Hilbert Complexes of Nonlinear Elasticity*†

Arzhang Angoshtari‡ Arash Yavari§

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Abstract

We introduce some Hilbert complexes involving second-order tensors on flat compact manifolds with boundary that describe the kinematics and the kinetics of motion in nonlinear elasticity. We then use the general framework of Hilbert complexes to write Hodge-type and Helmholtz-type orthogonal decompositions for second-order tensors. As some applications of these decompositions in nonlinear elasticity, we study the strain compatibility equations of linear and nonlinear elasticity in the presence of Dirichlet boundary conditions and the existence of stress functions on non-contractible bodies. As an application of these Hilbert complexes in computational mechanics, we briefly discuss the derivation of a new class of mixed finite element methods for nonlinear elasticity.

Contents

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‡Department of Civil and Environmental Engineering, The George Washington University, Washington, DC 20052. E-mail: aangoshtari@gwu.edu.

§School of Civil and Environmental Engineering & The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332. E-mail: arash.yavari@ce.gatech.edu.

1 Introduction

Suppose $\mathscr{L}$ is a linear (elliptic) differential operator that associates a tensor $\boldsymbol{\alpha} = \mathscr{L}(\boldsymbol{\gamma})$ to another tensor $\boldsymbol{\gamma}$. In this case, $\boldsymbol{\gamma}$ is called an $\mathscr{L}$-potential for $\boldsymbol{\alpha}$. Such potentials naturally arise in continuum mechanics. For example, a displacement field is a potential for a (linear or nonlinear) strain and a Beltrami stress function is a potential for a divergence-free Cauchy stress tensor. More discussions on various applications of such potentials in continuum mechanics can be found in Truesdell [38] and references therein. It is well-known that the necessary and sufficient conditions for the existence of these potentials are closely related to certain topological properties of the underlying bodies, e.g. see [27, 16, 28, 40, 2]. All these references directly or indirectly use the de Rham theorem [31, Theorem 18.14], which gives the necessary and sufficient conditions for the existence of d-potentials for sufficiently smooth differential forms, where d is the exterior derivative.

An alternative approach to potentials is offered by orthogonal decompositions in the following sense. Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary. Also suppose that the boundary $\partial\bar{\mathcal{B}}$ of $\bar{\mathcal{B}}$ can be written as $\partial\bar{\mathcal{B}} = \partial_1\bar{\mathcal{B}} \cup \partial_2\bar{\mathcal{B}}$, where $\partial_1\bar{\mathcal{B}}$ and $\partial_2\bar{\mathcal{B}}$ are disjoint compact surfaces without boundary. Then, a Hodge-type decomposition for differential 1-forms introduced by Gol'dshtein et al. [26, Theorem 4.3] implies that any smooth vector field $\boldsymbol{Y}$ on $\bar{\mathcal{B}}$ can be uniquely decomposed as

$$\boldsymbol{Y} = \boldsymbol{Y}_g + \boldsymbol{Y}_{\mathcal{H}} + \boldsymbol{Y}_c, \tag{1.1}$$

where the components are orthogonal with respect to the L^2-inner product $\langle\!\langle , \rangle\!\rangle_{L^2}$ of vector fields and satisfy $\boldsymbol{Y}_g, \boldsymbol{Y}_{\mathcal{H}} \perp \partial_1\bar{\mathcal{B}}$, and $\boldsymbol{Y}_c, \boldsymbol{Y}_{\mathcal{H}} \| \partial_2\bar{\mathcal{B}}$, with $\perp$ and $\|$ meaning "normal to" and "tangent to", respectively. Moreover, there exists a function f with $f|_{\partial_1\bar{\mathcal{B}}} = 0$, and a vector field $\boldsymbol{Z} \perp \partial_2\bar{\mathcal{B}}$ such that $\boldsymbol{Y}_g = \operatorname{grad} f$, and $\boldsymbol{Y}_c = \operatorname{curl} \boldsymbol{Z}$. The vector field $\boldsymbol{Y}_{\mathcal{H}}$ is a harmonic field, i.e. both $\operatorname{curl} \boldsymbol{Y}_{\mathcal{H}}$ and $\operatorname{div} \boldsymbol{Y}_{\mathcal{H}}$ vanish. Let $\mathcal{H}^{\mathfrak{X}}_{\mathrm{n_1,t_2}}(\bar{\mathcal{B}})$ be the finite-dimensional space of vector fields in $\ker \operatorname{div} \cap \ker \operatorname{curl}$ that satisfy the same boundary conditions as $\boldsymbol{Y}_{\mathcal{H}}$ does. By assuming $\partial_1\bar{\mathcal{B}} = \varnothing$, the decomposition (1.1) enables one to write the necessary and sufficient conditions for the existence of a grad-potential for $\boldsymbol{Y}$ as

$$\operatorname{curl} \boldsymbol{Y} = 0, \text{ and } \langle\!\langle \boldsymbol{Y}, \boldsymbol{H} \rangle\!\rangle_{L^2} = 0, \ \forall \boldsymbol{H} \in \mathcal{H}^{\mathfrak{X}}_{\mathrm{n_1,t_2}}(\bar{\mathcal{B}}). \tag{1.2}$$

Note that by using the de Rham theorem, the necessary and sufficient conditions for the existence of grad-potentials are

$$\operatorname{curl} \boldsymbol{Y} = 0, \text{ and } \int_\ell \boldsymbol{G}(\boldsymbol{Y}, \boldsymbol{t}_\ell) dS = 0, \ \forall \ell \subset \bar{\mathcal{B}}, \tag{1.3}$$

where $\boldsymbol{G}$ is the Riemannian metric of $\bar{\mathcal{B}}$, ℓ is an arbitrary closed curve in $\bar{\mathcal{B}}$, and $\boldsymbol{t}_\ell$ is the unit tangent vector field along ℓ. For sufficiently smooth vector fields, one can show that (1.2) and (1.3) are equivalent [35, Theorem 3.2.3].

On the other hand, if $\partial_1\bar{\mathcal{B}} \neq \varnothing$, then the decomposition (1.1) also allows one to study grad-potentials in the presence of Dirichlet boundary conditions. More specifically, consider the following problem:

> *Given a vector field $\boldsymbol{Y}$, determine the necessary and sufficient conditions for the existence of a function f such that $\boldsymbol{Y} = \operatorname{grad} f$, and $f|_{\partial_1\bar{\mathcal{B}}} = 0$.* (1.4)

For solving this problem, (1.1) tells us that the necessary and sufficient conditions sought for in (1.4) are

$$\operatorname{curl} \boldsymbol{Y} = 0, \boldsymbol{Y} \perp \partial_1\bar{\mathcal{B}}, \text{ and } \langle\!\langle \boldsymbol{X}, \boldsymbol{H} \rangle\!\rangle_{L^2} = 0, \ \forall \boldsymbol{H} \in \mathcal{H}^{\mathfrak{X}}_{\mathrm{n_1,t_2}}(\bar{\mathcal{B}}). \tag{1.5}$$

Note that depending on the topology of $\partial_1\bar{\mathcal{B}}$, the conditions (1.5) with the Dirichlet boundary condition

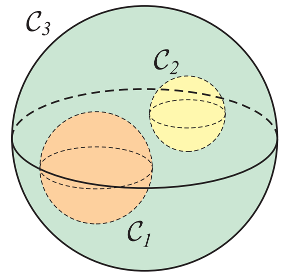

Figure 1: *A simply-connected 3D body $\bar{\mathcal{B}}$ with $\partial\bar{\mathcal{B}}$ being the union of the spheres $\mathcal{C}_i$, $i = 1, 2, 3$.*

on $\partial_1\bar{\mathcal{B}}$ and the conditions (1.2) without Dirichlet boundary conditions may be completely different. For example, consider the body $\bar{\mathcal{B}}$ shown in Fig. 1, with its boundary $\partial\bar{\mathcal{B}}$ consisting of three spherical surfaces $\mathcal{C}_i$, i=1,2,3. Since $\bar{\mathcal{B}}$ is simply-connected, (1.2) reads: $\operatorname{curl}\boldsymbol{Y} = 0$. If $\partial_1\bar{\mathcal{B}} = \mathcal{C}_1$, then (1.5) becomes $\operatorname{curl}\boldsymbol{Y} = 0$, and $\boldsymbol{Y} \perp \partial_1\bar{\mathcal{B}}$. For $\partial_1\bar{\mathcal{B}} = \mathcal{C}_1 \cup \mathcal{C}_2$, and $\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$, the space $\mathcal{H}^{\mathfrak{X}}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$ is 1- and 2-dimensional, respectively, and therefore, $\operatorname{curl}\boldsymbol{Y} = 0$, and $\boldsymbol{Y} \perp \partial_1\bar{\mathcal{B}}$ are no longer sufficient conditions for the existence of grad-potentials with these Dirichlet boundary conditions. Therefore, (1.1) allows one to study the effects of the topology of $\partial_1\bar{\mathcal{B}}$ on the solution of the problem (1.4).

Linear and nonlinear strain tensors, and stress tensors are of second order, and therefore, for extending the above results to potentials for strains and stresses one needs a tensorial analogue of (1.1). Orthogonal decompositions for second-order tensors have been studied by many authors in the literature. The motivation for most of these studies was the earlier works on decompositions of differential forms discussed in Morrey [32] and references therein. Cantor [15] proposed a general framework for deriving Helmholtz-type decompositions for tensor fields on compact manifolds using elliptic operators. By employing a similar approach, Berger and Ebin [10] obtained various Helmholtz-type decompositions for symmetric second-order tensors on compact manifolds. Ting [37] introduced a Helmholtz-type decomposition for symmetric tensors on compact manifolds with boundary, which is associated to linear strains. Geymonat and Krasucki [23] obtained a Hodge-type decomposition for symmetric second-order tensors on compact Lipschitz manifolds with boundary in the Euclidean space. Decompositions for divergence-free second-order tensors were studied by Gurtin [27] for symmetric tensors and Carlson [16] for non-symmetric tensors. In all the above works, boundary conditions on the components of decompositions are either ignored or are imposed on the whole boundary $\partial\bar{\mathcal{B}}$.

Contributions of this paper. By introducing appropriate Hilbert complexes that can describe the kinematics and the kinetics of motion in nonlinear elasticity, we establish orthogonal decompositions similar to (1.1) for second-order tensors on compact manifolds with boundary in $\mathbb{R}^n$, $n = 2, 3$. The main contributions of this paper can be summarized as follows.

- We write various Hilbert complexes in the sense of Brüning and Lesch [13] for different types of second-order tensors on 2D and 3D flat manifolds with boundary. We show that the Hilbert complexes for two-point tensors describe both the kinematics and the kinetics of motion in nonlinear elasticity (the Hilbert complex (3.22) for the 3D case and the complexes (3.24) and (3.25) for the 2D case). In these Hilbert complexes, boundary conditions can be imposed on the whole or only on a portion of the boundary. Let us also mention that in [2], by ignoring boundary conditions, we introduced some differential complexes for nonlinear elasticity that involve only C^∞ tensor fields. This C^∞ assumption is very unrealistic in practice. Moreover, unlike the above Hilbert complexes, these smooth complexes are not suitable for numerical analysis.

- By using the framework of Hilbert complexes, we derive the analogues of (1.1) for non-symmetric second-order tensors (Theorems 11 and 15) and two-point second-order tensors (Theorems 19 and 22). In these decompositions, one can impose boundary conditions on the whole or only on a portion of the boundary. For symmetric second-order tensors, we derive Helmholtz-type decompositions with proper boundary conditions on $\partial_1\bar{\mathcal{B}}$ and $\partial_2\bar{\mathcal{B}}$ (Theorems 30 and 35).

- As an application of the above decompositions, we study the strain compatibility equations of linear and nonlinear elasticity in the presence of Dirichlet boundary conditions (Theorems 36 and 42). In particular, we show that the tensorial analogue of the problem (1.4) gives one the nonlinear compatibility problem in terms of displacement (deformation) gradients. We will show that in the presence of Dirichlet boundary conditions on $\partial_1\bar{\mathcal{B}}$ and depending on the topologies of $\bar{\mathcal{B}}$ and $\partial_1\bar{\mathcal{B}}$, compatibility equations for displacement gradients can be different form the classical compatibility equations with no Dirichlet boundary conditions.

This paper is organized as follows. In §2 after reviewing some preliminaries, we use the machinery of Hilbert complexes to derive a Hodge-type decomposition analogous to (1.1) for $\mathbb{R}^n$-valued forms. In §3 we use the the results of §2 for deriving the analogues of (1.1) for non-symmetric and two-point second-order tensors. For symmetric second-order tensors, we use a basic fact for closed unbounded operators for establishing Helmholtz-type decompositions with appropriate boundary conditions on $\partial_1\bar{\mathcal{B}}$ and $\partial_2\bar{\mathcal{B}}$. In §4 we study some applications of the above decompositions in nonlinear elasticity including the linear and nonlinear compatibility problems with Dirichlet boundary conditions, and the existence of stress functions for non-contractible bodies. Finally, in §5 we briefly discuss the application of the Hilbert complexes introduced in this work for developing a new class of mixed finite element methods for large-deformation (finite-strain) analysis of solids.

2 Hilbert Complexes and Decompositions for Differential Forms

We shall show that some Hilbert complexes and their associated orthogonal decompositions for non-symmetric and two-point second-order tensors can be derived by using their counterparts for $\mathbb{R}^n$-valued differential forms. In this section we study Hilbert complexes of $\mathbb{R}^n$-valued differential forms with certain boundary conditions on flat, compact manifolds with boundary. In §2.1 and §2.2, we review some useful results for differential forms and the associated Hilbert complexes. Then, in §2.3 we derive a Hodge decomposition for L^2 forms by using the machinery of Hilbert complexes. Next, we will obtain similar results for symmetric second-order tensors by following similar approaches. In the following, unless stated otherwise, we assume the summation convention on repeated indices.

2.1 Hilbert Spaces of Differential Forms

Hilbert spaces of differential forms can be defined as completions of smooth forms with respect to certain inner products. To fix our notation, first we tersely review some notions related to smooth differential forms. More details can be found in [31, 35, 29, 1, 21]. We assume $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, is a smooth n-manifold with boundary, that is compact, connected, and orientable, where from now on, smooth signifies of class C^∞. The interior and boundary of $\bar{\mathcal{B}}$ are denoted by $\mathcal{B}$ and $\partial\bar{\mathcal{B}}$, respectively, where we assume that $\partial\bar{\mathcal{B}} = \partial_1\bar{\mathcal{B}} \cup \partial_2\bar{\mathcal{B}}$ with disjoint smooth $(n-1)$-manifolds $\partial_1\bar{\mathcal{B}}$ and $\partial_2\bar{\mathcal{B}}$ being closed (i.e. compact without boundary). Either $\partial_1\bar{\mathcal{B}}$ or $\partial_2\bar{\mathcal{B}}$ can be empty as well. $\Gamma^r(\mathcal{V})$ is the space of C^r-sections of a vector bundle $\mathcal{V} \to \bar{\mathcal{B}}$. Similarly, $\Gamma^{r,\mu}(\mathcal{V})$ is the Hölder space of $C^{r,\mu}$-sections of $\mathcal{V}$. It is customary to denote the spaces of smooth real-valued functions, smooth vector fields, and smooth differential k-forms on $\bar{\mathcal{B}}$ by $C^\infty(\bar{\mathcal{B}})$, $\mathfrak{X}(\bar{\mathcal{B}})$, and $\Omega^k(\bar{\mathcal{B}})$, respectively.

Let $\{\mathbf{E}_i\}$ and $\langle\!\langle , \rangle\!\rangle$ be the standard basis and the standard inner product of $\mathbb{R}^n$, respectively, and suppose $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k \in \mathfrak{X}(\bar{\mathcal{B}})$. The space of smooth $\mathbb{R}^n$-valued k-forms is denoted by $\Omega^k(\bar{\mathcal{B}}; \mathbb{R}^n)$. Any

$\boldsymbol{\alpha} \in \Omega^k(\bar{\mathcal{B}};\mathbb{R}^n)$ can be uniquely written as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^i \otimes \mathbf{E}_i$, where $\boldsymbol{\alpha}^i \in \Omega^k(\bar{\mathcal{B}})$, $i = 1,\ldots,n$, are given by $\boldsymbol{\alpha}^i(\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_k) := \langle\!\langle \boldsymbol{\alpha}(\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_k), \mathbf{E}_i \rangle\!\rangle$. The mapping $\boldsymbol{\alpha} \mapsto (\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^n)$ induces the isomorphism $\Omega^k(\bar{\mathcal{B}};\mathbb{R}^n) \approx \oplus_{i=1}^n \Omega^k(\bar{\mathcal{B}})$. Let $\{X^I\}$ be the global Euclidean coordinates and let $\boldsymbol{\xi},\boldsymbol{\eta} \in \Omega^k(\bar{\mathcal{B}})$. The standard Riemannian metric $\boldsymbol{G}$ on $\bar{\mathcal{B}}$ induces a Riemannian metric $\boldsymbol{G}^k$ on the wedge product $\Lambda^k T^*\bar{\mathcal{B}}$ given by

$$\boldsymbol{G}^k(X)(\boldsymbol{\xi},\boldsymbol{\eta}) := \sum_{I_1<\cdots<I_k} \xi_{I_1\cdots I_k}\eta^{I_1\cdots I_k},\ X \in \bar{\mathcal{B}},$$

where $\xi_{I_1\cdots I_k}$ are the components of $\boldsymbol{\xi}$ and $\eta^{I_1\cdots I_k} = G^{I_1J_1}\cdots G^{I_kJ_k}\eta_{J_1\cdots J_k}$. The exterior derivative $d : \Omega^k(\bar{\mathcal{B}}) \to \Omega^{k+1}(\bar{\mathcal{B}})$ is defined as

$$(d\boldsymbol{\xi})_{I_0\cdots I_k} = \sum_{i=0}^{k} (-1)^i \xi_{I_0\cdots \widehat{I_i}\cdots I_k, I_i},$$

where the hat over an index implies the elimination of that index and "$_{,I_i}$" indicates $\partial/\partial X^{I_i}$. Since $d \circ d = 0$, the exterior derivative induces a complex called the de Rham complex, which is denoted by $\big(\Omega(\bar{\mathcal{B}}), d\big)$.

