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Hamiltonian system based Saint Venant solutions for multi-layered composite plane anisotropic plates

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Abstract

This paper presents Hamiltonian system based Saint Venant solutions for the problem of multi-layered composite plane anisotropic plates. A mixed energy variational principle is proposed, and dual equations are derived in the symplectic space. The schemes of separation of variables and eigenfunction expansion, instead of the traditional semi-inverse method, are implemented, and compatibility conditions at interfaces are formulated by dual variables. By expending eigenfunctions in the subspace with zero eigenvalue, an analytical solution of a cantilever composite plate is presented to illustrate the proposed approach.

Keywords: Multi-layer; Anisotropic plate; Hamiltonian system; Saint Venant problem; Symplectic space

1. Introduction

Airy stress function (see, e.g., Timoshenko and Goodier, 1970) and other semi-inverse techniques are often used to solve the problem of elasticity, however for the case of multi-layered plate, the compatibility condition of displacement seems an obstacle for them to formulate, although compatibility condition of stress could be relatively easy to be described since each layer has an individual stress function.

Zhong and Zhong (1990) presented an analogy theory between computational structural mechanics and optimal control, the theory of Hamiltonian system can therefore be utilized for the solution of elasticity (see, e.g., Zhong, 1991), and a new solution system was established by using a translation from Euclidean space to symplectic space, in which, the schemes of separation of variables and eigenfunction expansion, instead of traditional semi-inverse methods are implemented (see, e.g., Zhong and Yao, 1997; Zhong and Yang, 1992). For the problem of elastic composite plate, compatibility conditions of displacements and stress at interfaces are easy to be described by dual variables in the symplectic space.

Variational method is a useful tool solving the problem of elasticity. Steele and Kim (1992) presented a Fourier transform based modified mixed variational principle from the Hellinger–Reissner mixed

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variational principle, and derived state-vector equations for both elastic bodies and shells of revolution. Further application of this principle can be found in the solution of plate bending (see, e.g., Kang et al., 1995). In this paper, a Hamiltonian mixed energy variational principle is established, and a group of corresponding dual equations are derived.

Saint Venant principle based solution, in the sense of static equivalence, is often employed to solve the problem with complex stress boundary conditions. Zhong and Yao (1997) presented a Saint Venant solution for multi-layered composite orthotropic plates via an eigenfunction expansion in the eigensubspace. This paper is a further extension of the above work, a solution of Saint Venant problem is given for multi-layered composite plane anisotropic plates.

2. Variational principles and Hamiltonian operator matrix

A composite plate with n layers is shown in Fig. 1 where $x_j \ll L$ $(j = 0, 1, \ldots, n)$ is longitudinal coordinate of the upper surface of jth layer. The boundary conditions at two side surfaces are specified by

$$\sigma_x = \overline{X}_1(z) \quad \text{and} \quad \tau_{xz} = \overline{Z}_1(z) \qquad x = 0 \tag{1a}$$

$$\sigma_x = \overline{X}_2(z) \quad \text{and} \quad \tau_{xz} = \overline{Z}_2(z) \qquad x = x_n \tag{1b}$$

The relationship between stress and strain for ith layer can be described as

$$\begin{Bmatrix} \sigma_{x,i} \\ \sigma_{z,i} \\ \tau_{xz,i} \end{Bmatrix} = \begin{bmatrix} \overline{c}_{1,i} & \overline{c}_{2,i} & \overline{c}_{3,i} \\ \overline{c}_{2,i} & \overline{c}_{4,i} & \overline{c}_{5,i} \\ \overline{c}_{3,i} & \overline{c}_{5,i} & \overline{c}_{6,i} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x,i} \\ \varepsilon_{z,i} \\ \gamma_{xz,i} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \varepsilon_{x,i} \\ \varepsilon_{z,i} \\ \gamma_{xz,i} \end{Bmatrix} = \begin{bmatrix} \overline{\beta}_{1,i} & \overline{\beta}_{2,i} & \overline{\beta}_{3,i} \\ \overline{\beta}_{2,i} & \overline{\beta}_{4,i} & \overline{\beta}_{5,i} \\ \overline{\beta}_{3,i} & \overline{\beta}_{5,i} & \overline{\beta}_{6,i} \end{bmatrix} \begin{Bmatrix} \sigma_{x,i} \\ \sigma_{z,i} \\ \tau_{xz,i} \end{Bmatrix}$$
$$(i = 0, 1, \ldots, n-1) \tag{2}$$

Following symbols are defined to simplify the derivation

$$\overline{d}_i = \det\left[\overline{c}_{j,i}\right]; \quad d_i = \frac{1}{\overline{\beta}_{4,i}}; \quad c_{j,i} = \frac{\overline{c}_{j,i}}{\overline{d}_i \overline{\beta}_{4,i}}; \quad \beta_{j,i} = \frac{\overline{\beta}_{j,i}}{\overline{\beta}_{4,i}} \qquad (j = 1, 2, \ldots, 6;\ i = 0, 1, \ldots, n-1) \tag{3}$$

where subscript i denotes ith layer, it will not appear in the derivation afterward except the case where it may cause confusion.

