{yz} n_y \tag{184c}$$

which can be written in terms of stress functions as

$$t_x^{(n)} = \left(\frac{\partial \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x \partial y} \right) n_x + \left(-\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial x^2} \right) n_y \tag{185a}$$

$$t_y^{(n)} = \left(-\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y^2} \right) n_x + \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x \partial y} \right) n_y \tag{185b}$$

$$m = \frac{\partial \Psi}{\partial x} n_x + \frac{\partial \Psi}{\partial y} n_y \tag{185c}$$

If the location on the surface is specified by the coordinate s along the boundary in a positive sense, we have

$$n_x = \frac{dy}{ds} \tag{186a}$$

$$n_y = -\frac{dx}{ds} \tag{186b}$$

Therefore

$$t_x^{(n)} = \frac{d}{ds}\left(\frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}\right) \tag{187a}$$

$$t_y^{(n)} = -\frac{d}{ds}\left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right) \tag{187b}$$

$$m = \frac{\partial \Psi}{\partial n} = 4\eta \frac{\partial \omega}{\partial n} \tag{187c}$$

8. Anti-plane deformation infinitesimal linear isotropic polar elasticity theory

We assume the displacement components are

$$u_1 = 0\,,\ \ u_2 = 0\,,\ \ u_3 = w(x, y) \tag{188a-c}$$

These are exactly the conditions for anti-plane deformation in Cauchy elasticity. The non-zero components of strains are

$$e_{zx} = \frac{1}{2}\frac{\partial w}{\partial x}\,,\ \ e_{zy} = \frac{1}{2}\frac{\partial w}{\partial y}\,, \tag{189a,b}$$

and the non-zero rotation components are

$$\omega_x = \frac{1}{2}\frac{\partial w}{\partial y} \quad , \quad \omega_y = -\frac{1}{2}\frac{\partial w}{\partial x} \tag{190}$$

Therefore, the only non-zero component of the mean curvature vector is

$$\kappa_z = \kappa_{yx} = \frac{1}{2}\left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) = -\frac{1}{4}\nabla^2 w \tag{191}$$

Then, the corresponding couple stress and force stress components can be written

$$\mu_z = \mu_{yx} = 2\eta \nabla^2 w \tag{192}$$

$$\sigma_{(zx)} = \mu \frac{\partial w}{\partial x}\,,\ \ \sigma_{(zy)} = \mu \frac{\partial w}{\partial y} \tag{193a,b}$$

$$\sigma_{[zx]} = \frac{1}{2}\frac{\partial \mu_{xy}}{\partial x} = -\eta \frac{\partial}{\partial x}\nabla^2 w\,,\ \ \sigma_{[zy]} = \frac{1}{2}\frac{\partial \mu_{yx}}{\partial y} = \eta \frac{\partial}{\partial y}\nabla^2 w \tag{193c,d}$$

Therefore

$$\sigma_{zx} = \mu \frac{\partial w}{\partial x} - \eta \frac{\partial}{\partial x} \nabla^2 w, \quad \sigma_{xz} = \mu \frac{\partial w}{\partial x} + \eta \frac{\partial}{\partial x} \nabla^2 w \tag{194a,b}$$

$$\sigma_{zy} = \mu \frac{\partial w}{\partial y} + \eta \frac{\partial}{\partial y} \nabla^2 w, \quad \sigma_{yz} = \mu \frac{\partial w}{\partial y} - \eta \frac{\partial}{\partial y} \nabla^2 w \tag{194c,d}$$

When there is no body force, these stresses satisfy the equilibrium equation

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \tag{195}$$

which in terms of displacement gives the single fourth order equation

$$\mu \nabla^2 w - \eta \nabla^2 \nabla^2 w = 0 \tag{196}$$

If couple stresses are zero on the boundary, the solution reduces to the classical Cauchy elasticity solution

$$\nabla^2 w = 0 \tag{197}$$

with

$$\mu_z = \mu_{yx} = 0 \tag{198a}$$

$$\sigma_{zx} = \sigma_{xz} = \mu \frac{\partial w}{\partial x} \tag{198b}$$

$$\sigma_{yz} = \sigma_{zy} = \mu \frac{\partial w}{\partial y} \tag{198c}$$

everywhere in the domain. However, geometrical boundary conditions, such as

$$w = 0, \quad \omega^{(\mathrm{ns})} = \frac{1}{2} \frac{\partial w}{\partial n} = 0 \qquad \text{on} \quad S \tag{199a,b}$$

create couple stresses in polar media. In that case, the classical solution cannot be used.

9. Some elementary linear isotropic polar elasticity problems

In this section, we reconsider several elementary problems in Cauchy elasticity within the framework of the present infinitesimal linear polar theory. Koiter (1964) considered all of these problems, but within the context of the indeterminate couple stress theory. Consequently, some differences appear.

Twist of a cylindrical bar

Consider the x_3-axis of our coordinate system along the axis of a cylindrical bar with constant cross section. We assume the displacement components are in the form as in the classical theory and examine the corresponding stress field in the couple stress theory. The assumed displacement components are

$$u_1 = -\theta x_2 x_3, \qquad u_2 = \theta x_1 x_3, \qquad u_3 = 0 \tag{200a-c}$$

where θ is the constant angle of twist per unit length. The non-zero components of the strain tensor and rotation vector are

$$e_{13} = -\frac{1}{2}\theta x_2, \qquad e_{23} = \frac{1}{2}\theta x_1 \tag{201a,b}$$

$$\omega_1 = \omega_{32} = -\frac{1}{2}\theta x_1, \qquad \omega_2 = \omega_{13} = -\frac{1}{2}\theta x_2, \qquad \omega_3 = \omega_{21} = \theta x_3 \tag{202a-c}$$

Interestingly, it is seen that the curvature vector vanishes

$$\boldsymbol{\kappa} = \frac{1}{2}\nabla \times \boldsymbol{\omega} = \mathbf{0} \tag{203}$$

Therefore, the force-stress distribution is the classical result

$$\sigma_{13} = -\mu\theta x_2, \qquad \sigma_{23} = \mu\theta x_1 \tag{204a,b}$$

and the twist of a cylindrical bar does not generate couple stresses. This is in contrast with the Koiter (1964) result, in which couple stresses appear.