The Hodge star operator $* : \Omega^k(\bar{\mathcal{B}}) \to \Omega^{n-k}(\bar{\mathcal{B}})$ is a linear isomorphism defined by $\boldsymbol{\xi} \wedge *\boldsymbol{\eta} = \boldsymbol{G}^k(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\mu_G}$, where $\boldsymbol{\mu_G}$ is the Riemannian volume element of $(\bar{\mathcal{B}},\boldsymbol{G})$. The codifferential operator $\delta : \Omega^k(\bar{\mathcal{B}}) \to \Omega^{k-1}(\bar{\mathcal{B}})$ is defined as $\delta := (-1)^{n(k+1)+1} * d *$. By using the Euclidean coordinates $\{X^I\}$ one can write $(\delta\boldsymbol{\beta})_{I_1\cdots I_{k-1}} = -\beta_{JI_1\cdots I_{k-1},J}$. We have $\delta \circ \delta = 0$, and the complex induced by δ is denoted by $\big(\Omega(\bar{\mathcal{B}}), \delta\big)$. The Laplace-de Rham operator $\Delta : \Omega^k(\bar{\mathcal{B}}) \to \Omega^k(\bar{\mathcal{B}})$ is an elliptic second-order operator defined as $\Delta := d \circ \delta + \delta \circ d$. The above operators for standard forms can be extended to $\mathbb{R}^n$-valued forms as follows: $\boldsymbol{d\alpha} := (d\boldsymbol{\alpha}^1,\ldots,d\boldsymbol{\alpha}^n)$, $\boldsymbol{*\alpha} := (*\boldsymbol{\alpha}^1,\ldots,*\boldsymbol{\alpha}^n)$, $\boldsymbol{\alpha} \wedge (\boldsymbol{*\omega}) := \sum_i \boldsymbol{\alpha}^i \wedge (*\boldsymbol{\omega}^i)$, $\boldsymbol{\delta\alpha} := (\delta\boldsymbol{\alpha}^1,\ldots,\delta\boldsymbol{\alpha}^n)$, and $\boldsymbol{\Delta\alpha} := (\Delta\boldsymbol{\alpha}^1,\ldots,\Delta\boldsymbol{\alpha}^n)$.

One can uniquely decompose $\boldsymbol{Y} \in \mathfrak{X}(\bar{\mathcal{B}})$ as $\boldsymbol{Y}|_{\partial\bar{\mathcal{B}}} = \mathsf{t}\boldsymbol{Y} + \mathsf{n}\boldsymbol{Y}$, where $\mathsf{t}\boldsymbol{Y}$ and $\mathsf{n}\boldsymbol{Y}$ are tangent and normal to $\partial\bar{\mathcal{B}}$, respectively. The tangential and normal parts $\mathsf{t}\boldsymbol{\xi}$ and $\mathsf{n}\boldsymbol{\xi}$ of $\boldsymbol{\xi} \in \Omega^k(\bar{\mathcal{B}})$ at $\partial\bar{\mathcal{B}}$ are defined as $\mathsf{t}\boldsymbol{\xi}(\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_k) := \boldsymbol{\xi}(\mathsf{t}\boldsymbol{Y}_1,\ldots,\mathsf{t}\boldsymbol{Y}_k)$, and $\mathsf{n}\boldsymbol{\xi} := \boldsymbol{\xi}|_{\partial\bar{\mathcal{B}}} - \mathsf{t}\boldsymbol{\xi}$ [35, page 27]. Note that these definitions imply that $\mathsf{t}\boldsymbol{\xi}$ and $\mathsf{n}\boldsymbol{\xi}$ are not differential forms on $\partial\bar{\mathcal{B}}$ in the sense that in general, the vector fields $\boldsymbol{Y}_i$ at $X \in \partial\bar{\mathcal{B}}$ belong to $T_X\bar{\mathcal{B}}$ and not necessarily to $T_X(\partial\bar{\mathcal{B}})$. Let $\mathrm{i}^*\boldsymbol{\xi}$ be the pull-back of $\boldsymbol{\xi}$ by the inclusion map $\mathrm{i} : \partial\bar{\mathcal{B}} \hookrightarrow \bar{\mathcal{B}}$. Then $\mathrm{i}^*\boldsymbol{\xi}$ is a form on $\partial\bar{\mathcal{B}}$ and we have $\mathrm{i}^*\boldsymbol{\xi} = \mathrm{i}^*(\mathsf{t}\boldsymbol{\xi})$. For $\boldsymbol{\xi} \in \Omega^0(\bar{\mathcal{B}})$, we have $\mathsf{t}\boldsymbol{\xi} = \boldsymbol{\xi}|_{\partial\bar{\mathcal{B}}}$, and $\mathsf{n}\boldsymbol{\xi} = 0$. A differential form $\boldsymbol{\xi}$ is called tangent (normal) to $\partial\bar{\mathcal{B}}$ if $\mathsf{n}\boldsymbol{\xi} = 0$ ($\mathsf{t}\boldsymbol{\xi} = 0$). The spaces of tangent and normal smooth forms are denoted by $\Omega^k_{\mathrm{t}}(\bar{\mathcal{B}})$ and $\Omega^k_{\mathrm{n}}(\bar{\mathcal{B}})$, respectively.

By using the natural isomorphism $\flat : \mathfrak{X}(\bar{\mathcal{B}}) \to \Omega^1(\bar{\mathcal{B}})$ induced by $\boldsymbol{G}$, i.e. $\big(\boldsymbol{Y}^\flat\big)_I = G_{IJ}Y^J$, it is straightforward to show that [35, Proposition 3.5.1]

$$\mathsf{t}(\boldsymbol{Y}^\flat) = (\mathsf{t}\boldsymbol{Y})^\flat, \text{ and } \quad \mathsf{n}(\boldsymbol{Y}^\flat) = (\mathsf{n}\boldsymbol{Y})^\flat. \tag{2.1}$$

For $\boldsymbol{\alpha} \in \Omega^k(\bar{\mathcal{B}};\mathbb{R}^n)$, we have $\mathsf{t}\boldsymbol{\alpha} = (\mathsf{t}\boldsymbol{\alpha}^1,\ldots,\mathsf{t}\boldsymbol{\alpha}^n)$, and $\mathsf{n}\boldsymbol{\alpha} = (\mathsf{n}\boldsymbol{\alpha}^1,\ldots,\mathsf{n}\boldsymbol{\alpha}^n)$. One can write [35, Proposition 1.2.6]

$$\boldsymbol{*}\,(\mathsf{n}\boldsymbol{\alpha}) = \mathsf{t}(\boldsymbol{*\alpha}), \qquad \boldsymbol{*}\,(\mathsf{t}\boldsymbol{\alpha}) = \mathsf{n}(\boldsymbol{*\alpha}), \tag{2.2}$$

$$\mathrm{i}^*\big(\mathsf{t}(\boldsymbol{d\alpha})\big) = \boldsymbol{d}\,(\mathrm{i}^*(\mathsf{t}\boldsymbol{\alpha}))\,, \quad \mathrm{i}^*\big(\,\boldsymbol{*}\,(\mathsf{n}\boldsymbol{\delta\alpha})\big) = (-1)^{(k+1)(n-k+1)}\boldsymbol{d}\,(\mathrm{i}^*(\boldsymbol{*}\mathsf{n}\boldsymbol{\alpha}))\,. \tag{2.3}$$

The space of $\mathbb{R}^n$-valued k-forms that are tangent (normal) to $\partial_j\bar{\mathcal{B}}$, $j = 1,2$, is denoted by $\Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n)$ ($\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n)$). The first relation in (2.3) suggests that if $\boldsymbol{\alpha} \in \Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n)$ then $\big(\mathsf{t}(\boldsymbol{d\alpha})\big)|_{\partial_j\bar{\mathcal{B}}} = 0$. This allows us to define the restriction $\boldsymbol{d}_{\mathrm{n}_j} : \Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \to \Omega^{k+1}_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n)$, $\boldsymbol{d}_{\mathrm{n}_j}\boldsymbol{\alpha} := \boldsymbol{d\alpha}$. Similarly, we have the restriction $\boldsymbol{\delta}_{\mathrm{t}_j} : \Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \to \Omega^{k-1}_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n)$. Therefore, the complexes $\big(\Omega(\bar{\mathcal{B}};\mathbb{R}^3), \boldsymbol{d}\big)$ and $\big(\Omega(\bar{\mathcal{B}};\mathbb{R}^3), \boldsymbol{\delta}\big)$

admit the following linear subcomplexes

$$
\begin{aligned}
&0 \longrightarrow \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} \cdots \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} \Omega^n_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \longrightarrow 0,\\
&0 \longleftarrow \Omega^0_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} \Omega^1_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} \cdots \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} \Omega^n_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n) \longleftarrow 0,
\end{aligned}
\tag{2.4}
$$

which are denoted by $\big(\Omega_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^n), \boldsymbol{d}_{\mathrm{n}_j}\big)$ and $\big(\Omega_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^n), \boldsymbol{\delta}_{\mathrm{t}_j}\big)$, respectively. In the terminology of Gilkey [25, §4.1], for $\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$, one recovers the relative de Rham complex $\big(\Omega_{\mathrm{n}}(\bar{\mathcal{B}};\mathbb{R}^3), \boldsymbol{d}_{\mathrm{n}}\big)$ and the absolute de Rham complex $\big(\Omega_{\mathrm{t}}(\bar{\mathcal{B}};\mathbb{R}^3), \boldsymbol{\delta}_{\mathrm{t}}\big)$. Clearly, for $\partial_1\bar{\mathcal{B}} = \varnothing$, we obtain the standard complexes $\big(\Omega(\bar{\mathcal{B}};\mathbb{R}^3), \boldsymbol{d}\big)$ and $\big(\Omega(\bar{\mathcal{B}};\mathbb{R}^3), \boldsymbol{\delta}\big)$, respectively.

The L^2-inner products $\langle\!\langle , \rangle\!\rangle_{L^2}$ on $\Omega^k(\bar{\mathcal{B}})$ and $\langle\!\langle , \rangle\!\rangle_{\boldsymbol{L}^2}$ on $\Omega^k(\bar{\mathcal{B}};\mathbb{R}^n)$ are defined as $\langle\!\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\!\rangle_{L^2} := \int_{\bar{\mathcal{B}}} \boldsymbol{G}^k(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\mu}_{\boldsymbol{G}}$, and $\langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle\!\rangle_{\boldsymbol{L}^2} := \sum_{i=1}^n \langle\!\langle \boldsymbol{\alpha}^i, \boldsymbol{\gamma}^i\rangle\!\rangle_{L^2}$. By using the L^2-inner products, we observe that $\boldsymbol{\alpha} \mapsto (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^n)$ and the Hodge star operator are isometries [35, page 40].

Let $\int_{\bar{\mathcal{B}}} \boldsymbol{\alpha} := \big(\int_{\bar{\mathcal{B}}} \boldsymbol{\alpha}^i\big)\mathbf{E}_i$, $\boldsymbol{\alpha} \in \Omega^n(\bar{\mathcal{B}};\mathbb{R}^n)$. Stokes' theorem for $\mathbb{R}^n$-valued forms reads $\int_{\bar{\mathcal{B}}} \boldsymbol{d\gamma} = \int_{\partial\bar{\mathcal{B}}} \mathrm{i}^*\boldsymbol{\gamma}$, $\boldsymbol{\gamma} \in \Omega^{n-1}(\bar{\mathcal{B}};\mathbb{R}^n)$, where the orientation of $\partial\bar{\mathcal{B}}$ is induced by that of $\bar{\mathcal{B}}$ [31, page 411]. Green's formula for $\mathbb{R}^n$-valued forms states that for any $\boldsymbol{\gamma} \in \Omega^{k-1}(\bar{\mathcal{B}};\mathbb{R}^n)$ and $\boldsymbol{\alpha} \in \Omega^k(\bar{\mathcal{B}};\mathbb{R}^n)$, we have [35, Proposition 2.1.2]

$$
\langle\!\langle \boldsymbol{d\gamma}, \boldsymbol{\alpha}\rangle\!\rangle_{\boldsymbol{L}^2} = \langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\delta\alpha}\rangle\!\rangle_{\boldsymbol{L}^2} + \int_{\partial\bar{\mathcal{B}}} \mathrm{i}^*(\mathsf{t}\boldsymbol{\gamma} \wedge *\mathsf{n}\boldsymbol{\alpha}). \tag{2.5}
$$

The infinite-dimensional linear space $\Omega^k(\bar{\mathcal{B}};\mathbb{R}^n)$ with its L^2-inner product is not complete. Let $\xi_{I_1\cdots I_k}$ be the components of a k-form $\boldsymbol{\xi}$ in the Cartesian coordinates $\{X^I\}$. The Hilbert space $H^s\Omega^k(\bar{\mathcal{B}})$ is the space of all k-forms $\boldsymbol{\xi}$ with all Cartesian components $\xi_{I_1\cdots I_k}$ belonging to the standard Sobolev space of $\mathbb{R}$-valued functions $\big(H^s(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^s}\big)$. The H^s-inner product of $H^s\Omega^k(\bar{\mathcal{B}})$ is given by

$$
\langle\!\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\!\rangle_{H^s} = \sum_{1\le I_1<\cdots<I_k\le n} \langle\!\langle \xi_{I_1\cdots I_k}, \eta_{I_1\cdots I_k}\rangle\!\rangle_{H^s}.
$$

Alternatively, $H^s\Omega^k(\bar{\mathcal{B}})$ can be defined as the completion of $\big(\Omega^k(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^s}\big)$. The L^2 space corresponds to the special case H^0.

The partly Sobolev spaces $H^d\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}})$ and $H^\delta\Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}})$ are defined as the completions of $\big(\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^d}\big)$ and $\big(\Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^\delta}\big)$, respectively, where

$$
\begin{aligned}
\langle\!\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\!\rangle_{H^d} &:= \langle\!\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\!\rangle_{L^2} + \langle\!\langle d\boldsymbol{\xi}, d\boldsymbol{\eta}\rangle\!\rangle_{L^2},\\
\langle\!\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\!\rangle_{H^\delta} &:= \langle\!\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\!\rangle_{L^2} + \langle\!\langle \delta\boldsymbol{\xi}, \delta\boldsymbol{\eta}\rangle\!\rangle_{L^2}.
\end{aligned}
$$

The Sobolev spaces $\boldsymbol{H}^s\Omega^k(\bar{\mathcal{B}})$ and $\boldsymbol{H}^d\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}})$ of $\mathbb{R}^n$-valued forms are defined as $\boldsymbol{H}^s\Omega^k(\bar{\mathcal{B}}) := \oplus_{i=1}^n H^s\Omega^k(\bar{\mathcal{B}})$, and $\boldsymbol{H}^d\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}}) := \oplus_{i=1}^n H^d\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}})$ with $\langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle\!\rangle_{\boldsymbol{H}^s} = \sum_i \langle\!\langle \boldsymbol{\alpha}^i, \boldsymbol{\gamma}^i\rangle\!\rangle_{H^s}$, and $\langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle\!\rangle_{\boldsymbol{H}^d} = \sum_i \langle\!\langle \boldsymbol{\alpha}^i, \boldsymbol{\gamma}^i\rangle\!\rangle_{H^d}$. For $\partial_1\bar{\mathcal{B}} = \varnothing$, and $\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$, we write $\boldsymbol{H}^d\Omega^k(\bar{\mathcal{B}}) := \boldsymbol{H}^d\Omega^k_{\mathrm{n}_1}(\bar{\mathcal{B}})$, and $\boldsymbol{H}^d\Omega^k_{\mathrm{n}}(\bar{\mathcal{B}}) := \boldsymbol{H}^d\Omega^k_{\mathrm{n}_1}(\bar{\mathcal{B}})$, respectively. Similarly, we also define $\boldsymbol{H}^\delta\Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}})$, $\boldsymbol{H}^\delta\Omega^k(\bar{\mathcal{B}})$, and $\boldsymbol{H}^\delta\Omega^k_{\mathrm{t}}(\bar{\mathcal{B}})$.

Theorems for standard Sobolev spaces such as the Sobolev and Rellich theorems extend to Sobolev spaces of differential forms as well, e.g. see [35, Theorem 1.3.6]. The Sobolev space $\boldsymbol{H}^1\Omega^k(\bar{\mathcal{B}})$ can be continuously embedded in both $\boldsymbol{H}^d\Omega^k(\bar{\mathcal{B}})$ and $\boldsymbol{H}^\delta\Omega^k(\bar{\mathcal{B}})$. Other useful properties of partly Sobolev spaces can be found in [29].