The Hellinger–Reissner variational principle for the composite plates can be written as (see, e.g., Hu, 1981)

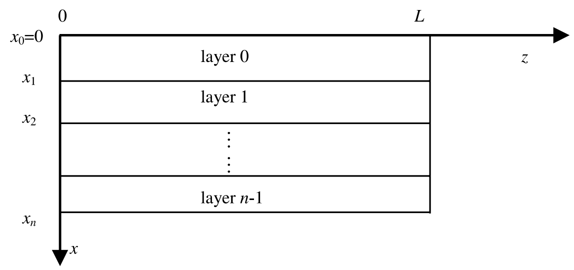

Fig. 1. A n-layer composite plate.

$$\delta\left\{\int_0^L\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left[\sigma_x\frac{\partial u}{\partial x}+\sigma_z\frac{\partial w}{\partial z}+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-U(\sigma_x,\sigma_z,\tau_{xz})-Xu-Zw\right]\mathrm{d}x\,\mathrm{d}z+U_e^1+U_e^2\right\}=0 \tag{4}$$

where

$$U(\sigma_x,\sigma_z,\tau_{xz})=\tfrac{1}{2}\left(\overline{\beta}_1\sigma_x^2+\overline{\beta}_4\sigma_z^2+\overline{\beta}_6\tau_{xz}^2\right)+\overline{\beta}_2\sigma_x\sigma_z+\overline{\beta}_3\sigma_x\tau_{xz}+\overline{\beta}_5\sigma_z\tau_{xz} \tag{5}$$

$$U_e^1=\int_0^L\left\{\left[\overline{X}_1u+\overline{Z}_1w\right]_{x=0}-\left[\overline{X}_2u+\overline{Z}_2w\right]_{x=x_n}\right\}\mathrm{d}z \tag{6a}$$

$$U_e^2=\begin{cases}-\int_0^{x_n}\left[\overline{Z}_3w+\overline{X}_3u\right]_0^L\mathrm{d}x & \text{for prescribed tractions}\\ -\int_0^{x_n}\left[(w-\overline{w})\sigma_z+(u-\overline{u})\tau_{xz}\right]_0^L\mathrm{d}x & \text{for prescribed displacements}\end{cases} \tag{6b}$$

$\overline{Z}_3$, $\overline{X}_3$ and $\overline{w}$, $\overline{u}$ are prescribed vectors of traction and displacement at the end surfaces of $z=0$ and $z=L$, respectively.

Coordinate z here is employed to simulate the time variable in the Hamiltonian system, x is a spatial coordinate. A symbol '.' in the following derivation will be used denoting the differential with respect to z, i.e. $(\cdot)=\partial/\partial z$.

By making stationary for Eq. (4) with respect to σ_x, σ_x can be expressed in terms of σ and τ as follows

$$\sigma_x=\frac{1}{\beta_1}\left(d\frac{\partial u}{\partial x}-\beta_2\sigma-\beta_3\tau\right) \tag{7}$$

where σ and τ represent σ_z and τ_{xz}, respectively. Substitution Eq. (7) for Eq. (4) can lead to a Hamiltonian mixed energy variational principle

$$\delta\left\{\int_0^L\left[\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left(\sigma\dot{w}+\tau\dot{u}-\mathscr{H}(w,u,\sigma,\tau)-Zw-Xu\right)\mathrm{d}x\right]\mathrm{d}z+U_e^1+U_e^2\right\}=0 \tag{8}$$

where $\mathscr{H}$ represents the Hamiltonian function (or density of mixed energy) having the form

$$\mathscr{H}(w,u,\sigma,\tau)=\frac{1}{2\beta_1}\left[c_6\sigma^2+c_4\tau^2-2c_5\sigma\tau+2\beta_2\sigma\frac{\partial u}{\partial x}+2\beta_3\tau\frac{\partial u}{\partial x}-d\left(\frac{\partial u}{\partial x}\right)^2-2\beta_1\tau\frac{\partial w}{\partial x}\right] \tag{9}$$

The state function vector can be described by $\mathbf{v}=\{w,u,\sigma,\tau\}^{\mathrm{T}}$, where σ and τ are dual variables of w and u in the symplectic space, respectively. The stationary requirement of Eq. (8) can yield a group of dual equations as follows: $\dot{\mathbf{v}}=\mathbf{H}\mathbf{v}+\mathbf{Q}$; where

$$\mathbf{H}=\frac{1}{\beta_1}\begin{bmatrix}0 & \beta_2\frac{\partial}{\partial x} & c_6 & -c_5\\ -\beta_1\frac{\partial}{\partial x} & \beta_3\frac{\partial}{\partial x} & -c_5 & c_4\\ 0 & 0 & 0 & -\beta_1\frac{\partial}{\partial x}\\ 0 & -d\frac{\partial^2}{\partial x^2} & \beta_2\frac{\partial}{\partial x} & \beta_3\frac{\partial}{\partial x}\end{bmatrix};\qquad \mathbf{Q}=\begin{Bmatrix}0\\0\\-Z\\-X\end{Bmatrix} \tag{10}$$

where $\mathbf{H}$ is a Hamiltonian operator matrix.