Cylindrical bending of a flat plate

Consider a flat polar material plate of thickness h bent into a cylindrical shell with generators parallel to the x_3-axis. Let R denote the radius of curvature of the middle plane $x_1 x_3$ in the deformed configuration. We assume the displacement components are similar to those in Cauchy elasticity. Thus,

$$u_1 = -\frac{1}{R}x_1 x_2, \qquad u_2 = \frac{1}{2}\frac{1}{R}x_1^2 + \frac{1}{2}\frac{\nu}{1-\nu}\frac{1}{R}x_2^2, \qquad u_3 = 0 \tag{205a-c}$$

The non-zero components of the strain tensor, rotation vector and mean curvature vector are

$$e_{11} = -\frac{1}{R}x_2, \qquad e_{22} = \frac{\nu}{1-\nu}\frac{1}{R}x_2 \tag{206a,b}$$

$$\omega_3 = \omega_{21} = \frac{x_1}{R} \tag{207}$$

$$\kappa_{31} = -\kappa_2 = \frac{1}{2R} \tag{208}$$

Therefore, the non-zero force and couple stresses are written as

$$\sigma_{11} = -\frac{2\mu}{1-\nu}\frac{x_2}{R}, \qquad \sigma_{33} = -\frac{2\mu\nu}{1-\nu}\frac{x_2}{R} \tag{209a.b}$$

$$\mu_2 = \mu_{13} = -\mu_{31} = 4\frac{\eta}{R} \tag{210}$$

Notice that unlike the previous example of twisting deformation, bending does produce couple stresses. This is due to the existence of non-zero mean curvature.

Pure bending of a bar with rectangular cross-section

We take the x_1-axis to coincide with the centerline of the rectangular beam and the other axes parallel to the sides of the cross section of the beam. Let *R* denote the radius of curvature of the central axis of the beam after bending in the $x_1 x_3$-plane . We assume the displacement components are the same as in the classical Cauchy elasticity theory as follows:

$$u_1 = \frac{1}{R}x_1x_3, \qquad u_2 = -\frac{\nu}{R}x_2x_3, \qquad u_3 = \frac{\nu}{2R}\left(x_2^2 - x_3^2\right) - \frac{1}{2R}x_1^2 \tag{211a-c}$$

Then, the strains, rotations and mean curvatures can be written

$$e_{11} = \frac{x_3}{R}, \qquad e_{22} = e_{33} = -\frac{\nu x_3}{R} \tag{212a,b}$$

$$\omega_1 = \omega_{32} = \frac{\nu x_2}{R}\ , \quad \omega_2 = \omega_{13} = \frac{x_1}{R} \tag{213a,b}$$

$$\kappa_1 = \kappa_{32} = \frac{1}{2}\left(\omega_{3,2} - \omega_{2,3}\right) = 0 \tag{214a}$$

$$\kappa_2 = \kappa_{13} = \frac{1}{2}\left(\omega_{1,3} - \omega_{3,1}\right) = 0 \tag{214b}$$

$$\kappa_3 = \kappa_{21} = \frac{1}{2}\left(\omega_{2,1} - \omega_{1,2}\right) = \frac{1-\nu}{2R} \tag{214c}$$

As a result, the non-zero force and couple stresses take the form

$$\sigma_{11} = 2\mu\left(1+\nu\right)\frac{x_3}{R} \tag{215a}$$

$$\mu_3 = \mu_{21} = -\mu_{12} = -4\eta\frac{1-\nu}{R} \tag{215b}$$

Again, for this problem, we find non-zero mean curvature and couple stresses.

10. Conclusions

By considering further the consequences of the kinematics of a continuum and the principle of virtual work, we find that the couple stress theory for polar media can be formulated as a practical theory without any ambiguity. In the resulting quasistatic theory, independent body couples cannot be specified in the volume and surface couples can only exist in the tangent plane at each boundary point. (Although not specifically addressed here, in the corresponding dynamical theory, it should be clear that spin inertia does not appear.) As a consequence, the couple stress tensor is found to be skew-symmetric and energetically conjugate to the mean curvature tensor, which also is skew-symmetrical. Then, for infinitesimal or small deformation linear polar elasticity, we can write constitutive relations for all of the components of the force stress and couple stress tensors. The most general anisotropic polar material is described by 27 independent constitutive coefficients, including six coefficients relating mean curvatures to couple stresses. At the other extreme, for isotropic materials, the two Lamé parameters and one length scale completely characterize the behavior. In addition, strain energy relations and uniqueness and reciprocal theorems have been developed for linear polar elasticity. General formulations for two-dimensional and anti-plane problems are also elucidated for the isotropic case. The former employs a pair of stress functions, as introduced previously by Mindlin for the indeterminate theory. Finally, several additional elementary problems are examined within the context of small deformation polar elasticity.

By resolving the indeterminacy of the previous couple stress theory, the present polar theory provides a fundamental basis for the development of scale-dependent material response. Additional aspects of the linear theory, including fundamental solutions and computational mechanics formulations, will be addressed in forthcoming work. Beyond this, the present polar theory should be useful for the development of nonlinear elastic, elastoplastic, viscoplastic and damage mechanics formulations that may govern the behavior of solid continua at the smallest scales.

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