The smooth operators d_{n_j} and δ_{t_j} can be extended to the continuous mappings $d_{\mathrm{n}_j} : H^d\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}}) \to H^d\Omega^{k+1}_{\mathrm{n}_j}(\bar{\mathcal{B}})$ and $\delta_{\mathrm{t}_j} : H^\delta\Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}}) \to H^\delta\Omega^{k-1}_{\mathrm{t}_j}(\bar{\mathcal{B}})$, where $d_{\mathrm{n}_j} \circ d_{\mathrm{n}_j} = 0$, and $\delta_{\mathrm{t}_j} \circ \delta_{\mathrm{t}_j} = 0$ [26, Proposition 3.8]. The extensions $\boldsymbol{d}_{\mathrm{n}_j} : \boldsymbol{H}^d\Omega^k_{\mathrm{n}_j}(\bar{\mathcal{B}}) \to \boldsymbol{H}^d\Omega^{k+1}_{\mathrm{n}_j}(\bar{\mathcal{B}})$ and $\boldsymbol{\delta}_{\mathrm{t}_j} : \boldsymbol{H}^\delta\Omega^k_{\mathrm{t}_j}(\bar{\mathcal{B}}) \to \boldsymbol{H}^\delta\Omega^{k-1}_{\mathrm{t}_j}(\bar{\mathcal{B}})$ are defined as

$\boldsymbol{d}_{\mathrm{n}_j}\boldsymbol{\alpha} := (d_{\mathrm{n}_j}\boldsymbol{\alpha}^1, \dots, d_{\mathrm{n}_j}\boldsymbol{\alpha}^n)$, and $\boldsymbol{\delta}_{\mathrm{t}_j}\boldsymbol{\alpha} := (\delta_{\mathrm{t}_j}\boldsymbol{\alpha}^1, \dots, \delta_{\mathrm{t}_j}\boldsymbol{\alpha}^n)$.

Green's formula (2.5) is still valid for $\boldsymbol{\gamma} \in \boldsymbol{H}^1\Omega^{k-1}(\bar{\mathcal{B}})$ and $\boldsymbol{\alpha} \in \boldsymbol{H}^1\Omega^k(\bar{\mathcal{B}})$ [35, Proposition 2.1.2]. The following special case of (2.5) is also valid [26, Theorem 4.2]: For any $\boldsymbol{\gamma} \in \boldsymbol{H}^d\Omega^{k-1}_{\mathrm{n}_1}(\bar{\mathcal{B}})$, and $\boldsymbol{\alpha} \in \boldsymbol{H}^\delta\Omega^k_{\mathrm{t}_2}(\bar{\mathcal{B}})$, we have

$$\langle\!\langle \boldsymbol{d}_{\mathrm{n}_1}\boldsymbol{\gamma}, \boldsymbol{\alpha}\rangle\!\rangle_{\boldsymbol{L}^2} = \langle\!\langle \boldsymbol{\gamma}, \boldsymbol{\delta}_{\mathrm{t}_2}\boldsymbol{\alpha}\rangle\!\rangle_{\boldsymbol{L}^2}. \tag{2.6}$$

2.2 Hilbert Complexes Induced by the de Rham Complex

We are now ready to study some Hilbert complexes induced by the $\mathbb{R}^n$-valued de Rham complex. First, we mention some basic properties of unbounded operators. Let H_1 and H_2 be Hilbert spaces and let $D : H_1 \to H_2$ be an unbounded operator, i.e. D is linear and its domain $\mathrm{D}(D) \subset H_1$ is a linear subspace. The operator D is called a closed operator if its graph $G(D) := \{(\mathbf{x}, D(\mathbf{x})) : \mathbf{x} \in \mathrm{D}(D)\}$ is a closed subset of $H_1 \times H_2$. Suppose D is densely-defined. Then, its adjoint operator $D^a : H_2 \to H_1$ is defined by using the relation $\langle\!\langle D^a(\mathbf{y}), \mathbf{x}\rangle\!\rangle_{H_1} = \langle\!\langle \mathbf{y}, D(\mathbf{x})\rangle\!\rangle_{H_2}$, $\forall \mathbf{x} \in \mathrm{D}(D)$. A densely-defined operator D is closed if and only if D^a is densely-defined, closed, and $(D^a)^a = D$ [41, §7.2].

A Hilbert complex is defined as follows [13]: Let $D_k : H_k \to H_{k+1}$, $0 \leq k \leq N$, be closed, densely-defined operators between Hilbert spaces H_k, with $H_{N+1} = \{0\}$. The domain and range of D_k are denoted by $\mathscr{D}_k := \mathrm{D}(D_k)$, and $\mathscr{R}_k := D_k(\mathscr{D}_k)$, respectively. Also assume that $\mathscr{R}_k \subset \mathscr{D}_{k+1}$, and $D_{k+1} \circ D_k = 0$. The Hilbert complex $(\mathscr{D}, D)$ is the complex induced by D_k, i.e. $0 \longrightarrow \mathscr{D}_0 \xrightarrow{D_0} \mathscr{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} \mathscr{D}_N \longrightarrow 0$.

The property of closed operators mentioned in the first paragraph implies that the operators D_k admit closed, densely-defined adjoint operators D^a_k, with $\mathscr{R}^a_k \subset \mathscr{D}^a_{k-1}$, and $D^a_{k-1} \circ D^a_k = 0$. The dual complex of $(\mathscr{D}, D)$ is then defined to be the complex $(\mathscr{D}^a, D^a)$, i.e. $0 \longleftarrow \mathscr{D}^a_0 \xleftarrow{D^a_0} \mathscr{D}^a_1 \xleftarrow{D^a_1} \cdots \xleftarrow{D^a_{N-1}} \mathscr{D}^a_N \longleftarrow 0$.

The k-th cohomology of $(\mathscr{D}, D)$ is defined as $\mathsf{H}_k := \ker D_k / \mathscr{R}_{k-1}$. A Hilbert complex is called Fredholm if $\dim \mathsf{H}_k < \infty$, for all k. One can show that $(\mathscr{D}, D)$ is Fredholm if and only if $(\mathscr{D}^a, D^a)$ is Fredholm and in this case, we have $\mathsf{H}_k \approx \mathcal{H}_k = \mathcal{H}^a_k \approx \mathsf{H}^a_k$, where $\mathcal{H}_k = \ker D_k \cap \ker D^a_{k-1}$, and $\mathsf{H}^a_k = \ker D^a_{k-1}/\mathscr{R}^a_k$ [13, Corollary 2.6].

The Laplacian $L_k : H_k \to H_k$ of $(\mathscr{D}, D)$ is a self-adjoint operator defined as $L_k := D^a_k \circ D_k + D_{k-1} \circ D^a_{k-1}$. Clearly, we have $\mathcal{H}_k \subset \ker L_k$. Because for any $\mathbf{x} \in \mathrm{D}(L_k)$, one can write

$$\langle\!\langle L_k(\mathbf{x}), \mathbf{x}\rangle\!\rangle_{H_k} = \langle\!\langle D_k(\mathbf{x}), D_k(\mathbf{x})\rangle\!\rangle_{H_{k+1}} + \langle\!\langle D^a_{k-1}(\mathbf{x}), D^a_{k-1}(\mathbf{x})\rangle\!\rangle_{H_{k-1}} \geq 0,$$

we also have $\ker L_k \subset \mathcal{H}_k$, and thus $\ker L_k = \mathcal{H}_k$.

The operator $\boldsymbol{d}_{\mathrm{n}_1} : \boldsymbol{H}^d\Omega^k_{\mathrm{n}_1}(\bar{\mathcal{B}}) \to \boldsymbol{H}^d\Omega^{k+1}_{\mathrm{n}_1}(\bar{\mathcal{B}})$ can be considered as a densely-defined closed operator $\boldsymbol{d}_{\mathrm{n}_1} : \boldsymbol{L}^2\Omega^k(\bar{\mathcal{B}}) \to \boldsymbol{L}^2\Omega^{k+1}(\bar{\mathcal{B}})$ [26, Theorem 4.2]. Consequently, one obtains the Hilbert complex $\big(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\big)$. The relation (2.6) suggests that the unbounded operator $\boldsymbol{\delta}_{\mathrm{t}_2} : \boldsymbol{L}^2\Omega^k(\bar{\mathcal{B}}) \to \boldsymbol{L}^2\Omega^{k-1}(\bar{\mathcal{B}})$, with $\mathrm{D}(\boldsymbol{\delta}_{\mathrm{t}_2}) = \boldsymbol{H}^\delta\Omega^k_{\mathrm{t}_2}(\bar{\mathcal{B}})$, is the adjoint operator of $\boldsymbol{d}_{\mathrm{n}_1}$, cf. [26, Theorem 4.2]. Hence, $\big(\boldsymbol{H}^\delta\Omega_{\mathrm{t}_2}(\bar{\mathcal{B}}), \boldsymbol{\delta}_{\mathrm{t}_2}\big)$ is the dual complex of $\big(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\big)$.

Next, we study the Fredholm property of $\big(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\big)$. Let $\mathsf{H}^k_{\mathrm{n}_1}(\bar{\mathcal{B}})$ be the k-th cohomology group of $\big(H^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), d_{\mathrm{n}_1}\big)$. Then, one can write

$$\mathsf{H}^k_{\mathrm{n}_1}(\bar{\mathcal{B}}) \approx \mathrm{H}^k_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}}), \tag{2.7}$$

where the finite-dimensional space $\mathrm{H}^k_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$ is the k-th relative de Rham cohomology of the pair $(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$ [26, Theorem 5.3]. Let $\boldsymbol{\mathsf{H}}^k_{\mathrm{n}_1}(\bar{\mathcal{B}})$ and $\boldsymbol{\mathsf{H}}^k_{\mathrm{t}_2}(\bar{\mathcal{B}})$ be the k-th cohomology groups of $\big(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\big)$ and $\big(\boldsymbol{H}^\delta\Omega_{\mathrm{t}_2}(\bar{\mathcal{B}}), \boldsymbol{\delta}_{\mathrm{t}_2}\big)$, respectively. By using (2.7), we conclude that $\big(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\big)$ is Fredholm and we have

$$\boldsymbol{\mathsf{H}}^k_{\mathrm{n}_1}(\bar{\mathcal{B}}) \approx \boldsymbol{\mathsf{H}}^k_{\mathrm{t}_2}(\bar{\mathcal{B}}) \approx \oplus_{i=1}^n \mathrm{H}^k_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}}). \tag{2.8}$$

Suppose $\boldsymbol{\Delta}_k^{\mathrm{n}_1,\mathrm{t}_2}$ is the Laplacian of $\left(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\right)$, where

$$\mathrm{D}(\boldsymbol{\Delta}_k^{\mathrm{n}_1,\mathrm{t}_2}) = \left\{\boldsymbol{\alpha} \in \boldsymbol{H}^d\Omega_{\mathrm{n}_1}^k(\bar{\mathcal{B}}) \cap \boldsymbol{H}^\delta\Omega_{\mathrm{t}_2}^k(\bar{\mathcal{B}}) : \boldsymbol{d}_{\mathrm{n}_1}\boldsymbol{\alpha} \in \boldsymbol{H}^\delta\Omega_{\mathrm{t}_2}^{k+1}(\bar{\mathcal{B}}), \boldsymbol{\delta}_{\mathrm{t}_2}\boldsymbol{\alpha} \in \boldsymbol{H}^d\Omega_{\mathrm{n}_1}^{k-1}(\bar{\mathcal{B}})\right\}.$$

Let $\boldsymbol{\mathcal{H}}_{\mathrm{n}_1,\mathrm{t}_2}^k(\bar{\mathcal{B}}) := \ker \boldsymbol{\Delta}_k^{\mathrm{n}_1,\mathrm{t}_2} = \ker \boldsymbol{d}_{\mathrm{n}_1} \cap \ker \boldsymbol{\delta}_{\mathrm{t}_2}$. The Hodge star operator induces an isomorphism between $\left(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\right)$ and $\left(\boldsymbol{H}^\delta\Omega_{\mathrm{t}_1}(\bar{\mathcal{B}}), \boldsymbol{\delta}_{\mathrm{t}_1}\right)$ [26, §3.4]. This isomorphism together with (2.8) allows one to write

$$\mathsf{H}_{\mathrm{n}_1}^k(\bar{\mathcal{B}}) \approx \boldsymbol{\mathcal{H}}_{\mathrm{n}_1,\mathrm{t}_2}^k(\bar{\mathcal{B}}) \approx \mathsf{H}_{\mathrm{n}_2}^{n-k}(\bar{\mathcal{B}}). \tag{2.9}$$

Remark 1. Differential forms with compact support in $\mathcal{B}$ induce the complex $(\Omega_c(\mathcal{B}), d)$. Let $\mathrm{H}_{dR_c}^k(\mathcal{B})$ be the k-th cohomology group of this complex. Poincaré duality [11, page 44] implies that $\mathrm{H}_{dR}^k(\mathcal{B}) \approx \mathrm{H}_{dR_c}^{n-k}(\mathcal{B})$. By using the analogue of (2.9) for standard forms with $\partial_1\bar{\mathcal{B}} = \varnothing$, one concludes that $\left(\Omega_{\mathrm{n}}(\bar{\mathcal{B}}), d_{\mathrm{n}}\right)$ and $(\Omega_c(\mathcal{B}), d)$ have the same cohomology groups. In fact, since $\Omega_c^k(\mathcal{B})$ is dense in $H^d\Omega_{\mathrm{n}}^k(\bar{\mathcal{B}})$ [29, Corollary 3.8], one can also use the completion of $(\Omega_c(\mathcal{B}), d)$ for deriving the Hilbert complex $\left(H^d\Omega_{\mathrm{n}}(\bar{\mathcal{B}}), d_{\mathrm{n}}\right)$. A similar conclusion holds for the complexes $(\Omega_c(\mathcal{B}), \delta)$ and $\left(H^\delta\Omega_{\mathrm{t}}(\bar{\mathcal{B}}), \delta_{\mathrm{t}}\right)$ as well.

Remark 2. The boundary-value problem

$$\begin{aligned}
&\boldsymbol{\Delta}(\boldsymbol{\alpha}) = \boldsymbol{\gamma}, \\
&\mathsf{t}\boldsymbol{\alpha} = 0,\ \mathsf{t}(\boldsymbol{\delta}\boldsymbol{\alpha}) = 0,\ \text{on } \partial_1\bar{\mathcal{B}},\ \mathsf{n}\boldsymbol{\alpha} = 0,\ \mathsf{n}(\boldsymbol{d}\boldsymbol{\alpha}) = 0,\ \text{on } \partial_2\bar{\mathcal{B}},
\end{aligned} \tag{2.10}$$

is elliptic [9], where $\boldsymbol{\Delta} = \boldsymbol{\delta} \circ \boldsymbol{d} + \boldsymbol{d} \circ \boldsymbol{\delta}$ is the Laplace-de Rham operator. Therefore, some standard regularity theorems apply to solutions of (2.10), e.g. see [36, Propositions 5.11.2, 5.11.16]. In particular, since $\boldsymbol{\mathcal{H}}_{\mathrm{n}_1,\mathrm{t}_2}^k(\bar{\mathcal{B}})$ corresponds to $\boldsymbol{\gamma} = 0$, one concludes that $\boldsymbol{\mathcal{H}}_{\mathrm{n}_1,\mathrm{t}_2}^k(\bar{\mathcal{B}})$ merely consists of smooth forms and is finite dimensional. It is well known that on a compact manifold with non-empty boundary, the spaces of harmonic forms $\ker \boldsymbol{\Delta}$ and harmonic fields $\ker \boldsymbol{d} \cap \ker \boldsymbol{\delta}$ do not coincide and are infinite dimensional [35, Theorem 3.4.2]. The space of harmonic forms that satisfy the boundary conditions of (2.10) are finite dimensional.