Compatibility conditions of displacement and stress at interfaces are specified by

$$\frac{1}{\beta_{1,i}}\left(d_i\frac{\partial u_i}{\partial x}-\beta_{2,i}\sigma_i-\beta_{3,i}\tau_i\right)=\frac{1}{\beta_{1,i-1}}\left(d_{i-1}\frac{\partial u_{i-1}}{\partial x}-\beta_{2,i-1}\sigma_{i-1}-\beta_{3,i-1}\tau_{i-1}\right);$$
$$\tau_i=\tau_{i-1}w_i=w_{i-1}\quad\text{and}\quad u_i=u_{i-1}\qquad x=x_i\qquad(i=1,2,\ldots,n-1) \tag{11}$$

The boundary conditions at two side surfaces can be rewritten as

$$\frac{1}{\beta_{1,0}}\left(d_0\frac{\partial u_0}{\partial x}-\beta_{2,0}\sigma_0-\beta_{3,0}\tau_0\right)=\overline{X}_1\quad \tau_0=\overline{Z}_1\ \ x=0 \tag{12a}$$

$$\frac{1}{\beta_{1,n-1}}\left(d_{n-1}\frac{\partial u_{n-1}}{\partial x}-\beta_{2,n-1}\sigma_{n-1}-\beta_{3,n-1}\tau_{n-1}\right)=\overline{X}_2\quad \tau_{n-1}=\overline{Z}_2\ \ x=x_n \tag{12b}$$

3. Hamiltonian eigenvalue problem and symplectic adjoint orthogonal relationship

The homogeneous equation corresponding to Eq. (10) can be written as

$$\dot{\mathbf{v}}=\mathbf{H}\mathbf{v} \tag{13}$$

For the free boundary conditions

$$\frac{1}{\beta_1}\left(d\frac{\partial u}{\partial x}-\beta_2\sigma-\beta_3\tau\right)=0\quad\text{and}\quad\tau=0\qquad x=0\text{ or }x_n \tag{14}$$

Eq. (13) can be solved by using a scheme of separation of variables, i.e.

$$\mathbf{v}=\exp(\mu z)\mathbf{\Psi}(x)\quad\text{and}\quad\mathbf{H\Psi}=\mu\mathbf{\Psi} \tag{15}$$

where μ is an eigenvalue, and $\mathbf{\Psi}$, satisfying the requirements of Eqs. (11) and (14), is an eigenfunction vector only related with x.

To describe the behavior of Hamiltonian operator, following unit symplectic matrix is introduced

$$\mathbf{J}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix};\quad \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{16}$$

and a symplectic inner product is defined as

$$\langle\mathbf{v}_1,\mathbf{v}_2\rangle\equiv\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}(\mathbf{v}_1^{\mathrm{T}}\mathbf{J}\mathbf{v}_2)\,\mathrm{d}x \tag{17}$$

If $\mathbf{v}_1$ and $\mathbf{v}_2$ meet the requirement of Eqs. (11) and (14), it can be proved that

$$\langle\mathbf{v}_1,\mathbf{H}\mathbf{v}_2\rangle\equiv\langle\mathbf{v}_2,\mathbf{H}\mathbf{v}_1\rangle \tag{18}$$

$\mathbf{H}$ is therefore a Hamiltonian operator matrix in the symplectic space. The Hamiltonian eigenvalue problem is not self-adjoint; however, it can be termed as an symplectic adjoint eigenvalue problem, that is to say, if μ is an eigenvalue, then $-\mu$ must also be one. Thus the eigenvalues can be divided into two groups, i.e. group A and group B

$$\mu_{\mathrm{A}j}(j=1,2,\ldots);\quad \mathrm{Re}(\mu_{\mathrm{A}j})\geqslant 0\qquad\text{in the group A} \tag{19a}$$

$$\mu_{\mathrm{B}j}(j=1,2,\ldots);\quad \mu_{\mathrm{B}j}=-\mu_{\mathrm{A}j}\qquad\text{in the group B} \tag{19b}$$

In addition, there must exist two groups of eigenfunction vectors $\mathbf{\Psi}_{\mathrm{A}j}$ and $\mathbf{\Psi}_{\mathrm{B}j}$ corresponding to eigenvalue $\mu_{\mathrm{A}j}$ and $\mu_{\mathrm{B}j}$ respectively, in which there must exist some of $\mathbf{\Psi}_{\mathrm{A}j}$ and $\mathbf{\Psi}_{\mathrm{B}j}$ satisfying an symplectic adjoint orthogonal relationship

$$\langle\mathbf{\Psi}_{\mathrm{A}i},\mathbf{\Psi}_{\mathrm{B}j}\rangle=\delta_{ij};\qquad \langle\mathbf{\Psi}_{\mathrm{A}i},\mathbf{\Psi}_{\mathrm{A}j}\rangle=\langle\mathbf{\Psi}_{\mathrm{B}i},\mathbf{\Psi}_{\mathrm{B}j}\rangle=0 \tag{20}$$

where δ_{ij} is Kronecker symbol.

So the solution of Eq. (10) can be obtained by eigenfunction expansion (Yao et al., 1999) with following combination

$$\mathbf{v} = \sum_{j=1}^{\infty} \left[A_j(z)\mathbf{\Psi}_{\mathrm{A}j} + B_j(z)\mathbf{\Psi}_{\mathrm{B}j} \right] \tag{21}$$

4. Eigenfunction vectors for eigenvalue zero

In the Hamiltonian eigenvalue problem, zero, if it is an eigenvalue, must be multiple with even number, and there often exist subsidiary eigenfunction vectors with various orders in Jordan normal form (Zhong and Yao, 1997; Xu et al., 1997; Van Loan, 1984). A combination of eigenfunction vectors for zero eigenvalue with relevant eigenfunction vectors in Jordan form can be exploited to describe Saint Venant solutions.

Two fundamental solutions for equation $\mathbf{H\Psi} = \mathbf{0}$ with conditions (11) and (14) can be written as

$$\mathbf{\Psi}_1^{(0)} = \{ w = 1, \quad u = 0, \quad \sigma = 0, \quad \tau = 0 \}^{\mathrm{T}} \quad \text{and} \quad \mathbf{v}_1^{(0)} = \mathbf{\Psi}_1^{(0)} \tag{22}$$

$$\mathbf{\Psi}_2^{(0)} = \{ w = 0, \quad u = 1, \quad \sigma = 0, \quad \tau = 0 \}^{\mathrm{T}} \quad \text{and} \quad \mathbf{v}_2^{(0)} = \mathbf{\Psi}_2^{(0)} \tag{23}$$

$\mathbf{\Psi}_1^{(0)}$ and $\mathbf{\Psi}_2^{(0)}$ describe rigid body translations along z and x axes, respectively, and are located at the heads of two Jordan chains.

The governing equation of Jordan normal form eigenfunction for eigenvalue zero can be written as

$$\mathbf{H\Psi}^{(k)} = \mathbf{\Psi}^{(k-1)} \tag{24}$$

where k refers to kth order Jordan normal form ($k = 1, 2, \ldots$).

Solving equation $\mathbf{H\Psi}_1^{(1)} = \mathbf{\Psi}_1^{(0)}$ with conditions (11) and (14) can give an eigenfunction in the first order Jordan form in the chain 1

$$\mathbf{\Psi}_{1,i}^{(1)} = \left\{ \beta_{5,i}x + f_i + D_0; \quad \beta_{2,i}x + g_i + D_1; \quad d_i \quad 0 \right\}^{\mathrm{T}} \quad (i = 0, 1, \ldots, n-1) \tag{25}$$

where

$$f_0 = 0; \qquad f_i = f_{i-1} - x_i(\beta_{5,i} - \beta_{5,i-1}) \quad (i = 1, 2, \ldots, n-1) \tag{26}$$

$$g_0 = 0; \qquad g_i = g_{i-1} - x_i(\beta_{2,i} - \beta_{2,i-1}) \quad (i = 1, 2, \ldots, n-1) \tag{27}$$

$\mathbf{\Psi}_1^{(1)}$ itself is not the solution of the original Eq. (13); however by combining with $\mathbf{\Psi}_1^{(0)}$ a solution can be obtain to describe a simple extension in the form

$$\mathbf{v}_1^{(1)} = \mathbf{\Psi}_1^{(1)} + z\mathbf{\Psi}_1^{(0)} \tag{28}$$

Because $\mathbf{\Psi}_1^{(1)}$ is symplectic adjoint with $\mathbf{\Psi}_1^{(0)}$, eigenfunction in the second order Jordan form in the chain 1 will not exist.

Similarly, $\mathbf{v}_2^{(1)}$, a solution for rigid body rotation can be obtained by solving equation $\mathbf{H\Psi}_2^{(1)} = \mathbf{\Psi}_2^{(0)}$ and combining $\mathbf{\Psi}_2^{(1)}$ with $\mathbf{\Psi}_2^{(0)}$, where

$$\mathbf{\Psi}_2^{(1)} = \{ D_2 - x, \quad D_3, \quad 0, \quad 0 \}^{\mathrm{T}} \quad \text{and} \quad \mathbf{v}_2^{(1)} = \mathbf{\Psi}_2^{(1)} + z\mathbf{\Psi}_2^{(0)} \tag{29}$$

$\mathbf{\Psi}_2^{(1)}$ is an eigenfunction in the first order Jordan form in the chain 2. Because $\mathbf{\Psi}_2^{(1)}$ is symplectic orthogonal with the eigenfunctions $\mathbf{\Psi}_1^{(0)}$ and $\mathbf{\Psi}_2^{(0)}$, eigenfunction in the second order Jordan form in the chain 2 will exist.

The eigenfunction in second order Jordan form in the chain 2 is

$$\mathbf{\Psi}_{2,i}^{(2)} = \begin{Bmatrix} w_{2,i}^{(2)} \\ u_{2,i}^{(2)} \\ \sigma_{2,i}^{(2)} \\ \tau_{2,i}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{2}\beta_{5,i}(x-D_2)^2 - D_3(x-D_2) + p_i + D_4 \\ -\frac{1}{2}\beta_{2,i}(x-D_2)^2 + q_i + D_5 \\ -d_i(x-D_2) \\ 0 \end{Bmatrix} \quad (i = 0,1,\ldots,n-1) \tag{30}$$

where

$$p_0 = 0; \quad p_i = p_{i-1} + \tfrac{1}{2}(x_i - D_2)^2(\beta_{5,i} - \beta_{5,i-1}) \quad (i = 1,2,\ldots,n-1) \tag{31}$$

$$q_0 = 0; \quad q_i = q_{i-1} + \tfrac{1}{2}(x_i - D_2)^2(\beta_{2,i} - \beta_{2,i-1}) \quad (i = 1,2,\ldots,n-1) \tag{32}$$

A solution of pure bending can be given by

$$\mathbf{v}_2^{(2)} = \mathbf{\Psi}_2^{(2)} + z\mathbf{\Psi}_2^{(1)} + \tfrac{1}{2}z^2\mathbf{\Psi}_2^{(0)} \tag{33}$$

$\mathbf{\Psi}_2^{(2)}$ is also symplectic orthogonal with the eigenfunctions $\mathbf{\Psi}_1^{(0)}$ and $\mathbf{\Psi}_2^{(0)}$ by choosing appropriate constant D_2 (see Appendix A), eigenfunction in the third order Jordan form in the chain 2 will exist.