2.3 Hodge Decompositions and Potentials

Any Hilbert complex $(\mathscr{D}, D)$ endows the underlying Hilbert spaces H_i with a Hodge decomposition as follows [13, Lemma 2.1]: Since D_i is a closed operator, $\ker D_i$ is closed and $(\ker D_i)^\perp = \overline{\mathscr{R}_i^a}$, where $(\ker D_i)^\perp$ is the orthogonal complement of $\ker D_i$ and $\overline{\mathscr{R}_i^a}$ is the closure of $\mathscr{R}_i^a$ in H_i. The orthogonal projection theorem implies that $H_i = \ker D_i \oplus (\ker D_i)^\perp$. We also have $\ker D_i = \overline{\mathscr{R}}_{i-1} \oplus \mathcal{H}_i$. Therefore, one obtains the following orthogonal decomposition

$$H_i = \overline{\mathscr{R}}_{i-1} \oplus \mathcal{H}_i \oplus \overline{\mathscr{R}_i^a}, \tag{2.11}$$

which is called the weak Hodge decomposition. If all $\mathscr{R}_i$ are closed, by using the closed range theorem, one concludes that all $\mathscr{R}_i^a$ are also closed and (2.11) gives the (strong) Hodge decomposition

$$H_i = \mathscr{R}_{i-1} \oplus \mathcal{H}_i \oplus \mathscr{R}_i^a. \tag{2.12}$$

If $(\mathscr{D}, D)$ is Fredholm, then it admits the Hodge decomposition (2.12) [13, Corollary 2.5]. Therefore, by using (2.8), one concludes that $\left(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_1}\right)$ induces the $\boldsymbol{L}^2$-orthogonal Hodge decomposition

$$\boldsymbol{L}^2\Omega^k(\bar{\mathcal{B}}) = \boldsymbol{d}\left(\boldsymbol{H}^d\Omega_{\mathrm{n}_1}^{k-1}(\bar{\mathcal{B}})\right) \oplus \boldsymbol{\mathcal{H}}_{\mathrm{n}_1,\mathrm{t}_2}^k(\bar{\mathcal{B}}) \oplus \boldsymbol{\delta}\left(\boldsymbol{H}^\delta\Omega_{\mathrm{t}_2}^{k+1}(\bar{\mathcal{B}})\right). \tag{2.13}$$

This decomposition with $\partial_1\bar{\mathcal{B}} = \varnothing$ ($\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$) is called the Hodge decomposition for the absolute (relative) boundary condition, cf. [25, Lemma 4.1.3]. Equivalently, one can also write the following Helmholtz decompositions

$$\boldsymbol{L}^2\Omega^k(\bar{\mathcal{B}}) = \boldsymbol{d}\Big(\boldsymbol{H}^d\Omega^{k-1}_{\mathrm{n}_1}(\bar{\mathcal{B}})\Big) \oplus \ker\boldsymbol{\delta}_{\mathrm{t}_2} = \ker\boldsymbol{d}_{\mathrm{n}_1} \oplus \boldsymbol{\delta}\Big(\boldsymbol{H}^\delta\Omega^{k+1}_{\mathrm{t}_2}(\bar{\mathcal{B}})\Big), \tag{2.14}$$

where

$$\begin{aligned} \ker\boldsymbol{d}_{\mathrm{n}_1} &= \boldsymbol{d}\Big(\boldsymbol{H}^d\Omega^{k-1}_{\mathrm{n}_1}(\bar{\mathcal{B}})\Big) \oplus \boldsymbol{\mathcal{H}}^k_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}), \\ \ker\boldsymbol{\delta}_{\mathrm{t}_2} &= \boldsymbol{\delta}\Big(\boldsymbol{H}^\delta\Omega^{k+1}_{\mathrm{t}_2}(\bar{\mathcal{B}})\Big) \oplus \boldsymbol{\mathcal{H}}^k_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \end{aligned} \tag{2.15}$$

Thus, any $\boldsymbol{\alpha} \in \boldsymbol{L}^2\Omega^k(\bar{\mathcal{B}})$ can be uniquely decomposed as

$$\boldsymbol{\alpha} = \boldsymbol{d\alpha}^{\mathrm{n}_1} + \boldsymbol{\alpha}^{\mathrm{n}_1,\mathrm{t}_2}_{\mathcal{H}} + \boldsymbol{\delta\alpha}^{\mathrm{t}_2}, \tag{2.16}$$

where $\boldsymbol{\alpha}^{\mathrm{n}_1} \in \boldsymbol{H}^d\Omega^{k-1}_{\mathrm{n}_1}(\bar{\mathcal{B}})$, $\boldsymbol{\alpha}^{\mathrm{n}_1,\mathrm{t}_2}_{\mathcal{H}} \in \boldsymbol{\mathcal{H}}^k_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$, $\boldsymbol{\alpha}^{\mathrm{t}_2} \in \boldsymbol{H}^\delta\Omega^{k+1}_{\mathrm{t}_2}(\bar{\mathcal{B}})$, and $\boldsymbol{d\alpha}^{\mathrm{n}_1}$, $\boldsymbol{\alpha}^{\mathrm{n}_1,\mathrm{t}_2}_{\mathcal{H}}$, and $\boldsymbol{\delta\alpha}^{\mathrm{t}_2}$ are unique and mutually orthogonal with respect to the inner product $\langle\!\langle,\rangle\!\rangle_{\boldsymbol{L}^2}$. The decomposition (1.1) for vector fields corresponds to the analogue of (2.13) for standard forms.

To compare the Hodge decomposition (2.13) and the standard Hodge-Morrey decomposition [32, page 312], let

$$\begin{aligned} \boldsymbol{H}^1\Omega^k_{\mathrm{n}}(\bar{\mathcal{B}}) &:= \boldsymbol{H}^1\Omega^k(\bar{\mathcal{B}}) \cap \boldsymbol{H}^d\Omega^k_{\mathrm{n}}(\bar{\mathcal{B}}), \\ \boldsymbol{H}^1\Omega^k_{\mathrm{t}}(\bar{\mathcal{B}}) &:= \boldsymbol{H}^1\Omega^k(\bar{\mathcal{B}}) \cap \boldsymbol{H}^\delta\Omega^k_{\mathrm{t}}(\bar{\mathcal{B}}), \\ \boldsymbol{\mathcal{H}}^k(\bar{\mathcal{B}}) &:= \Big\{\boldsymbol{\alpha} \in \boldsymbol{H}^1\Omega^k(\bar{\mathcal{B}}) : \boldsymbol{d\alpha} = 0,\ \boldsymbol{\delta\alpha} = 0\Big\}. \end{aligned}$$

The space $\boldsymbol{\mathcal{H}}^k(\bar{\mathcal{B}})$ is the direct sum of n-copies of the infinite-dimensional space of harmonic fields on $\bar{\mathcal{B}}$, see Remark 2. The standard Hodge-Morrey decomposition induces the following decomposition for $\mathbb{R}^n$-valued forms:

$$\boldsymbol{L}^2\Omega^k(\bar{\mathcal{B}}) = \boldsymbol{d}\Big(\boldsymbol{H}^1\Omega^{k-1}_{\mathrm{n}}(\bar{\mathcal{B}})\Big) \oplus \boldsymbol{\mathcal{H}}^k(\bar{\mathcal{B}}) \oplus \boldsymbol{\delta}\Big(\boldsymbol{H}^1\Omega^{k+1}_{\mathrm{t}}(\bar{\mathcal{B}})\Big). \tag{2.17}$$

In contrary to (2.13), all the spaces in the decomposition (2.17) are infinite dimensional.

Remark 3. Arguments similar to [32, Theorem 7.7.8] leads to the following results. In the decomposition (2.16), if $\boldsymbol{\alpha}$ is of class $C^{r,\mu}$ with $r \geq 0$ and $0 < \mu < 1$, then $\boldsymbol{d\alpha}^{\mathrm{n}_1}$, $\boldsymbol{\alpha}^{\mathrm{n}_1,\mathrm{t}_2}_{\mathcal{H}}$, and $\boldsymbol{\delta\alpha}^{\mathrm{t}_2}$ are of class $C^{r,\mu}$ and $\boldsymbol{\alpha}^{\mathrm{n}_1}$ and $\boldsymbol{\alpha}^{\mathrm{t}_2}$ are of class $C^{r+1,\mu}$. If $\boldsymbol{\alpha}$ is smooth, then its components in (2.13) and (2.17) will be smooth as well.

Remark 4. By using an abstract version of the Hodge theory together with the fact that $\mathrm{D}(d_{\mathrm{n}_1}) \cap \mathrm{D}(\delta_{\mathrm{t}_2})$ compactly embeds into $L^2\Omega^k(\bar{\mathcal{B}})$, Gol'dshtein et al. [26] proved that (2.13) is valid for standard forms on weakly Lipschitz subdomains of compact Lipschitz manifolds.

Remark 5. The form $\boldsymbol{\alpha}^{\mathrm{n}_1}$ in (2.16) is not unique and can be replaced with $\boldsymbol{\alpha}^{\mathrm{n}_1} + \boldsymbol{\zeta}$, $\forall\boldsymbol{\zeta} \in \ker\boldsymbol{d}_{\mathrm{n}_1}$. It can be uniquely chosen if one further requires that $\boldsymbol{\alpha}^{\mathrm{n}_1} \in (\ker\boldsymbol{d}_{\mathrm{n}_1})^\perp$, i.e. if $\boldsymbol{\alpha}^{\mathrm{n}_1}$ admits a $\boldsymbol{\delta}_{\mathrm{t}_2}$-potential, see [35, Lemma 2.4.7]. Similarly, $\boldsymbol{\alpha}^{\mathrm{t}_2}$ can be uniquely chosen if it admits a $\boldsymbol{d}_{\mathrm{n}_1}$-potential.

Remark 6. If $\boldsymbol{\alpha}$ admits a $\boldsymbol{d}_{\mathrm{n}_1}$-potential, then by using (2.15), one can write

$$\boldsymbol{\alpha} \in \boldsymbol{H}^d\Omega^k_{\mathrm{n}_1}(\bar{\mathcal{B}}),\ \boldsymbol{d\alpha} = 0,\ \langle\!\langle\boldsymbol{\alpha},\boldsymbol{\chi}\rangle\!\rangle_{\boldsymbol{L}^2} = 0,\ \forall\boldsymbol{\chi} \in \boldsymbol{\mathcal{H}}^k_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{2.18}$$

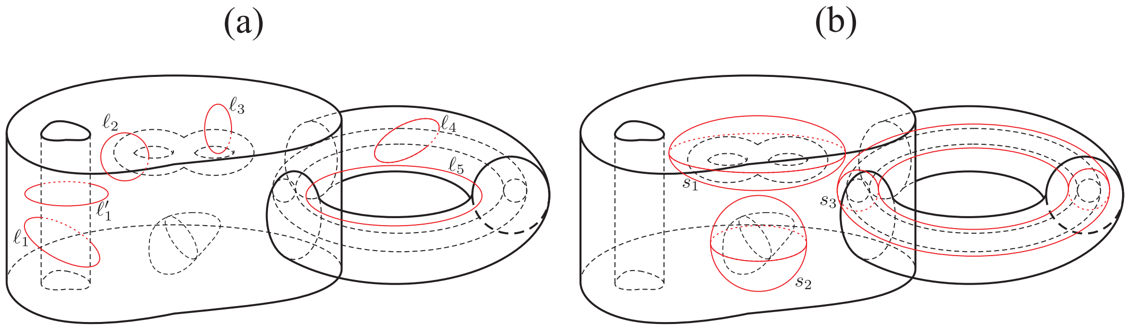

Figure 2: *(a) Loops $\ell_1, \ldots, \ell_5$ induce the generators of $\mathrm{H}^1_{dR}(\mathcal{B})$, and (b) closed surfaces s_1, s_2, s_3 induce the generators of $\mathrm{H}^2_{dR}(\mathcal{B})$. Loops ℓ_1 and ℓ_1' are not independent.*

Conversely, if $\boldsymbol{\alpha}$ satisfies (2.18), then (2.15) implies that there is a $\boldsymbol{d}_{\mathrm{n}_1}$-potential for $\boldsymbol{\alpha}$. Thus, the conditions (2.18) are the necessary and sufficient conditions for the existence of a $\boldsymbol{d}_{\mathrm{n}_1}$-potential for an $\boldsymbol{L}^2$-form $\boldsymbol{\alpha}$. Similarly, the analogue of (2.18) for $\boldsymbol{\delta}_{\mathrm{t}_2}$-potentials reads

$$\boldsymbol{\alpha} \in \boldsymbol{H}^{\delta}\Omega^k_{\mathrm{t}_2}(\bar{\mathcal{B}}),\ \boldsymbol{\delta\alpha} = 0,\ \langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\chi} \rangle\!\rangle_{\boldsymbol{L}^2} = 0,\ \forall \boldsymbol{\chi} \in \boldsymbol{\mathcal{H}}^k_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{2.19}$$

Remark 7. The de Rham theorem implies that $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^n) \in \Omega^k(\bar{\mathcal{B}}; \mathbb{R}^n)$ admits a $\boldsymbol{d}$-potential if and only if

$$\boldsymbol{d\alpha} = 0,\ \text{and}\ \int_{c_k} \boldsymbol{\alpha} = \left(\int_{c_k} \boldsymbol{\alpha}^1, \ldots, \int_{c_k} \boldsymbol{\alpha}^n \right) = 0,\ \forall c_k \in Z_k(\bar{\mathcal{B}}), \tag{2.20}$$

where $Z_k(\bar{\mathcal{B}})$ is the space of all k-cycles in $\bar{\mathcal{B}}$ [12, §5.9]. Since the k-th Betti number $b_k(\bar{\mathcal{B}}) := \dim \mathrm{H}^k_{dR}(\bar{\mathcal{B}})$ is finite, it suffices to calculate (2.20) merely for $b_k(\bar{\mathcal{B}})$ independent k-cycles called generators of the k-th (singular) homology group. On the other hand, if $\partial_1\bar{\mathcal{B}} = \varnothing$, (2.18) becomes

$$\boldsymbol{\alpha} \in \boldsymbol{H}^{d}\Omega^k(\bar{\mathcal{B}}),\ \boldsymbol{d\alpha} = 0,\ \langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\chi} \rangle\!\rangle_{\boldsymbol{L}^2} = 0,\ \forall \boldsymbol{\chi} \in \boldsymbol{\mathcal{H}}^k_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{2.21}$$

One can show that for smooth forms, (2.20) and (2.21) are equivalent [35, Theorem 3.2.3]. By replacing $Z_k(\bar{\mathcal{B}})$ with the space of relative cycles $Z_k(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$, one can also include boundary conditions in (2.20), see [19] and [24, page 599].

Remark 8. A standard tool for calculating de Rham cohomologies is the Mayer-Vietoris sequence [11, page 22]. For example, the dimensions of the first and the second de Rham cohomologies of the body in Fig. 2 are 5 and 3, respectively. As mentioned in Remark 7, the number of "independent" closed manifolds for the integral condition (2.20) is the same as the dimension of the corresponding cohomology group. Roughly speaking, two closed k-manifolds in $\bar{\mathcal{B}}$ are independent if they do not constitute the boundary of a $(k+1)$-manifold in $\bar{\mathcal{B}}$. Fig. 2 shows some possible choices for these closed manifolds. Standard methods for calculating relative cohomology groups can be found in standard texts on algebraic topology such as [33].

3 Hilbert Complexes and Decompositions for Second-Order Tensors

In this section, we derive orthogonal decompositions for various types of second-order tensors on flat, compact 2- and 3-manifolds with boundary by using the proper Hilbert complexes. The Hilbert complexes for non-symmetric and two-point tensors directly follow from the Hilbert complexes for differential forms studied earlier. The Hilbert complex for symmetric second-order tensors follows from the linear elasticity (Kröner) complex. From now on, we assume that $\{X^I\}$ is the Cartesian coordinates on $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, and the Riemannian metric $\boldsymbol{G}$ on $\bar{\mathcal{B}}$ is induced by the standard metric of $\mathbb{R}^n$.

3.1 Non-symmetric Tensors

Let $n = 3$ and suppose $\Gamma(\otimes^2 T\bar{\mathcal{B}})$ is the space of smooth $\binom{2}{0}$-tensors on $\bar{\mathcal{B}}$, i.e. $\boldsymbol{T} \in \Gamma(\otimes^2 T\bar{\mathcal{B}})$ has C^∞-components T^{IJ}, $I, J = 1, \dots, n$. One can show that the following diagram commutes [2].

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathfrak{X}(\bar{\mathcal{B}}) & \xrightarrow{\mathbf{grad}} & \Gamma(\otimes^2 T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{curl}^{\mathsf{T}}} & \Gamma(\otimes^2 T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{div}} & \mathfrak{X}(\bar{\mathcal{B}}) & \longrightarrow & 0 \\ & & \downarrow \boldsymbol{\imath}_0 & & \downarrow \boldsymbol{\imath}_1 & & \downarrow \boldsymbol{\imath}_2 & & \downarrow \boldsymbol{\imath}_3 & & \\ 0 & \longrightarrow & \Omega^0(\bar{\mathcal{B}}; \mathbb{R}^3) & \xrightarrow{\boldsymbol{d}} & \Omega^1(\bar{\mathcal{B}}; \mathbb{R}^3) & \xrightarrow{\boldsymbol{d}} & \Omega^2(\bar{\mathcal{B}}; \mathbb{R}^3) & \xrightarrow{\boldsymbol{d}} & \Omega^3(\bar{\mathcal{B}}; \mathbb{R}^3) & \longrightarrow & 0 \end{array} \tag{3.1}$$

The complex in the first row of the above diagram is called the **gcd** complex on $\bar{\mathcal{B}}$ and its operators are defined by

$$\begin{aligned} &\mathbf{grad} : \mathfrak{X}(\bar{\mathcal{B}}) \to \Gamma(\otimes^2 T\bar{\mathcal{B}}), && (\mathbf{grad}\, \boldsymbol{Y})^{IJ} = Y^I{}_{,J}, \\ &\mathbf{curl}^{\mathsf{T}} : \Gamma(\otimes^2 T\bar{\mathcal{B}}) \to \Gamma(\otimes^2 T\bar{\mathcal{B}}), && (\mathbf{curl}^{\mathsf{T}} \boldsymbol{T})^{IJ} = \varepsilon_{JKL} T^{IL}{}_{,K}, \\ &\mathbf{div} : \Gamma(\otimes^2 T\bar{\mathcal{B}}) \to \mathfrak{X}(\bar{\mathcal{B}}), && (\mathbf{div}\, \boldsymbol{T})^I = T^{IJ}{}_{,J}, \end{aligned}$$

where $_{,J}$ indicates $\partial/\partial X^J$, and ε_{JKL} is the standard permutation symbol. For $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n) \in \Omega^k(\bar{\mathcal{B}}; \mathbb{R}^n)$, let $[\boldsymbol{\alpha}]^i{}_{I_1 \cdots I_k}$ denote components of $\boldsymbol{\alpha}^i$, $i = 1, \dots, n$. The vertical isomorphisms in (3.1) are given by

$$\begin{aligned} &\boldsymbol{\imath}_0 : \mathfrak{X}(\bar{\mathcal{B}}) \to \Omega^0(\bar{\mathcal{B}}; \mathbb{R}^3), && [\boldsymbol{\imath}_0(\boldsymbol{Y})]^i = \delta_{iI} Y^I, \\ &\boldsymbol{\imath}_1 : \Gamma(\otimes^2 T\bar{\mathcal{B}}) \to \Omega^1(\bar{\mathcal{B}}; \mathbb{R}^3), && [\boldsymbol{\imath}_1(\boldsymbol{T})]^i{}_J = \delta_{iI} T^{IJ}, \\ &\boldsymbol{\imath}_2 : \Gamma(\otimes^2 T\bar{\mathcal{B}}) \to \Omega^2(\bar{\mathcal{B}}; \mathbb{R}^3), && [\boldsymbol{\imath}_2(\boldsymbol{T})]^i{}_{JK} = \delta_{iI} \varepsilon_{JKL} T^{IL}, \\ &\boldsymbol{\imath}_3 : \mathfrak{X}(\bar{\mathcal{B}}) \to \Omega^3(\bar{\mathcal{B}}; \mathbb{R}^3), && [\boldsymbol{\imath}_3(\boldsymbol{Y})]^i{}_{123} = \delta_{iI} Y^I, \end{aligned}$$

where δ_{iI} is the Kronecker delta.