$$\mathbf{\Psi}_{2,i}^{(3)} = \left\{ w_{2,i}^{(3)}, \quad u_{2,i}^{(3)}, \quad \sigma_{2,i}^{(3)}, \quad \tau_{2,i}^{(3)} \right\}^{\mathrm{T}} \quad (i = 0,1,\ldots,n-1) \tag{34}$$

where

$$\begin{aligned} w_{2,i}^{(3)} = {} & \tfrac{1}{6}\left(\beta_{2,i} + \beta_{6,i} - 2\beta_{5,i}^2\right)(x-D_2)^3 - \tfrac{1}{2}\beta_{5,i}D_3(x-D_2)^2 + (x-D_2)\left[\beta_{5,i}(p_i + D_4) \right. \\ & \left. + c_{1,i}r_i - q_i - D_5\right] + t_i + D_6 \end{aligned} \tag{35a}$$

$$u_{2,i}^{(3)} = -\tfrac{1}{6}\left(\beta_{2,i}\beta_{5,i} + c_{3,i}d_i\right)(x-D_2)^3 - \tfrac{1}{2}\beta_{2,i}D_3(x-D_2)^2 + (x-D_2)\left[\beta_{2,i}(p_i + D_4) - c_{3,i}r_i\right] + s_i + D_7 \tag{35b}$$

$$\sigma_{2,i}^{(3)} = -\beta_{5,i}d_i(x-D_2)^2 - d_iD_3(x-D_2) + d_i(p_i + D_4) - \beta_{5,i}r_i \tag{35c}$$

$$\tau_{2,i}^{(3)} = \tfrac{1}{2}d_i(x-D_2)^2 + r_i \tag{35d}$$

where

$$r_0 = -\tfrac{1}{2}d_0D_2^2; \qquad r_i = r_{i-1} - \tfrac{1}{2}(x_i - D_2)^2(d_i - d_{i-1}) \quad (i = 1,2,\ldots,n-1) \tag{36}$$

$$\begin{aligned} s_0 = 0; \quad s_i = {} & s_{i-1} + \tfrac{1}{6}(x_i - D_2)^3\left(\beta_{2,i}\beta_{5,i} + c_{3,i}d_i - \beta_{2,i-1}\beta_{5,i-1} - c_{3,i-1}d_{i-1}\right) + \tfrac{1}{2}D_3(x_i - D_2)^2 \\ & \times \left(\beta_{2,i} - \beta_{2,i-1}\right) - (x_i - D_2)\left[\beta_{2,i}(p_i + D_4) - c_{3,i}r_i - \beta_{2,i-1}(p_{i-1} + D_4) + c_{3,i-1}r_{i-1}\right] \\ & (i = 1,2,\ldots,n-1) \end{aligned} \tag{37}$$

$$\begin{aligned} t_0 = 0; \quad t_i = {} & t_{i-1} - \tfrac{1}{6}(x_i - D_2)^3\left(\beta_{2,i} + \beta_{6,i} - 2\beta_{5,i}^2 - \beta_{2,i-1} - \beta_{6,i-1} + 2\beta_{5,i-1}^2\right) + \tfrac{1}{2}D_3(x_i - D_2)^2 \\ & \times \left(\beta_{5,i} - \beta_{5,i-1}\right) - (x_i - D_2)\left[\beta_{5,i}(p_i + D_4) + c_{1,i}r_i - q_i - \beta_{5,i-1}(p_{i-1} + D_4) - c_{1,i-1}r_{i-1} + q_{i-1}\right] \\ & (i = 1,2,\ldots,n-1) \end{aligned} \tag{38}$$

A solution of a bending problem with constant shear force can be given by

$$\mathbf{v}_2^{(3)} = \mathbf{\Psi}_2^{(3)} + z\mathbf{\Psi}_2^{(2)} + \tfrac{1}{2}z^2\mathbf{\Psi}_2^{(1)} + \tfrac{1}{6}z^3\mathbf{\Psi}_2^{(0)} \tag{39}$$

Table 1
Adjoint symplectic orthogonal relationship of eigenfunctions for eigenvalue zero

	$\Psi_1^{(0)}$	$\Psi_1^{(1)}$	$\Psi_2^{(0)}$	$\Psi_2^{(1)}$	$\Psi_2^{(2)}$	$\Psi_2^{(3)}$
$\Psi_1^{(0)}$	0	*	0	0	D_2	D_4
$\Psi_1^{(1)}$		0	0	D_2	D_4	D_1, D_6
$\Psi_2^{(0)}$			0	0	0	*
$\Psi_2^{(1)}$				0	*	0
$\Psi_2^{(2)}$					0	D_5
$\Psi_2^{(3)}$						0

Because $\Psi_2^{(3)}$ is symplectic adjoint with the $\Psi_2^{(0)}$, eigenfunction in the fourth order Jordan form in the chain 2 will not exist.

The above six eigenfunctions constitute symplectic adjoint and orthogonal relationships, as shown in Table 1 where 0 represents a symplectic orthogonal relationship, $*$ represents a symplectic adjoint relationship, and D_i $(i = 1, 2, 4, 5, 6)$ represent a set of parameters. Proper choice of D_i $(i = 1, 2, 4, 5, 6)$ can make two eigenfunction vector symplectic orthogonal. The expressions of D_i $(i = 1, 2, 4, 5, 6)$ are listed in Appendix A, where k_1 and k_2 denote extensional and bending stiffness of a cross-section, respectively, and D_2 represents the position of centroidal axis of a cross-section in the pure bending problem.