Let $\langle \boldsymbol{T}, \boldsymbol{Y} \rangle := T^{IJ} Y^J \mathbf{E}_I$ be the traction vector of the tensor field $\boldsymbol{T}$ in the $\boldsymbol{Y}$ direction. We say that $\boldsymbol{T}$ is normal to $\partial_j \bar{\mathcal{B}}$ and write $\boldsymbol{T} \perp \partial_j \bar{\mathcal{B}}$ if $\langle \boldsymbol{T}, \boldsymbol{Y} \rangle = 0$, for all vector fields $\boldsymbol{Y} \| \partial_j \bar{\mathcal{B}}$. Similarly, we say that $\boldsymbol{T}$ is tangent to $\partial_j \bar{\mathcal{B}}$ and write $\boldsymbol{T} \| \partial_j \bar{\mathcal{B}}$ if the traction vector of $\boldsymbol{T}$ on $\partial_j \bar{\mathcal{B}}$ vanishes, that is, $\langle \boldsymbol{T}, \boldsymbol{N}_j \rangle = 0$, where $\boldsymbol{N}_j$ is the outward unit normal of $\partial_j \bar{\mathcal{B}}$. By imposing certain boundary conditions on $\partial_j \bar{\mathcal{B}}$, $j = 1, 2$, we define the following linear subspaces of $\mathfrak{X}(\bar{\mathcal{B}})$ and $\Gamma(\otimes^2 T\bar{\mathcal{B}})$:

$$\begin{aligned} \mathfrak{X}_j(\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{Y} \in \mathfrak{X}(\bar{\mathcal{B}}) : \boldsymbol{Y}|_{\partial_j \bar{\mathcal{B}}} = 0 \right\}, \\ \Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{T} \in \Gamma(\otimes^2 T\bar{\mathcal{B}}) : \boldsymbol{T} \perp \partial_j \bar{\mathcal{B}} \right\}, \\ \Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{T} \in \Gamma(\otimes^2 T\bar{\mathcal{B}}) : \boldsymbol{T} \| \partial_j \bar{\mathcal{B}} \right\}. \end{aligned} \tag{3.2}$$

Let $\{\mathbf{E}_I\}$ be the standard basis of $\mathbb{R}^n$ and let $\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} := T^{IJ} \mathbf{E}_J \in \mathfrak{X}(\bar{\mathcal{B}})$, that is, the Cartesian components of $\boldsymbol{T}$ can be arranged as follows:

$$\boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_1} \\ \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_2} \\ \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_3} \end{pmatrix}.$$

Then, it is straightforward to check that $\boldsymbol{T} \perp \partial_j \bar{\mathcal{B}}$ ($\boldsymbol{T} \| \partial_j \bar{\mathcal{B}}$) if and only if $\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} \perp \partial_j \bar{\mathcal{B}}$ ($\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} \| \partial_j \bar{\mathcal{B}}$), $I = 1, \ldots, n$. The relation between the spaces defined in (3.2) and the spaces of tangent and normal differential forms can be stated as follows.

Lemma 9. *We have* $\boldsymbol{\imath}_0\big(\mathfrak{X}_j(\bar{\mathcal{B}})\big) = \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3)$, $\boldsymbol{\imath}_1\big(\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}})\big) = \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3)$, *and* $\boldsymbol{\imath}_2\big(\Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}})\big) = \Omega^2_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3)$. *The restrictions of* **grad**, $\mathbf{curl}^{\mathsf{T}}$, *and* **div** *to the above subspaces are the operators* $\mathbf{grad}_j : \mathfrak{X}_j(\bar{\mathcal{B}}) \to \Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}})$, $\mathbf{curl}^{\mathsf{T}}_j : \Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}) \to \Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}})$, *and* $\mathbf{div}_j : \Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}) \to \mathfrak{X}(\bar{\mathcal{B}})$.

Proof. The first relation for $\boldsymbol{\imath}_0$ is trivial. Since $\boldsymbol{\imath}_1(\boldsymbol{T}) = \big(\overrightarrow{\boldsymbol{T}}^{\flat}_{\mathbf{E}_1}, \overrightarrow{\boldsymbol{T}}^{\flat}_{\mathbf{E}_2}, \overrightarrow{\boldsymbol{T}}^{\flat}_{\mathbf{E}_3}\big)$, by using (2.1), we conclude that $\mathsf{t}(\boldsymbol{\imath}_1(\boldsymbol{T})) = \big((\mathsf{t}\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_1})^{\flat}, (\mathsf{t}\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_2})^{\flat}, (\mathsf{t}\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_3})^{\flat}\big)$, which gives the relation for $\boldsymbol{\imath}_1$. Similarly, the relation for $\boldsymbol{\imath}_2$ follows from $\boldsymbol{\imath}_2(\boldsymbol{T}) = \big(* \overrightarrow{\boldsymbol{T}}^{\flat}_{\mathbf{E}_1}, * \overrightarrow{\boldsymbol{T}}^{\flat}_{\mathbf{E}_2}, * \overrightarrow{\boldsymbol{T}}^{\flat}_{\mathbf{E}_3}\big)$, together with (2.2) and (2.1). By using the relations that we just established, (2.4), and diagram (3.1), we conclude that the restrictions $\mathbf{grad}_j$, $\mathbf{curl}^{\mathsf{T}}_j$, and $\mathbf{div}_j$ are well-defined. □

Using the above lemma, one can introduce boundary conditions in diagram (3.1) as follows.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathfrak{X}_j(\bar{\mathcal{B}}) & \xrightarrow{\mathbf{grad}_j} & \Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{curl}^{\mathsf{T}}_j} & \Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{div}_j} & \mathfrak{X}(\bar{\mathcal{B}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \boldsymbol{\imath}_0} & & \downarrow{\scriptstyle \boldsymbol{\imath}_1} & & \downarrow{\scriptstyle \boldsymbol{\imath}_2} & & \downarrow{\scriptstyle \boldsymbol{\imath}_3} & & \\ 0 & \longrightarrow & \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^2_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^3_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \longrightarrow & 0 \end{array} \tag{3.3}$$

Similarly, we can show that the following diagram commutes.

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \mathfrak{X}(\bar{\mathcal{B}}) & \xleftarrow{\mathbf{div}_j} & \Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{curl}^{\mathsf{T}}_j} & \Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{grad}_j} & \mathfrak{X}_j(\bar{\mathcal{B}}) & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle -\boldsymbol{\imath}_0} & & \downarrow{\scriptstyle \boldsymbol{\imath}_1} & & \downarrow{\scriptstyle \boldsymbol{\imath}_2} & & \downarrow{\scriptstyle -\boldsymbol{\imath}_3} & & \\ 0 & \longleftarrow & \Omega^0_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^1_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^2_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^3_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \longleftarrow & 0 \end{array} \tag{3.4}$$

Using the global orthonormal coordinate system $\{X^I\}$, the $\boldsymbol{L}^2$-inner product $\langle\!\langle , \rangle\!\rangle_{\boldsymbol{L}^2}$ can be written as

$$\langle\!\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle\!\rangle_{\boldsymbol{L}^2} = \int_{\bar{\mathcal{B}}} \sum_{\substack{1 \le I_1 < \cdots < I_k \le n \\ 1 \le i \le n}} [\boldsymbol{\alpha}]^i{}_{I_1 \cdots I_k} \cdot [\boldsymbol{\gamma}]^i{}_{I_1 \cdots I_k} dX^1 \wedge \cdots \wedge dX^n.$$

The L^2-inner products on $\mathfrak{X}(\bar{\mathcal{B}})$ and $\Gamma(\otimes^2 T\bar{\mathcal{B}})$ are given by

$$\langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{L^2} = \int_{\bar{\mathcal{B}}} \sum_{I=1}^{n} Y^I \cdot Z^I dX^1 \wedge \cdots \wedge dX^n,$$
$$\langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{L^2} = \int_{\bar{\mathcal{B}}} \sum_{I,J=1}^{n} S^{IJ} \cdot T^{IJ} dX^1 \wedge \cdots \wedge dX^n.$$

It is straightforward to see that with these L^2-inner products, the isomorphisms $\boldsymbol{\imath}_k$ are also isometries, i.e. $\langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{\imath}_k(\boldsymbol{Y}), \boldsymbol{\imath}_k(\boldsymbol{Z}) \rangle\!\rangle_{\boldsymbol{L}^2}$, $k = 0, 3$, and $\langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{\imath}_k(\boldsymbol{S}), \boldsymbol{\imath}_k(\boldsymbol{T}) \rangle\!\rangle_{\boldsymbol{L}^2}$, $k = 1, 2$. Next, consider the standard H^1-inner product on $\mathfrak{X}(\bar{\mathcal{B}})$, which can be written as

$$\langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{H^1} := \langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{grad}\, \boldsymbol{Y}, \mathbf{grad}\, \boldsymbol{Z} \rangle\!\rangle_{L^2}. \tag{3.5}$$

Also consider the following inner products

$$\begin{aligned} \langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{H^{\mathbf{c}}} &:= \langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{curl}^{\mathsf{T}} \boldsymbol{S}, \mathbf{curl}^{\mathsf{T}} \boldsymbol{T} \rangle\!\rangle_{L^2}, \\ \langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{H^{\mathbf{d}}} &:= \langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{div}\, \boldsymbol{S}, \mathbf{div}\, \boldsymbol{T} \rangle\!\rangle_{L^2}, \end{aligned} \tag{3.6}$$

on $\Gamma(\otimes^2 T\bar{\mathcal{B}})$. The Hilbert spaces $L^2\mathfrak{X}(\bar{\mathcal{B}})$ and $H^1\mathfrak{X}_j(\bar{\mathcal{B}})$ are the completions of $\big(\mathfrak{X}(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{L^2}\big)$ and $\big(\mathfrak{X}_j(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^1}\big)$, respectively. Analogously, the Hilbert spaces $H^{\mathbf{c}}\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}})$ and $H^{\mathbf{d}}\Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}})$ are defined to be the completions of $\big(\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{c}}}\big)$ and $\big(\Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{d}}}\big)$, respectively. Lemma 9 and diagram (3.3) imply that the isometries $\boldsymbol{\imath}_0, \ldots, \boldsymbol{\imath}_3$ also induce isometries between these Hilbert spaces and the corresponding partly Sobolev spaces, e.g. we have $\langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{H^{\mathbf{c}}} = \langle\!\langle \boldsymbol{\imath}_1(\boldsymbol{S}), \boldsymbol{\imath}_1(\boldsymbol{T}) \rangle\!\rangle_{\boldsymbol{H}^{\boldsymbol{d}}}$ that gives an isometry between $H^{\mathbf{c}}\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}})$ and $\boldsymbol{H}^{\boldsymbol{d}}\Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}})$. The operators $\mathbf{grad}_j$, $\mathbf{curl}_j^{\mathsf{T}}$, and $\mathbf{div}_j$ can be considered as densely-defined, closed operators between the associated L^2-spaces, and therefore, we can write the following Hilbert complex for non-symmetric second-order tensors:

$$0 \longrightarrow H^1\mathfrak{X}_j(\bar{\mathcal{B}}) \xrightarrow{\mathbf{grad}_j} H^{\mathbf{c}}\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}) \xrightarrow{\mathbf{curl}_j^{\mathsf{T}}} H^{\mathbf{d}}\Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}) \xrightarrow{\mathbf{div}_j} L^2\mathfrak{X}(\bar{\mathcal{B}}) \longrightarrow 0. \tag{3.7}$$

The above discussions suggest that this Hilbert complex is isomorphic to $(\boldsymbol{H}^{\boldsymbol{d}}\Omega_{\mathrm{n}_j}(\bar{\mathcal{B}}), \boldsymbol{d}_{\mathrm{n}_j})$. Let $\mathrm{H}^k_{\mathbf{gcd}_j}(\bar{\mathcal{B}})$ and $\mathsf{H}^k_{\mathbf{gcd}_j}(\bar{\mathcal{B}})$ be the k-th cohomology groups of the smooth **gcd** complex in (3.3) and the Hilbert complex (3.7), respectively. The following is a consequence of (2.8).

Corollary 10. *The Hilbert complex (3.7) is Fredholm and*

$$\mathsf{H}^k_{\mathbf{gcd}_j}(\bar{\mathcal{B}}) \approx \mathrm{H}^k_{\mathbf{gcd}_j}(\bar{\mathcal{B}}) \approx \oplus_{i=1}^3 \mathrm{H}^k_{dR}(\bar{\mathcal{B}}, \partial_j \bar{\mathcal{B}}).$$

Using the diagram (3.4), one can find the dual complex of (3.7). In particular, the dual of (3.7) for $j = 1$, reads

$$0 \longleftarrow L^2\mathfrak{X}(\bar{\mathcal{B}}) \xleftarrow{-\mathbf{div}_2} H^{\mathbf{d}}\Gamma_{\mathrm{t}_2}(\otimes^2 T\bar{\mathcal{B}}) \xleftarrow{\mathbf{curl}_2^{\mathsf{T}}} H^{\mathbf{c}}\Gamma_{\mathrm{n}_2}(\otimes^2 T\bar{\mathcal{B}}) \xleftarrow{-\mathbf{grad}_2} H^1\mathfrak{X}_2(\bar{\mathcal{B}}) \longleftarrow 0. \tag{3.8}$$

Let $\mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) := \ker \mathbf{curl}_1^{\mathsf{T}} \cap \ker \mathbf{div}_2$. Note that this space coincides with the kernel of the tensor Laplacian $L_{\otimes}(\boldsymbol{T}) := \mathbf{curl}^{\mathsf{T}} \circ \mathbf{curl}^{\mathsf{T}} \boldsymbol{T} - \mathbf{grad} \circ \mathbf{div}\, \boldsymbol{T}$, subject to the boundary conditions

$$\begin{aligned} &\boldsymbol{T} \in H^{\mathbf{c}}\Gamma_{\mathrm{n}_1}(\otimes^2 T\bar{\mathcal{B}}) \cap H^{\mathbf{d}}\Gamma_{\mathrm{t}_2}(\otimes^2 T\bar{\mathcal{B}}), \\ &\mathbf{curl}^{\mathsf{T}} \boldsymbol{T} \in H^{\mathbf{c}}\Gamma_{\mathrm{n}_2}(\otimes^2 T\bar{\mathcal{B}}),\ \mathbf{div}\, \boldsymbol{T} \in H^1\mathfrak{X}_1(\bar{\mathcal{B}}). \end{aligned}$$

Remark 2 implies that $\mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$ only consists of smooth harmonic tensor fields. Moreover, since (3.7) is Fredholm, we have $\mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \approx \mathsf{H}^1_{\mathbf{gcd}_1}(\bar{\mathcal{B}})$. The next theorem follows from the discussions in §2.3 and gives the analogues of decompositions (2.13) and (2.14) for non-symmetric second-order tensors.

Theorem 11. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary. The Hilbert complex (3.7)* *induces the following L^2-orthogonal decompositions: The Hodge decomposition*

$$L^2\Gamma(\otimes^2 T\bar{\mathcal{B}}) = \mathbf{grad}\big(H^1\mathfrak{X}_1(\bar{\mathcal{B}})\big) \oplus \mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \oplus \mathbf{curl}^{\mathsf{T}}\left(H^{\mathbf{c}}\Gamma_{\mathrm{n}_2}(\otimes^2 T\bar{\mathcal{B}})\right),$$

and, equivalently, the Helmholtz decompositions

$$\begin{aligned} L^2\Gamma(\otimes^2 T\bar{\mathcal{B}}) &= \mathbf{grad}\big(H^1\mathfrak{X}_1(\bar{\mathcal{B}})\big) \oplus \ker \mathbf{div}_2 \\ &= \ker \mathbf{curl}_1^{\mathsf{T}} \oplus \mathbf{curl}^{\mathsf{T}}\left(H^{\mathbf{c}}\Gamma_{\mathrm{n}_2}(\otimes^2 T\bar{\mathcal{B}})\right), \end{aligned}$$

where

$$\begin{aligned} \ker \mathbf{div}_2 &= \mathbf{curl}^{\mathsf{T}}\left(H^{\mathbf{c}}\Gamma_{\mathrm{n}_2}(\otimes^2 T\bar{\mathcal{B}})\right) \oplus \mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}), \\ \ker \mathbf{curl}_1^{\mathsf{T}} &= \mathbf{grad}\big(H^1\mathfrak{X}_1(\bar{\mathcal{B}})\big) \oplus \mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \end{aligned}$$

If in addition $\boldsymbol{T} \in L^2\Gamma(\otimes^2 T\bar{\mathcal{B}})$ is of class $C^{r,\mu}$ (C^∞), then the components of $\boldsymbol{T}$ in the above decompositions are of class $C^{r,\mu}$ (C^∞).

Remark 12. In contrary to the decomposition (1.1), which is valid on arbitrary manifolds, diagram (3.1) and the decompositions in Theorem 11 are valid only on flat manifolds. By using the above Hodge decomposition for a second-order tensor $\boldsymbol{T}$, one obtains the decomposition $\boldsymbol{T} = \mathbf{grad}_1\boldsymbol{Y}^1 + \boldsymbol{T}_{\mathcal{H}} + \mathbf{curl}_2^{\mathsf{T}}\boldsymbol{T}^{\mathrm{n}_2}$. Remark 5 implies that $\boldsymbol{Y}^1$ and $\boldsymbol{T}^{\mathrm{n}_2}$ can be uniquely chosen if one further assumes that $\boldsymbol{Y}^1$ admits a $\mathbf{div}_2$-potential and $\boldsymbol{T}^{\mathrm{n}_2}$ admits a $\mathbf{curl}_1^{\mathsf{T}}$-potential.

Corollary 13. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary. The necessary and sufficient conditions for the existence of a $\mathbf{grad}_1$-potential for $\boldsymbol{T} \in L^2\Gamma(\otimes^2 T\bar{\mathcal{B}})$ are*

$$\boldsymbol{T} \in H^{\mathbf{c}}\Gamma_{\mathrm{n}_1}(\otimes^2 T\bar{\mathcal{B}}),\ \mathbf{curl}^{\mathsf{T}}\boldsymbol{T} = 0,\ \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{Q} \in \mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{3.9}$$

Similarly, the necessary and sufficient conditions for the existence of a $\mathbf{curl}_2^{\mathsf{T}}$-potential for $\boldsymbol{T}$ are

$$\boldsymbol{T} \in H^{\mathbf{d}}\Gamma_{\mathrm{t}_2}(\otimes^2 T\bar{\mathcal{B}}),\ \mathbf{div}\,\boldsymbol{T} = 0,\ \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{Q} \in \mathcal{H}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{3.10}$$

Remark 14. Suppose ℓ is an arbitrary closed curve in $\bar{\mathcal{B}}$ with $\boldsymbol{t}_\ell$ the unit tangent vector field along ℓ and let $\mathcal{C}$ be an arbitrary closed surface in $\bar{\mathcal{B}}$ with $\boldsymbol{N}_{\mathcal{C}}$ its unit outward normal vector field. Using diagram (3.1) and the de Rham theorem for manifolds with boundary, one can show the following [2]: $\boldsymbol{T} \in \Gamma(\otimes^2 T\bar{\mathcal{B}})$ is the gradient of a vector field if and only if

$$\mathbf{curl}^{\mathsf{T}}\boldsymbol{T} = 0,\ \text{and} \qquad \int_\ell \langle \boldsymbol{T}, \boldsymbol{t}_\ell \rangle dS = 0,\ \forall \ell \subset \bar{\mathcal{B}}, \tag{3.11}$$

and $\boldsymbol{T}$ admits a $\mathbf{curl}^{\mathsf{T}}$-potential if and only if

$$\mathbf{div}\,\boldsymbol{T} = 0,\ \text{and} \qquad \int_{\mathcal{C}} \langle \boldsymbol{T}, \boldsymbol{N}_{\mathcal{C}} \rangle dA = 0,\ \forall \mathcal{C} \subset \bar{\mathcal{B}}. \tag{3.12}$$

Remark 7 suggests that (3.11) and (3.12) are equivalent to (3.9) with $\partial_1\bar{\mathcal{B}} = \varnothing$, and (3.10) with $\partial_2\bar{\mathcal{B}} = \varnothing$, respectively.

Next, we write the analogues of the above decompositions for a 2-manifold $\bar{\mathcal{B}} \subset \mathbb{R}^2$. One can show that the following diagrams commute [2].

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{X}(\bar{\mathcal{B}}) & \xrightarrow{\mathbf{grad}} & \Gamma(\otimes^2 T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{c}} & \mathfrak{X}(\bar{\mathcal{B}}) & \longrightarrow & 0 \\
 & & \Big\downarrow \boldsymbol{\jmath}_0 & & \Big\downarrow \boldsymbol{\jmath}_1 & & \Big\downarrow \boldsymbol{\jmath}_2 & & \\
0 & \longrightarrow & \Omega^0(\bar{\mathcal{B}};\mathbb{R}^2) & \xrightarrow{\boldsymbol{d}} & \Omega^1(\bar{\mathcal{B}};\mathbb{R}^2) & \xrightarrow{\boldsymbol{d}} & \Omega^2(\bar{\mathcal{B}};\mathbb{R}^2) & \longrightarrow & 0
\end{array} \tag{3.13}$$

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathfrak{X}(\bar{\mathcal{B}}) & \xleftarrow{\mathbf{div}} & \Gamma(\otimes^2 T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{s}} & \mathfrak{X}(\bar{\mathcal{B}}) & \longleftarrow & 0 \\
 & & \Big\downarrow -\boldsymbol{\jmath}_0 & & \Big\downarrow \boldsymbol{\jmath}_1 & & \Big\downarrow \boldsymbol{\jmath}_2 & & \\
0 & \longleftarrow & \Omega^0(\bar{\mathcal{B}};\mathbb{R}^2) & \xleftarrow{\boldsymbol{\delta}} & \Omega^1(\bar{\mathcal{B}};\mathbb{R}^2) & \xleftarrow{\boldsymbol{\delta}} & \Omega^2(\bar{\mathcal{B}};\mathbb{R}^2) & \longleftarrow & 0
\end{array} \tag{3.14}$$

The first rows of (3.13) and (3.14) are called the **gc** complex and the **sd** complex, respectively, where the linear operators $\mathbf{c}$ and $\mathbf{s}$ are given by

$$\begin{aligned}
&\mathbf{c} : \Gamma(\otimes^2 T\bar{\mathcal{B}}) \to \mathfrak{X}(\bar{\mathcal{B}}), \quad (\mathbf{c}(\boldsymbol{T}))^I = T^{I2}{}_{,1} - T^{I1}{}_{,2}, \\
&\mathbf{s} : \mathfrak{X}(\bar{\mathcal{B}}) \to \Gamma(\otimes^2 T\bar{\mathcal{B}}), \quad (\mathbf{s}(\boldsymbol{Y}))^{IJ} = \delta^{1J} Y^I{}_{,2} - \delta^{2J} Y^I{}_{,1}.
\end{aligned} \tag{3.15}$$

The vertical isomorphisms $\boldsymbol{\jmath}_0$, $\boldsymbol{\jmath}_1$, and $\boldsymbol{\jmath}_2$ are defined as

$$\begin{aligned}
&\boldsymbol{\jmath}_0 : \mathfrak{X}(\bar{\mathcal{B}}) \to \Omega^0(\bar{\mathcal{B}};\mathbb{R}^2), && [\boldsymbol{\jmath}_0(\boldsymbol{Y})]^i = \delta_{iI} Y^I, \\
&\boldsymbol{\jmath}_1 : \Gamma(\otimes^2 T\bar{\mathcal{B}}) \to \Omega^1(\bar{\mathcal{B}};\mathbb{R}^2), && [\boldsymbol{\jmath}_1(\boldsymbol{T})]^i{}_J = \delta_{iI} T^{IJ}, \\
&\boldsymbol{\jmath}_2 : \mathfrak{X}(\bar{\mathcal{B}}) \to \Omega^2(\bar{\mathcal{B}};\mathbb{R}^2), && [\boldsymbol{\jmath}_2(\boldsymbol{Y})]^i{}_{12} = \delta_{iI} Y^I.
\end{aligned}$$

By imposing boundary conditions on (3.13) and (3.14), one obtains the following commutative diagrams.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{X}_j(\bar{\mathcal{B}}) & \xrightarrow{\mathbf{grad}_j} & \Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{c}_j} & \mathfrak{X}(\bar{\mathcal{B}}) & \longrightarrow & 0 \\
 & & \Big\downarrow \boldsymbol{\jmath}_0 & & \Big\downarrow \boldsymbol{\jmath}_1 & & \Big\downarrow \boldsymbol{\jmath}_2 & & \\
0 & \longrightarrow & \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^2_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathfrak{X}(\bar{\mathcal{B}}) & \xleftarrow{\mathbf{div}_j} & \Gamma_{\mathrm{t}_j}(\otimes^2 T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{s}_j} & \mathfrak{X}_j(\bar{\mathcal{B}}) & \longleftarrow & 0 \\
 & & \Big\downarrow -\boldsymbol{\jmath}_0 & & \Big\downarrow \boldsymbol{\jmath}_1 & & \Big\downarrow \boldsymbol{\jmath}_2 & & \\
0 & \longleftarrow & \Omega^0_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^1_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^2_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \longleftarrow & 0
\end{array}$$

Note that $\langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{H^1} = \langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{s}(\boldsymbol{Y}), \mathbf{s}(\boldsymbol{Z}) \rangle\!\rangle_{L^2}$, and let

$$\langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{\bar{H}^{\mathbf{c}}} := \langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{c}(\boldsymbol{S}), \mathbf{c}(\boldsymbol{T}) \rangle\!\rangle_{L^2}.$$

The Hilbert spaces $\bar{H}^{\mathbf{c}}\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}})$ and $H^1\mathfrak{X}_j(\bar{\mathcal{B}})$ are the completions of $\big(\Gamma_{\mathrm{n}_j}(\otimes^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{\bar{H}^{\mathbf{c}}}\big)$ and $\big(\mathfrak{X}_j(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^1}\big)$, respectively. Then, one can write the following Hilbert complex

$$0 \longrightarrow H^1\mathfrak{X}_1(\bar{\mathcal{B}}) \xrightarrow{\mathbf{grad}_1} \bar{H}^{\mathbf{c}}\Gamma_{\mathrm{n}_1}(\otimes^2 T\bar{\mathcal{B}}) \xrightarrow{\mathbf{c}_1} L^2\mathfrak{X}(\bar{\mathcal{B}}) \longrightarrow 0. \tag{3.16}$$

The dual of this Hilbert complex is

$$0 \longleftarrow L^2\mathfrak{X}(\bar{\mathcal{B}}) \overset{-\mathbf{div}_2}{\longleftarrow} H^{\mathbf{d}}\Gamma_{\mathrm{t}_2}(\otimes^2 T\bar{\mathcal{B}}) \overset{\mathbf{s}_2}{\longleftarrow} H^1\mathfrak{X}_2(\bar{\mathcal{B}}) \longleftarrow 0. \tag{3.17}$$

Similar to Corollary 10, one can show that (3.16) is Fredholm and $\mathsf{H}^k_{\mathbf{gc}_1}(\bar{\mathcal{B}}) \approx \oplus_{i=1}^2 \mathrm{H}^k_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$, where $\mathsf{H}^k_{\mathbf{gc}_1}(\bar{\mathcal{B}})$ is the k-th cohomology group of (3.16). Let $\bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) := \ker \mathbf{c}_1 \cap \ker \mathbf{div}_2$, which coincides with the kernel of $\bar{L}_{\otimes}(\boldsymbol{T}) := \mathbf{s} \circ \mathbf{c}(\boldsymbol{T}) - \mathbf{grad} \circ \mathbf{div}\, \boldsymbol{T}$ subject to the proper boundary conditions. We have $\bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \approx \mathsf{H}^1_{\mathbf{gc}_1}(\bar{\mathcal{B}})$ and $\bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$ merely consists of smooth harmonic tensor fields. We also obtain the following decompositions for non-symmetric second-order tensors.

Theorem 15. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^2$ be a smooth, compact 2-manifold with boundary. The Hilbert complex (3.16) induces the following L^2-orthogonal decompositions: The Hodge decomposition*

$$L^2\Gamma(\otimes^2 T\bar{\mathcal{B}}) = \mathbf{grad}\big(H^1\mathfrak{X}_1(\bar{\mathcal{B}})\big) \oplus \bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \oplus \mathbf{s}\big(H^1\mathfrak{X}_2(\bar{\mathcal{B}})\big)\,,$$

and, equivalently, the Helmholtz decompositions

$$L^2\Gamma(\otimes^2 T\bar{\mathcal{B}}) = \mathbf{grad}\big(H^1\mathfrak{X}_1(\bar{\mathcal{B}})\big) \oplus \ker \mathbf{div}_2 = \ker \mathbf{c}_1 \oplus \mathbf{s}\big(H^1\mathfrak{X}_2(\bar{\mathcal{B}})\big)\,,$$

where

$$\begin{aligned} \ker \mathbf{div}_2 &= \mathbf{s}\big(H^1\mathfrak{X}_2(\bar{\mathcal{B}})\big) \oplus \bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}), \\ \ker \mathbf{c}_1 &= \mathbf{grad}\big(H^1\mathfrak{X}_1(\bar{\mathcal{B}})\big) \oplus \bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \end{aligned}$$

If in addition $\boldsymbol{T} \in L^2\Gamma(\otimes^2 T\bar{\mathcal{B}})$ is of class $C^{r,\mu}$ (C^∞), then the components of $\boldsymbol{T}$ in the above decompositions are of class $C^{r,\mu}$ (C^∞).

Corollary 16. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^2$ be a smooth, compact 2-manifold with boundary. The necessary and sufficient conditions for the existence of a $\mathbf{grad}_1$-potential for $\boldsymbol{T} \in L^2\Gamma(\otimes^2 T\bar{\mathcal{B}})$ are*

$$\boldsymbol{T} \in \bar{H}^{\mathbf{c}}\Gamma_{\mathrm{n}_1}(\otimes^2 T\bar{\mathcal{B}}),\ \mathbf{c}(\boldsymbol{T}) = 0,\ \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{Q} \in \bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{3.18}$$

Similarly, the necessary and sufficient conditions for the existence of an $\mathbf{s}_2$-potential for $\boldsymbol{T}$ are

$$\boldsymbol{T} \in H^{\mathbf{d}}\Gamma_{\mathrm{t}_2}(\otimes^2 T\bar{\mathcal{B}}),\ \mathbf{div}\,\boldsymbol{T} = 0,\ \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{Q} \in \bar{\mathcal{H}}^{\otimes}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \tag{3.19}$$

Remark 17. Remark 12 readily extends to tensors on 2-manifolds with boundary. Let $\boldsymbol{N}_\ell$ be a unit vector field along a closed curve $\ell \subset \bar{\mathcal{B}}$, which is normal to the tangent vector field $\boldsymbol{t}_\ell$ of ℓ, such that $\{\boldsymbol{t}_\ell, \boldsymbol{N}_\ell\}$ has the same orientation as $\{\mathbf{E}_1, \mathbf{E}_2\}$ does. Then, the following results hold [2]: $\boldsymbol{T} \in \Gamma(\otimes^2 T\bar{\mathcal{B}})$ is the gradient of a vector field if and only if

$$\mathbf{c}(\boldsymbol{T}) = 0,\ \text{and} \qquad \int_\ell \langle \boldsymbol{T}, \boldsymbol{t}_\ell \rangle dS = 0,\ \forall \ell \subset \bar{\mathcal{B}}, \tag{3.20}$$

and $\boldsymbol{T}$ admits an $\mathbf{s}$-potential if and only if

$$\mathbf{div}\,\boldsymbol{T} = 0,\ \text{and} \qquad \int_\ell \langle \boldsymbol{T}, \boldsymbol{N}_\ell \rangle dS = 0,\ \forall \ell \subset \bar{\mathcal{B}}. \tag{3.21}$$

The conditions (3.20) and (3.21) are equivalent to (3.18) with $\partial_1\bar{\mathcal{B}} = \varnothing$, and (3.19) with $\partial_2\bar{\mathcal{B}} = \varnothing$, respectively.