5. The Saint Venant solutions

Combining eigenfunction vectors for zero eigenvalue with eigenfunction vectors in Jordan form, an analytic solution of Saint Venant problem can be obtained via expanding eigenfunction in the subspace for zero eigenvalue, having the form

$$\mathbf{v} = a_1(z)\Psi_1^{(0)} + a_2(z)\Psi_2^{(0)} + a_3(z)\Psi_1^{(1)} + a_4(z)\Psi_2^{(1)} + a_5(z)\Psi_2^{(2)} + a_6(z)\Psi_2^{(3)} \tag{40}$$

Substituting Eq. (40) for Eq. (8) and choosing proper D_i $(i = 0, 3, 7)$, listed in Appendix A, then yields

$$\begin{aligned}\delta\Bigg\{ \int_0^L \Big(& k_1 a_3\dot{a}_1 + k_2 a_5\dot{a}_4 - k_2 a_6\dot{a}_2 - \tfrac{1}{2}k_1 a_3^2 - \tfrac{1}{2}k_2 a_5^2 + k_2 a_4 a_6 - N(z)a_1 - Q(z)a_2 - W(z)a_3 \\ & - M(z)a_4 - \theta(z)a_5 - V(z)a_6\Big)\mathrm{d}z + (k_3 a_3 a_5 + k_4 a_5 a_6)\big|_{z=0}^{L} + U_e^2 \Bigg\} = 0\end{aligned} \tag{41}$$

The expressions of k_i $(1, 2, 3, 4)$, $N(z)$, and $Q(z)$ etc. are listed in Appendix A.

The implement of variations for Eq. (41) with respect to a_i $(i = 1, 2, \ldots, 6)$ leads to following differential equations

$$\dot{a}_3 = -N(z)/k_1 \quad \text{with respect to } \delta a_1 \tag{42a}$$

$$\dot{a}_1 = a_3 + W(z)/k_1 \quad \text{with respect to } \delta a_3 \tag{42b}$$

$$\dot{a}_6 = Q(z)/k_2 \quad \text{with respect to } \delta a_2 \tag{42c}$$

$$\dot{a}_5 = a_6 - M(z)/k_2 \quad \text{with respect to } \delta a_4 \tag{42d}$$

$$\dot{a}_4 = a_5 + \theta(z)/k_2 \quad \text{with respect to } \delta a_5 \tag{42e}$$

$$\dot{a}_2 = a_4 - V(z)/k_2 \quad \text{with respect to } \delta a_6 \tag{42f}$$

Usually it is difficult to describe the required boundary conditions on two ends exactly with these six a_i, a group of variational principles (Eq. (41)) based boundary conditions of tractions are therefore presented, having the form

$$a_3 = \overline{N}/k_1; \quad a_5 = (\overline{M} + D_3\overline{Q})/k_2; \quad a_6 = -\overline{Q}/k_2 \quad \text{when } z = 0 \text{ or } L \tag{43}$$

where

$$\overline{N} = \int_0^{x_n} \overline{Z}_3 \mathrm{d}x; \qquad \overline{Q} = \int_0^{x_n} \overline{X}_3 \mathrm{d}x; \qquad \overline{M} = \int_0^{x_n} (D_2 - x)\overline{Z}_3 \mathrm{d}x \tag{44}$$

On the other hand, on the basis of Eq. (41), the boundary conditions of displacement can be rewritten as

$$k_1 a_1 + k_3 a_5 = \overline{W}; \qquad k_2 a_4 + k_3 a_3 + k_4 a_6 = \overline{\theta}; \qquad k_2 a_2 - k_4 a_5 = -\overline{V} \tag{45}$$

where

$$\overline{W} = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \overline{w} d_i \,\mathrm{d}x; \qquad \overline{\theta} = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \overline{w}\sigma_{2,i}^{(2)} \,\mathrm{d}x; \qquad \overline{V} = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[\overline{w}\sigma_{2,i}^{(3)} + \overline{u}\tau_{2,i}^{(3)}\right] \mathrm{d}x \tag{46}$$

$\overline{W}$, $\overline{\theta}$ and $\overline{V}$ represent equivalent displacements.

For clamped end, boundary conditions can be described by Eq. (45) or

$$w = u = \partial w/\partial x = 0 \quad \text{when } z = 0 \text{ or } L \quad \text{and} \quad x = \bar{x} \ (0 \leqslant \bar{x} \leqslant x_n) \tag{47}$$

Assuming $\bar{x} = 0$, analytic solutions of Saint Venant problem can be given by integrating Eq. (42) with boundary conditions (43) and (45) or Eq. (47).

As a example, a cantilever plate is considered, which is clamped at the end $z = 0$, and subjected to a load P in the direction of x axis at the another end $z = L$. By integrating Eq. (42) with conditions (43) and (45) (or Eq. (47)), the distribution of stress can be described as

$$\sigma_i = P\left[(L + D_3 - z)\sigma_{2,i}^{(2)} - \sigma_{2,i}^{(3)}\right] \Big/ k_2; \quad \tau_i = -P\tau_{2,i}^{(3)}/k_2 \qquad (i = 0, 1, \ldots, n-1) \tag{48}$$

If the distribution of load P is described as Eq. (48), can be considered as an elastic analytic solution since the requirements of both Eq. (10) and two side boundary conditions (1) are satisfied. The difference between two solutions of taking boundary condition (45) or (47) can be proved to be only a minute rigid body displacement.

6. Conclusions

The major objective of this paper is to present a Saint Venant solution for elastic multi-layered composite plane anisotropic plates in the proposed Hamiltonian system. The merits of using the proposed approach lie in

1. being convenient for the application of conventional schemes, such as separation of variables, and eigenfunction expansion etc.,
2. facilitating to describe compatibility conditions at interfaces for displacements and stress.