3.2 Two-Point Tensors

Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary. Also let $\mathcal{S} = \mathbb{R}^3$, with Cartesian coordinates $\{x^i\}$ and suppose $\varphi : \bar{\mathcal{B}} \to \mathcal{S}$ is a smooth mapping. We assume that $T_X\varphi(\bar{\mathcal{B}}) := T_{\varphi(X)}\mathcal{S}$, and thus, the dimension of $T_X\varphi(\bar{\mathcal{B}})$ is 3 even if φ is not an embedding. Let $\Gamma(T\varphi(\bar{\mathcal{B}}))$ and $\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ be the spaces of two-point tensors over φ with components U^i and F^{iI}, respectively. Then, the following diagram commutes [2].

$$
\begin{array}{ccccccccccc}
0 \longrightarrow & \Gamma(T\varphi(\bar{\mathcal{B}})) & \xrightarrow{\mathbf{Grad}} & \Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Curl}^{\mathsf{T}}} & \Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Div}} & \Gamma(T\varphi(\bar{\mathcal{B}})) & \longrightarrow 0 \\
& \downarrow \boldsymbol{I}_0 & & \downarrow \boldsymbol{I}_1 & & \downarrow \boldsymbol{I}_2 & & \downarrow \boldsymbol{I}_3 & \\
0 \longrightarrow & \Omega^0(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}} & \Omega^1(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}} & \Omega^2(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}} & \Omega^3(\bar{\mathcal{B}};\mathbb{R}^3) & \longrightarrow 0
\end{array}
$$

The complex in the first row is called the **GCD** complex on $\bar{\mathcal{B}}$ and its operators are given by

$$
\begin{aligned}
\mathbf{Grad} &: \Gamma(T\varphi(\bar{\mathcal{B}})) \to \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), & (\mathbf{Grad}\,\boldsymbol{U})^{iI} &= U^i{}_{,I}, \\
\mathbf{Curl}^{\mathsf{T}} &: \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) \to \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), & (\mathbf{Curl}^{\mathsf{T}}\boldsymbol{F})^{iI} &= \varepsilon_{IKL}F^{iL}{}_{,K}, \\
\mathbf{Div} &: \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) \to \Gamma(T\varphi(\bar{\mathcal{B}})), & (\mathbf{Div}\,\boldsymbol{F})^{i} &= F^{iI}{}_{,I}.
\end{aligned}
$$

The vertical isomorphisms are defined as

$$
\begin{aligned}
\boldsymbol{I}_0 &: \Gamma(T\varphi(\bar{\mathcal{B}})) \to \Omega^0(\bar{\mathcal{B}};\mathbb{R}^3), & [\boldsymbol{I}_0(\boldsymbol{U})]^i &= U^i, \\
\boldsymbol{I}_1 &: \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) \to \Omega^1(\bar{\mathcal{B}};\mathbb{R}^3), & [\boldsymbol{I}_1(\boldsymbol{F})]^i{}_J &= F^{iJ}, \\
\boldsymbol{I}_2 &: \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) \to \Omega^2(\bar{\mathcal{B}};\mathbb{R}^3), & [\boldsymbol{I}_2(\boldsymbol{F})]^i{}_{JK} &= \varepsilon_{JKL}F^{iL}, \\
\boldsymbol{I}_3 &: \Gamma(T\varphi(\bar{\mathcal{B}})) \to \Omega^3(\bar{\mathcal{B}};\mathbb{R}^3), & [\boldsymbol{I}_3(\boldsymbol{U})]^i{}_{123} &= U^i.
\end{aligned}
$$

Let $\{\mathbf{E}_I\}$ and $\{\mathbf{e}_i\}$ be two copies of the standard basis of $\mathbb{R}^3$. For $\boldsymbol{F} \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$, we define $\overrightarrow{\boldsymbol{F}}_{\mathbf{e}_i} = F^{iI}\mathbf{E}_I \in \mathfrak{X}(\bar{\mathcal{B}})$. We also define the following linear subspaces of $\Gamma(T\varphi(\bar{\mathcal{B}}))$ and $\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$:

$$
\begin{aligned}
\Gamma_j(T\varphi(\bar{\mathcal{B}})) &:= \left\{ \boldsymbol{U} \in \Gamma(T\varphi(\bar{\mathcal{B}})) : \boldsymbol{U}|_{\partial_j \bar{\mathcal{B}}} = 0 \right\}, \\
\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{F} \in \Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) : \overrightarrow{\boldsymbol{F}}_{\mathbf{e}_i} \perp \partial_j\bar{\mathcal{B}},\ i = 1,\ldots,n \right\}, \\
\Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{F} \in \Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) : \overrightarrow{\boldsymbol{F}}_{\mathbf{e}_i} \| \partial_j\bar{\mathcal{B}},\ i = 1,\ldots,n \right\}.
\end{aligned}
$$

Similar to Lemma 9, the operators **Grad**, $\mathbf{Curl}^{\mathsf{T}}$, and **Div** can be restricted to the above subspaces which allows one to impose boundary conditions on the **GCD** complex. The upshot is the following commutative diagrams.

$$
\begin{array}{ccccccccccc}
0 \longrightarrow & \Gamma_j(T\varphi(\bar{\mathcal{B}})) & \xrightarrow{\mathbf{Grad}_j} & \Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Curl}^{\mathsf{T}}_j} & \Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Div}_j} & \Gamma(T\varphi(\bar{\mathcal{B}})) & \longrightarrow 0 \\
& \downarrow \boldsymbol{I}_0 & & \downarrow \boldsymbol{I}_1 & & \downarrow \boldsymbol{I}_2 & & \downarrow \boldsymbol{I}_3 & \\
0 \longrightarrow & \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^2_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^3_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \longrightarrow 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 \longleftarrow \Gamma(T\varphi(\bar{\mathcal{B}})) & \xleftarrow{\mathbf{Div}_j} & \Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Curl}_j^{\mathsf{T}}} & \Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Grad}_j} & \Gamma_j(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0 \\
\big\downarrow -\boldsymbol{I}_0 & & \big\downarrow \boldsymbol{I}_1 & & \big\downarrow \boldsymbol{I}_2 & & \big\downarrow -\boldsymbol{I}_3 \\
0 \longleftarrow \Omega^0_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^1_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^2_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^3_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^3) \longleftarrow 0
\end{array}
$$

For $\boldsymbol{U},\boldsymbol{V}\in\Gamma(T\varphi(\bar{\mathcal{B}}))$ and $\boldsymbol{F},\boldsymbol{P}\in\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$, we define the following inner products:

$$
\begin{aligned}
\langle\!\langle \boldsymbol{U},\boldsymbol{V}\rangle\!\rangle_{L^2} &:= \int_{\bar{\mathcal{B}}}\sum_{i=1}^{n} U^i\cdot V^i dX^1\wedge\cdots\wedge dX^n,\\
\langle\!\langle \boldsymbol{F},\boldsymbol{P}\rangle\!\rangle_{L^2} &:= \int_{\bar{\mathcal{B}}}\sum_{i,I=1}^{n} F^{iI}\cdot P^{iI} dX^1\wedge\cdots\wedge dX^n,\\
\langle\!\langle \boldsymbol{U},\boldsymbol{V}\rangle\!\rangle_{H^1} &:= \langle\!\langle \boldsymbol{U},\boldsymbol{V}\rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{Grad}\,\boldsymbol{U},\mathbf{Grad}\,\boldsymbol{V}\rangle\!\rangle_{L^2},\\
\langle\!\langle \boldsymbol{F},\boldsymbol{P}\rangle\!\rangle_{H^{\mathbf{C}}} &:= \langle\!\langle \boldsymbol{F},\boldsymbol{P}\rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{Curl}^{\mathsf{T}}\boldsymbol{F},\mathbf{Curl}^{\mathsf{T}}\boldsymbol{P}\rangle\!\rangle_{L^2},\\
\langle\!\langle \boldsymbol{F},\boldsymbol{P}\rangle\!\rangle_{H^{\mathbf{D}}} &:= \langle\!\langle \boldsymbol{F},\boldsymbol{P}\rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{Div}\,\boldsymbol{F},\mathbf{Div}\,\boldsymbol{P}\rangle\!\rangle_{L^2}.
\end{aligned}
$$

The isomorphisms $\boldsymbol{I}_0,\cdots,\boldsymbol{I}_3$ are L^2-isometries. The Hilbert spaces $L^2\Gamma(T\varphi(\bar{\mathcal{B}}))$, $H^1\Gamma_j(T\varphi(\bar{\mathcal{B}}))$, $H^{\mathbf{C}}\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$, and $H^{\mathbf{D}}\Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$ are the completions of $\big(\Gamma(T\varphi(\bar{\mathcal{B}})),\langle\!\langle,\rangle\!\rangle_{L^2}\big)$, $\big(\Gamma_j(T\varphi(\bar{\mathcal{B}})),\langle\!\langle,\rangle\!\rangle_{H^1}\big)$, $\big(\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}),\langle\!\langle,\rangle\!\rangle_{H^{\mathbf{C}}}\big)$, and $\big(\Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}),\langle\!\langle,\rangle\!\rangle_{H^{\mathbf{D}}}\big)$, respectively. These Hilbert spaces allow one to write the following Hilbert complex for two-point tensors:

$$
\begin{aligned}
0 \longrightarrow H^1\Gamma_1(T\varphi(\bar{\mathcal{B}})) \xrightarrow{\mathbf{Grad}_1} H^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \xrightarrow{\mathbf{Curl}_1^{\mathsf{T}}}\\
H^{\mathbf{D}}\Gamma_{\mathrm{t}_1}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \xrightarrow{\mathbf{Div}_1} L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \longrightarrow 0
\end{aligned}
\tag{3.22}
$$

The dual of this Hilbert complex reads:

$$
\begin{aligned}
\xleftarrow{\mathbf{Curl}_2^{\mathsf{T}}} H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) \xleftarrow{-\mathbf{Grad}_2} H^1\Gamma_2(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0\\
0 \longleftarrow L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \xleftarrow{-\mathbf{Div}_2} H^{\mathbf{D}}\Gamma_{\mathrm{t}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})
\end{aligned}
\tag{3.23}
$$

Remark 18. If φ is of class $C^{r,\mu}$, then we do not have the smooth **GCD** complex anymore. However, one can still write the Hilbert complex (3.22) by considering completions of $C^{r,\mu}$-sections. This case is similar to defining Hilbert complexes for less smooth manifolds. See Gol'dshtein et al. [26] for the definition of partly Sobolev spaces on less smooth manifolds.

The complex (3.22) is isomorphic to $(\boldsymbol{H^d}\Omega_{\mathrm{n}_1}(\bar{\mathcal{B}}),\boldsymbol{d}_{\mathrm{n}_1})$, and hence, it is Fredholm with $\mathsf{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}})\approx \mathrm{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}})\approx\oplus_{i=1}^3\mathrm{H}^k_{dR}(\bar{\mathcal{B}},\partial_1\mathcal{B})$, where $\mathrm{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}})$ and $\mathsf{H}^k_{\mathbf{GCD}_1}(\bar{\mathcal{B}})$ are the k-th cohomologies of the smooth **GCD** complex (with boundary conditions on $\partial_1\bar{\mathcal{B}}$) and the Hilbert complex (3.22), respectively. Let $\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) := \ker\mathbf{Curl}_1^{\mathsf{T}}\cap\ker\mathbf{Div}_2$ be the kernel of the Laplacian L_φ associated to (3.22) and (3.23). Then, $\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$ only consists of smooth harmonic two-point tensors and $\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})\approx\mathsf{H}^1_{\mathbf{GCD}_1}(\bar{\mathcal{B}})$. The next theorem is the analogue of Theorem 11 for two-point second-order tensors.

Theorem 19. *Let $\bar{\mathcal{B}}\subset\mathbb{R}^3$ be a smooth, compact 3-manifold with boundary and suppose $\varphi:\bar{\mathcal{B}}\to\mathbb{R}^3$ is a smooth mapping. The Hilbert complex (3.22) induces the following L^2-orthogonal decompositions: The Hodge decomposition*

$$
\begin{aligned}
&L^2\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) =\\
&\mathbf{Grad}\big(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\big)\oplus\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})\oplus\mathbf{Curl}^{\mathsf{T}}\big(H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\big),
\end{aligned}
$$

and, equivalently, the Helmholtz decompositions

$$\begin{aligned} L^2\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) &= \mathbf{Grad}\left(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\right)\oplus \ker \mathbf{Div}_2 \\ &= \ker \mathbf{Curl}_1^{\mathsf{T}} \oplus \mathbf{Curl}^{\mathsf{T}}\left(H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\right), \end{aligned}$$

where

$$\begin{aligned} \ker \mathbf{Div}_2 &= \mathbf{Curl}^{\mathsf{T}}\left(H^{\mathbf{C}}\Gamma_{\mathrm{n}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\right)\oplus \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}), \\ \ker \mathbf{Curl}_1^{\mathsf{T}} &= \mathbf{Grad}\left(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\right)\oplus \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \end{aligned}$$

If in addition a two-point tensor is of class $C^{r,\mu}$ (C^∞), then its components in the above decompositions are of class $C^{r,\mu}$ (C^∞) as well.

Corollary 20. *Let $\bar{\mathcal{B}}\subset\mathbb{R}^3$ be a smooth, compact 3-manifold with boundary and suppose $\varphi:\bar{\mathcal{B}}\to\mathbb{R}^3$ is a smooth mapping. The necessary and sufficient conditions for the existence of a $\mathbf{Grad}_1$-potential for $\boldsymbol{F}\in L^2\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$ are*

$$\boldsymbol{F}\in H^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}),\ \mathbf{Curl}^{\mathsf{T}}\boldsymbol{F}=0,\ \langle\!\langle \boldsymbol{F},\boldsymbol{K}\rangle\!\rangle_{L^2}=0,\ \forall \boldsymbol{K}\in\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).$$

Similarly, the necessary and sufficient conditions for the existence of a $\mathbf{Curl}_2^{\mathsf{T}}$-potential for $\boldsymbol{F}$ are

$$\boldsymbol{F}\in H^{\mathbf{D}}\Gamma_{\mathrm{t}_2}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}),\ \mathbf{Div}\,\boldsymbol{F}=0,\ \langle\!\langle \boldsymbol{F},\boldsymbol{K}\rangle\!\rangle_{L^2}=0,\ \forall \boldsymbol{K}\in\mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).$$

Remark 21. The smooth diagrams for the **GCD** complex discussed in this section are valid only for a flat $\mathcal{S}$. Similar to Remark 12, it is straightforward to extend Remark 5 to two-point tensors. By using the notation of Remark 14, one can write the following integral conditions that are equivalent to the conditions in Corollary 20 (with $\partial_j\bar{\mathcal{B}}=\varnothing$) [2]: $\boldsymbol{F}\in\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})$ admits a **Grad**-potential if and only if

$$\mathbf{Curl}^{\mathsf{T}}\boldsymbol{F}=0,\ \text{and}\qquad \int_\ell\langle \boldsymbol{F},\boldsymbol{t}_\ell\rangle dS=0,\ \forall\ell\subset\bar{\mathcal{B}},$$

and $\boldsymbol{F}$ admits a $\mathbf{Curl}^{\mathsf{T}}$-potential if and only if

$$\mathbf{Div}\,\boldsymbol{F}=0,\ \text{and}\qquad \int_{\mathcal{C}}\langle \boldsymbol{F},\boldsymbol{N}_{\mathcal{C}}\rangle dA=0,\ \forall\mathcal{C}\subset\bar{\mathcal{B}}.$$

Similar results are valid for a 2-manifold $\bar{\mathcal{B}}\subset\mathbb{R}^2$. For two-point tensors over a smooth mapping $\varphi:\bar{\mathcal{B}}\to\mathbb{R}^2$ one can define the **GC** and the **SD** complexes as follows [2]. Consider the linear differential operators

$$\begin{aligned} &\mathbf{C}:\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\to\Gamma(T\varphi(\bar{\mathcal{B}})), && (\mathbf{C}(\boldsymbol{F}))^i=F^{i2}{}_{,1}-F^{i1}{}_{,2},\\ &\mathbf{S}:\Gamma(T\varphi(\bar{\mathcal{B}}))\to\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}), && (\mathbf{S}(\boldsymbol{U}))^{iI}=\delta^{1I}U^i{}_{,2}-\delta^{2I}U^i{}_{,1}, \end{aligned}$$

and linear isomorphisms

$$\begin{aligned} &\boldsymbol{J}_0:\Gamma(T\varphi(\bar{\mathcal{B}}))\to\Omega^0(\bar{\mathcal{B}};\mathbb{R}^2), && [\boldsymbol{J}_0(\boldsymbol{U})]^i=U^i,\\ &\boldsymbol{J}_1:\Gamma(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}})\to\Omega^1(\bar{\mathcal{B}};\mathbb{R}^2), && [\boldsymbol{J}_1(\boldsymbol{F})]^i{}_J=F^{iJ},\\ &\boldsymbol{J}_2:\Gamma(T\varphi(\bar{\mathcal{B}}))\to\Omega^2(\bar{\mathcal{B}};\mathbb{R}^2), && [\boldsymbol{J}_2(\boldsymbol{U})]^i{}_{12}=U^i. \end{aligned}$$

By replacing $\jmath_0$, $\jmath_1$, $\jmath_2$, **grad**, **c**, **s**, and **div** with $\boldsymbol{J}_0$, $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, **Grad**, **C**, **S**, and **Div**, respectively, in diagrams (3.13) and (3.14), one obtains the corresponding diagrams for two-point tensors. In particular, one has the following commutative diagrams that include boundary conditions.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Gamma_j(T\varphi(\bar{\mathcal{B}})) & \xrightarrow{\mathbf{Grad}_j} & \Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{C}_j} & \Gamma(T\varphi(\bar{\mathcal{B}})) & \longrightarrow 0 \\ & \downarrow \boldsymbol{J}_0 & & \downarrow \boldsymbol{J}_1 & & \downarrow \boldsymbol{J}_2 & \\ 0 \longrightarrow & \Omega^0_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^1_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xrightarrow{\boldsymbol{d}_{\mathrm{n}_j}} & \Omega^2_{\mathrm{n}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \longrightarrow 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 \longleftarrow & \Gamma(T\varphi(\bar{\mathcal{B}})) & \xleftarrow{\mathbf{Div}_j} & \Gamma_{\mathrm{t}_j}(T\varphi(\bar{\mathcal{B}})\otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{S}_j} & \Gamma_j(T\varphi(\bar{\mathcal{B}})) & \longleftarrow 0 \\ & \downarrow -\boldsymbol{J}_0 & & \downarrow \boldsymbol{J}_1 & & \downarrow \boldsymbol{J}_2 & \\ 0 \longleftarrow & \Omega^0_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^1_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \xleftarrow{\boldsymbol{\delta}_{\mathrm{t}_j}} & \Omega^2_{\mathrm{t}_j}(\bar{\mathcal{B}};\mathbb{R}^2) & \longleftarrow 0 \end{array}$$

Let the Hilbert spaces $\bar{H}^{\mathbf{C}}\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ and $H^1\Gamma_j(T\varphi(\bar{\mathcal{B}}))$ be the completions of $\big(\Gamma_{\mathrm{n}_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{\bar{H}^{\mathbf{C}}}\big)$ and $\big(\Gamma_j(T\varphi(\bar{\mathcal{B}})), \langle\!\langle , \rangle\!\rangle_{H^1}\big)$, respectively, where

$$\begin{aligned} \langle\!\langle \boldsymbol{F}, \boldsymbol{P} \rangle\!\rangle_{\bar{H}^{\mathbf{C}}} &:= \langle\!\langle \boldsymbol{F}, \boldsymbol{P} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{C}(\boldsymbol{F}), \mathbf{C}(\boldsymbol{P}) \rangle\!\rangle_{L^2}, \\ \langle\!\langle \boldsymbol{U}, \boldsymbol{V} \rangle\!\rangle_{H^1} &:= \langle\!\langle \boldsymbol{U}, \boldsymbol{V} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{S}(\boldsymbol{U}), \mathbf{S}(\boldsymbol{V}) \rangle\!\rangle_{L^2}. \end{aligned}$$

Then, one obtains the following Hilbert complex

$$0 \longrightarrow H^1\Gamma_1(T\varphi(\bar{\mathcal{B}})) \xrightarrow{\mathbf{Grad}_1} \bar{H}^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) \xrightarrow{\mathbf{C}_1} L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \longrightarrow 0, \tag{3.24}$$

with the dual complex

$$0 \longleftarrow L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \xleftarrow{-\mathbf{Div}_2} H^{\mathbf{D}}\Gamma_{\mathrm{t}_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) \xleftarrow{\mathbf{S}_2} H^1\Gamma_2(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0. \tag{3.25}$$

Let $\mathsf{H}^k_{\mathbf{GC}_1}(\bar{\mathcal{B}})$ be the k-th cohomology group of (3.24). The Hilbert complex (3.24) is Fredholm and $\mathsf{H}^k_{\mathbf{GC}_1}(\bar{\mathcal{B}}) \approx \oplus_{i=1}^2 \mathrm{H}^k_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$. Also one has $\bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \approx \mathsf{H}^1_{\mathbf{GC}_1}(\bar{\mathcal{B}})$, where $\bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) := \ker \mathbf{C}_1 \cap \ker \mathbf{Div}_2$. The decompositions associated to the 2D case are as follows.