In some cases, such as interlaminates stresses analysis (see, e.g., Pipes and Pagano, 1970), the end effects must be taken into account, it is definitely necessary to consider non-zero eigenvalues and their eigenfunctions to give more exact description. Due to the limited capacity of a paper, this issue is not discussed here. In fact, the major effect of the addition of non-zero eigenfunctions will appear at the neighbor areas of the ends, these non-zero eigenfunctions in Eq. (21) will decay from two ends quickly, the solution with zero eigenvalue is still a fairly good description of stress distribution in the region far enough from ends.

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Appendix A

(1) Expressions of constants D_0, D_2 and k_1

$$D_0 = -\frac{b_0}{k_1}; \quad D_2 = \frac{b_1}{k_1}; \qquad k_1 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i \, \mathrm{d}x \tag{A.1}$$

where

$$b_0 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i \left(\beta_{5,i} x + f_i\right) \mathrm{d}x, \qquad b_1 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i x \, \mathrm{d}x \tag{A.2}$$

(2) Expressions of constants D_3, D_4, k_2 and k_3

$$D_3 = -\frac{b_3}{k_2}; \qquad D_4 = \frac{k_3 - b_2}{k_1}; \tag{A.3}$$

$$k_2 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i (x - D_2)^2 \mathrm{d}x; \qquad k_3 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i (D_2 - x)\left(\beta_{5,i} x + f_i\right) \mathrm{d}x \tag{A.4}$$

where

$$b_2 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i \left[-\frac{1}{2} \beta_{5,i} (x - D_2)^2 + p_i \right] \mathrm{d}x; \qquad b_3 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i \left[\tfrac{1}{2} \beta_{5,i} (x - D_2)^3 - p_i (x - D_2) \right] \mathrm{d}x \tag{A.5}$$

(3) Expressions of constants D_1, D_5, D_6, D_7 and k_4

$$D_1 = \frac{b_4}{k_2}; \quad D_5 = \frac{b_6 + b_7}{2k_2}; \quad D_6 = -\frac{b_5}{k_1}; \quad D_7 = D_3 D_5 + \frac{b_8}{k_2}; \quad k_4 = \tfrac{1}{2}(b_7 - b_6) \tag{A.6}$$

where

$$\begin{aligned} b_4 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} & \Big\{ \tfrac{1}{2} d_i \left(\beta_{2,i} - 2\beta_{5,i}^2 \right) (x - D_2)^3 + \tfrac{1}{2} d_i \left[\beta_{2,i} D_2 + g_i - 2\beta_{5,i} \left(\beta_{5,i} D_2 + D_3 + f_i \right) \right] (x - D_2)^2 \\ & + \left[r_i \left(\beta_{2,i} - \beta_{5,i}^2 \right) + d_i \beta_{5,i} (p_i + D_4 - D_2 D_3) - d_i D_3 f_i \right] (x - D_2) + \big\lfloor r_i \left(\beta_{2,i} D_2 + g_i \right) \\ & + \left(\beta_{5,i} D_2 + f_i \right) \left(d_i D_4 + d_i p_i - \beta_{5,i} r_i \right) \big\rfloor \Big\} \mathrm{d}x \end{aligned} \tag{A.7}$$

$$\begin{aligned} b_5 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i & \Big\{ \tfrac{1}{6} \left(\beta_{2,i} + \beta_{6,i} - 2\beta_{5,i}^2 \right) (x - D_2)^3 - \tfrac{1}{2} \beta_{5,i} D_3 (x - D_2)^2 + \big[\beta_{5,i} (p_i + D_4) \\ & + c_{1,i} r_i - q_i \big] (x - D_2) + t_i \Big\} \mathrm{d}x \end{aligned} \tag{A.8}$$

$$b_6 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} d_i \Big\{ \tfrac{1}{6}\Big(\beta_{2,i} + \beta_{6,i} - 2\beta_{5,i}^2\Big)(x - D_2)^4 - \tfrac{1}{2}\beta_{5,i} D_3 (x - D_2)^3 + \big[\beta_{5,i}(p_i + D_4) + c_{1,i} r_i - q_i\big](x - D_2)^2 + t_i (x - D_2)\Big\} \mathrm{d}x \tag{A.9}$$

$$b_7 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \Big\{ \tfrac{1}{4} d_i \Big(2\beta_{5,i}^2 - \beta_{2,i}\Big)(x - D_2)^4 + \tfrac{3}{2} d_i \beta_{5,i} D_3 (x - D_2)^3 + \tfrac{1}{2}\Big[d_i\big(q_i + 2D_3^2 - 3\beta_{5,i} p_i - \beta_{5,i} D_4\big) + r_i\Big(\beta_{5,i}^2 - \beta_{2,i}\Big)\Big](x - D_2)^2 - D_3\big(2 d_i p_i - \beta_{5,i} r_i\big)(x - D_2) + d_i p_i (p_i + D_4) + r_i\big(q_i - \beta_{5,i} p_i\big)\Big\} \mathrm{d}x \tag{A.10}$$

$$\begin{aligned} b_8 = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \Big\{ & -\frac{1}{12} d_i \Big[2\beta_{5,i}(2\beta_{2,i} + \beta_{6,i} - 2\beta_{5,i}^2) - \beta_{3,i}\Big](x - D_2)^5 + \frac{1}{12} d_i D_3 \Big(10\beta_{5,i}^2 - 5\beta_{2,i} - 2\beta_{6,i}\Big) \\ & \times (x - D_2)^4 + \frac{1}{6}\Big[d_i (p_i + D_4)\Big(\beta_{6,i} + 4\beta_{2,i} - 8\beta_{5,i}^2\Big) + 3 d_i \beta_{5,i}\big(D_3^2 + 2q_i + 2D_5\big) \\ & - r_i \Big(2\beta_{2,i}\beta_{5,i} + 4c_{3,i} d_i - 8\beta_{5,i}^3 + 7\beta_{5,i}\beta_{6,i}\Big)\Big](x - D_2)^3 + \frac{1}{2}\Big[r_i D_3 \Big(3\beta_{5,i}^2 - \beta_{2,i} - 2\beta_{6,i}\Big) \\ & + d_i D_3 \big(2q_i - 3\beta_{5,i} p_i - 3\beta_{5,i} D_4\big) + s_i d_i - 2\beta_{5,i} d_i t_i\Big](x - D_2)^2 \\ & + \Big[d_i (p_i + D_4)\big(\beta_{5,i} p_i + \beta_{5,i} D_4 + c_{1,i} r_i - q_i - D_5\big) - d_i t_i D_3 - r_i^2\big(c_{3,i} + c_{1,i}\beta_{5,i}\big) \\ & + r_i \beta_{5,i}(q_i + D_5) + r_i (p_i + D_4)\Big(\beta_{2,i} - \beta_{5,i}^2\Big)\Big](x - D_2) \\ & + s_i r_i + t_i d_i (p_i + D_4) - t_i r_i \beta_{5,i}\Big\} \mathrm{d}x \end{aligned} \tag{A.11}$$

(4) Expressions of functions $N(z)$, $Q(z)$, $M(z)$, $W(z)$, $\theta(z)$, and $V(z)$ in Eq. (41)

$$N(z) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} Z(x,z)\mathrm{d}x + \overline{Z}_2(z) - \overline{Z}_1(z) \tag{A.12}$$

$$Q(z) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} X(x,z)\mathrm{d}x + \overline{X}_2(z) - \overline{X}_1(z) \tag{A.13}$$

$$M(z) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} [(D_2 - x)Z + D_3 X]\,\mathrm{d}x + (D_2 - x_n)\overline{Z}_2 + D_3\overline{X}_2 - D_2\overline{Z}_1 - D_3\overline{X}_1 \tag{A.14}$$

$$W(z) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \Big[w_{1,i}^{(1)} Z + u_{1,i}^{(1)} X\Big]\,\mathrm{d}x + \Big[w_{1,n-1}^{(1)}\overline{Z}_2 + u_{1,n-1}^{(1)}\overline{X}_2\Big]_{x=x_n} - \Big[w_{1,0}^{(1)}\overline{Z}_1 + u_{1,0}^{(1)}\overline{X}_1\Big]_{x=0} \tag{A.15}$$

$$\theta(z) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[w_{2,i}^{(2)} Z + u_{2,i}^{(2)} X \right] \mathrm{d}x + \left[w_{2,n-1}^{(2)} \overline{Z}_2 + u_{2,n-1}^{(2)} \overline{X}_2 \right]_{x=x_n} - \left[w_{2,0}^{(2)} \overline{Z}_1 + u_{2,0}^{(2)} \overline{X}_1 \right]_{x=0} \tag{A.16}$$

$$V(z) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[w_{2,i}^{(3)} Z + u_{2,i}^{(3)} X \right] \mathrm{d}x + \left[w_{2,n-1}^{(3)} \overline{Z}_2 + u_{2,n-1}^{(3)} \overline{X}_2 \right]_{x=x_n} - \left[w_{2,0}^{(3)} \overline{Z}_1 + u_{2,0}^{(3)} \overline{X}_1 \right]_{x=0} \tag{A.17}$$

References

Hu, H.C., 1981. Variational principle of elasticity and its application. Science Press, Beijing, China (in Chinese).

Kang, L.C., Wu, C.H., Steele, C.R., 1995. Fourier series for polygonal plate bending: a very large plate element. Applied Mathematics and Computation 67, 157–225.

Pipes, R.B., Pagano, N.J., 1970. Interlaminar stresses in composite laminates under uniform axial extension. Journal of Composite Materials 4, 538–548.

Steele, C.R., Kim, Y.Y., 1992. Modified mixed variational principle and the state-vector equation for elastic bodies and shells of revolution. Journal of Applied Mechanics 59 (3), 587–595.

Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity, third ed., McGraw-Hill, New York.

Van Loan, C.F., 1984. A symplectic method for approximating all the eigenvalues of a Hamiltonian matrix. Linear Algebra and Applications 61, 233–251.

Xu, X.S., Zhong, W.X., Zhang, H.W., 1997. The Saint-Venant problem and principle in elasticity. International Journal of Solids and Structures 34 (22), 2815–2827.

Yao, W.A., Zhong, W.X., Su, B., 1999. New solution system for circular sector plate bending and its application. ACTA Mechanics Solida Sinica 12 (4), 307–315.

Zhong, W.X., 1991. Plane elasticity problem in strip domain and Hamiltonian system. Journal of Dalian University of Technology 31 (4), 373–384 (in Chinese).

Zhong, W.X., Yang, Z.S., 1992. Partial differential equations and Hamiltonian system. In: Cheng, F.Y., Zizhi, Fu. (Eds.), Computational Mechanics in Structural Engineering, Elsevier, Amsterdam, pp. 32–48.

Zhong, W.X., Yao W.A, 1997. The Saint Venant solutions of multi-layered composite plates. Advances in Structural Engineering 1 (2), 127–133.

Zhong, W.X., Zhong, X.X., 1990. Computational structural mechanics, optimal control and semi-analytical method for PDE. Computers and Structures 37 (6), 993–1004.