Theorem 22. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^2$ be a smooth, compact 2-manifold with boundary and suppose $\varphi : \bar{\mathcal{B}} \to \mathbb{R}^2$ is a smooth mapping. The Hilbert complex (3.24) induces the following L^2-orthogonal decompositions: The Hodge decomposition*

$$L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) = \mathbf{Grad}\left(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\right) \oplus \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) \oplus \mathbf{S}\left(H^1\Gamma_2(T\varphi(\bar{\mathcal{B}}))\right),$$

and, equivalently, the Helmholtz decompositions

$$\begin{aligned} L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) &= \mathbf{Grad}\left(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\right) \oplus \ker \mathbf{Div}_2 \\ &= \ker \mathbf{C}_1 \oplus \mathbf{S}\left(H^1\Gamma_2(T\varphi(\bar{\mathcal{B}}))\right), \end{aligned}$$

where

$$\begin{aligned} \ker \mathbf{Div}_2 &= \mathbf{S}\left(H^1\Gamma_2(T\varphi(\bar{\mathcal{B}}))\right) \oplus \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}), \\ \ker \mathbf{C}_1 &= \mathbf{Grad}\big(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))\big) \oplus \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}). \end{aligned}$$

If in addition a two-point tensor is of class $C^{r,\mu}$ (C^∞), then its components in the above decompositions *are of class $C^{r,\mu}$ (C^∞) as well.*

Corollary 23. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^2$ be a smooth, compact 2-manifold with boundary and suppose $\varphi : \bar{\mathcal{B}} \to \mathbb{R}^2$ is a smooth mapping. The necessary and sufficient conditions for the existence of a* $\mathbf{Grad}_1$*-potential for $\boldsymbol{F} \in L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ are*

$$\boldsymbol{F} \in \bar{H}^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}),\ \mathbf{C}(\boldsymbol{F}) = 0,\ \langle\!\langle \boldsymbol{F}, \boldsymbol{K} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{K} \in \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).$$

Similarly, the necessary and sufficient conditions for the existence of an $\mathbf{S}_2$*-potential for $\boldsymbol{F}$ are*

$$\boldsymbol{F} \in H^{\mathbf{D}}\Gamma_{\mathrm{t}_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}),\ \mathbf{Div}\, \boldsymbol{F} = 0,\ \langle\!\langle \boldsymbol{F}, \boldsymbol{K} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{K} \in \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}).$$

Remark 24. Remark 21 applies to the 2D case as well. In particular, using the notation of Remark 17, the integral conditions equivalent to the conditions of Corollary 23 (with $\partial_j \bar{\mathcal{B}} = \varnothing$) read as follows [2]: $\boldsymbol{F} = \mathbf{Grad}\, \boldsymbol{U}$, if and only if

$$\mathbf{C}(\boldsymbol{F}) = 0,\ \text{and} \qquad \int_\ell \langle \boldsymbol{F}, \boldsymbol{t}_\ell \rangle dS = 0,\ \forall \ell \subset \bar{\mathcal{B}},$$

and we have $\boldsymbol{F} = \mathbf{S}(\boldsymbol{U})$, if and only if

$$\mathbf{Div}\, \boldsymbol{F} = 0,\ \text{and} \qquad \int_\ell \langle \boldsymbol{F}, \boldsymbol{N}_\ell \rangle dS = 0,\ \forall \ell \subset \bar{\mathcal{B}}.$$

3.3 Symmetric Tensors

It is possible to employ the framework of Hilbert complexes for deriving orthogonal decompositions for symmetric second-order tensors. To this end, we use the linear elasticity complex, also called the Kröner complex [30], which is equivalent to a more general complex introduced by Calabi [14]. More discussions on this equivalence can be found in [2, 20]. Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a compact 3-manifold with boundary and let $\Gamma(S^2 T\bar{\mathcal{B}})$ be the space of smooth symmetric $\binom{2}{0}$-tensors. The linear elasticity complex on $\bar{\mathcal{B}}$ reads

$$0 \longrightarrow \mathfrak{X}(\bar{\mathcal{B}}) \xrightarrow{\mathbf{grad}^s} \Gamma(S^2 T\bar{\mathcal{B}}) \xrightarrow{\mathbf{curl}\circ\mathbf{curl}} \Gamma(S^2 T\bar{\mathcal{B}}) \xrightarrow{\mathbf{div}} \mathfrak{X}(\bar{\mathcal{B}}) \longrightarrow 0, \tag{3.26}$$

where

$$(\mathbf{grad}^s \boldsymbol{Y})^{IJ} = \frac{1}{2}\left(Y^I{}_{,J} + Y^J{}_{,I}\right),\ \ (\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{T})^{IJ} = \varepsilon_{IKL}\varepsilon_{JMN} T^{LN}{}_{,KM}.$$

Consider the inner products $\langle\!\langle , \rangle\!\rangle_{H^{\mathbf{g}}_s}$ on $\mathfrak{X}(\bar{\mathcal{B}})$ and $\langle\!\langle , \rangle\!\rangle_{H^{\mathbf{cc}}}$ on $\Gamma(S^2 T\bar{\mathcal{B}})$ given by

$$\begin{aligned}
\langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{H^{\mathbf{g}}_s} &:= \langle\!\langle \boldsymbol{Y}, \boldsymbol{Z} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{grad}^s \boldsymbol{Y}, \mathbf{grad}^s \boldsymbol{Z} \rangle\!\rangle_{L^2}, \\
\langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{H^{\mathbf{cc}}} &:= \langle\!\langle \boldsymbol{S}, \boldsymbol{T} \rangle\!\rangle_{L^2} + \langle\!\langle \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{S}, \mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{T} \rangle\!\rangle_{L^2}.
\end{aligned}$$

The Hilbert spaces $L^2\Gamma(S^2 T\bar{\mathcal{B}})$, $H^{\mathbf{g}}_s \mathfrak{X}(\bar{\mathcal{B}})$, $H^{\mathbf{cc}}\Gamma(S^2 T\bar{\mathcal{B}})$, and $H^{\mathbf{d}}\Gamma(S^2 T\bar{\mathcal{B}})$ are defined as the completions of $\big(\Gamma(S^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{L^2}\big)$, $\big(\mathfrak{X}(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{g}}_s}\big)$, $\big(\Gamma(S^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{cc}}}\big)$, and $\big(\Gamma(S^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{d}}}\big)$, respectively, where $\langle\!\langle , \rangle\!\rangle_{H^{\mathbf{d}}}$ was introduced in (3.6). Korn's inequality (e.g. see [17]) suggests that $H^{\mathbf{g}}_s \mathfrak{X}(\bar{\mathcal{B}})$ can be continuously embedded in $H^1 \mathfrak{X}(\bar{\mathcal{B}})$. However, note that these spaces are not isometric. By using appropriate closed extensions of the operators of the smooth linear elasticity complex, one obtains the following Hilbert complex

$$0 \longrightarrow H^{\mathbf{g}}_s\mathfrak{X}(\bar{\mathcal{B}}) \xrightarrow{\mathbf{grad}^s} H^{\mathbf{cc}}\Gamma(S^2T\bar{\mathcal{B}}) \xrightarrow{\mathbf{curl}\circ\mathbf{curl}} H^{\mathbf{d}}\Gamma(S^2T\bar{\mathcal{B}}) \xrightarrow{\mathbf{div}} L^2\mathfrak{X}(\bar{\mathcal{B}}) \longrightarrow 0. \tag{3.27}$$

For determining the dual of this Hilbert complex, we use the following Green's formulae. Let $\boldsymbol{Y} \in \mathfrak{X}(\bar{\mathcal{B}})$, $\boldsymbol{T} \in \Gamma(S^2T\bar{\mathcal{B}})$, and suppose $\boldsymbol{N}$ is the unit outward normal vector field of $\partial\bar{\mathcal{B}}$. Then, it is straightforward to show that (e.g. see [37, Eq. (3.8)])

$$\langle\!\langle \mathbf{grad}^s\boldsymbol{Y}, \boldsymbol{T}\rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{Y}, -\mathbf{div}\,\boldsymbol{T}\rangle\!\rangle_{L^2} + \int_{\partial\bar{\mathcal{B}}} \boldsymbol{G}\left(\boldsymbol{Y}, \langle \boldsymbol{T}, \boldsymbol{N}\rangle\right) dA. \tag{3.28}$$

The Green's formula for $\mathbf{curl}\circ\mathbf{curl}$ is more complicated and we derive it here. For $\boldsymbol{T} \in \Gamma(\otimes^2T\bar{\mathcal{B}})$, let $\overleftarrow{\boldsymbol{T}}_{\mathbf{E}_I} := T^{JI}\mathbf{E}_J \in \mathfrak{X}(\bar{\mathcal{B}})$, that is, the Cartesian components of $\boldsymbol{T}$ can be rearranged as

$$\boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} \overleftarrow{\boldsymbol{T}}_{\mathbf{E}_1} & \overleftarrow{\boldsymbol{T}}_{\mathbf{E}_2} & \overleftarrow{\boldsymbol{T}}_{\mathbf{E}_3} \end{pmatrix}.$$

Also consider the linear operator

$$\mathbf{f} : \Gamma(\otimes^2T\bar{\mathcal{B}}) \to \Gamma(\otimes^2T\bar{\mathcal{B}}), \ \ (\mathbf{f}(\boldsymbol{T}))^{IJ} = \varepsilon_{JKL}T^{LI}{}_{,K}. \tag{3.29}$$

Note that $\mathbf{f}\circ\mathbf{f} = \mathbf{curl}\circ\mathbf{curl}$. The Green's formula for $\mathbf{curl}\circ\mathbf{curl}$ can be written as follows.

Lemma 25. *For arbitrary $\boldsymbol{T}, \boldsymbol{S} \in \Gamma(\otimes^2T\bar{\mathcal{B}})$, we have*

$$\langle\!\langle \mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{T}, \boldsymbol{S}\rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{T}, \mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{S}\rangle\!\rangle_{L^2} + \mathrm{BC}_1 + \mathrm{BC}_2, \tag{3.30}$$

where

$$\mathrm{BC}_1 = \sum_{I=1}^{3}\int_{\partial\bar{\mathcal{B}}} \mathrm{i}^*\Big(\mathsf{t}\,\overrightarrow{\mathbf{curl}\,\boldsymbol{T}}_{\mathbf{E}_I}\Big)^\flat \wedge \mathrm{i}^*\Big(\mathsf{t}\,\overleftarrow{\boldsymbol{S}}_{\mathbf{E}_I}\Big)^\flat,$$
$$\mathrm{BC}_2 = \sum_{I=1}^{3}\int_{\partial\bar{\mathcal{B}}} \mathrm{i}^*\Big(\mathsf{t}\,\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I}\Big)^\flat \wedge \mathrm{i}^*\Big(\mathsf{t}\,\overleftarrow{\mathbf{f}(\boldsymbol{S})}_{\mathbf{E}_I}\Big)^\flat.$$

Proof. Consider the isomorphism $\bar{\boldsymbol{\imath}}_2 : \Gamma(\otimes^2T\bar{\mathcal{B}}) \to \Omega^2(\bar{\mathcal{B}};\mathbb{R}^3)$ given by $\bar{\boldsymbol{\imath}}_2(\boldsymbol{T}) = \big(*\overleftarrow{\boldsymbol{T}}^\flat_{\mathbf{E}_1}, *\overleftarrow{\boldsymbol{T}}^\flat_{\mathbf{E}_2}, *\overleftarrow{\boldsymbol{T}}^\flat_{\mathbf{E}_3}\big)$. One can show that $\boldsymbol{d}\circ\boldsymbol{\imath}_1 = \bar{\boldsymbol{\imath}}_2\circ\mathbf{curl}$, and $\boldsymbol{\delta}\circ\bar{\boldsymbol{\imath}}_2 = \boldsymbol{\imath}_1\circ\mathbf{f}$. Since $\boldsymbol{\imath}_1$ and $\bar{\boldsymbol{\imath}}_2$ are L^2-isometries, by using Green's formula (2.5), we can write

$$\begin{aligned} \langle\!\langle \mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{T}, \boldsymbol{S}\rangle\!\rangle_{L^2} &= \langle\!\langle \boldsymbol{d}\left(\boldsymbol{\imath}_1(\mathbf{curl}\,\boldsymbol{T})\right), \bar{\boldsymbol{\imath}}_2(\boldsymbol{S})\rangle\!\rangle_{\boldsymbol{L}^2} \\ &= \langle\!\langle \mathbf{curl}\,\boldsymbol{T}, \mathbf{f}(\boldsymbol{S})\rangle\!\rangle_{L^2} + \overline{\mathrm{BC}}_1 \\ &= \langle\!\langle \boldsymbol{d}\left(\boldsymbol{\imath}_1(\boldsymbol{T})\right), \bar{\boldsymbol{\imath}}_2\left(\mathbf{f}(\boldsymbol{S})\right)\rangle\!\rangle_{\boldsymbol{L}^2} + \overline{\mathrm{BC}}_1 \\ &= \langle\!\langle \boldsymbol{T}, \mathbf{f}\circ\mathbf{f}(\boldsymbol{S})\rangle\!\rangle_{L^2} + \overline{\mathrm{BC}}_1 + \overline{\mathrm{BC}}_2, \end{aligned}$$

where

$$\overline{\mathrm{BC}}_1 = \int_{\partial\bar{\mathcal{B}}} \mathrm{i}^*\left(\mathsf{t}\left(\boldsymbol{\imath}_1(\mathbf{curl}\,\boldsymbol{T})\right)\right) \wedge \mathrm{i}^*\left(*\mathsf{n}\left(\bar{\boldsymbol{\imath}}_2(\boldsymbol{S})\right)\right),$$
$$\overline{\mathrm{BC}}_2 = \int_{\partial\bar{\mathcal{B}}} \mathrm{i}^*\left(\mathsf{t}\left(\boldsymbol{\imath}_1(\boldsymbol{T})\right)\right) \wedge \mathrm{i}^*\left(*\mathsf{n}\left(\bar{\boldsymbol{\imath}}_2\circ\mathbf{f}(\boldsymbol{S})\right)\right).$$

The relations (2.1) and (2.2) imply that $\overline{\mathrm{BC}}_i = \mathrm{BC}_i$, $i = 1, 2$. This completes the proof. □

Let $\mathfrak{X}_c(\mathcal{B})$ and $\Gamma_c(S^2T\mathcal{B})$ be the spaces of smooth vector fields and smooth symmetric $\binom{2}{0}$-tensors on $\mathcal{B}$ with compact supports. By restricting the smooth linear elasticity complex to these spaces, one obtains the following smooth complex

$$0 \longrightarrow \mathfrak{X}_c(\mathcal{B}) \xrightarrow{\mathbf{grad}^s_c} \Gamma_c(S^2T\mathcal{B}) \xrightarrow{\mathbf{curl}\circ\mathbf{curl}_c} \Gamma_c(S^2T\mathcal{B}) \xrightarrow{\mathbf{div}_c} \mathfrak{X}_c(\mathcal{B}) \longrightarrow 0. \tag{3.31}$$

Suppose $H^{\mathbf{g}}_s\mathfrak{X}_c(\mathcal{B})$, $H^{\mathbf{cc}}\Gamma_c(S^2T\mathcal{B})$, and $H^{\mathbf{d}}\Gamma_c(S^2T\mathcal{B})$ are the completions of $\big(\mathfrak{X}_c(\mathcal{B}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{g}}_s}\big)$, $\big(\Gamma_c(S^2T\mathcal{B}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{cc}}}\big)$, and $\big(\Gamma_c(S^2T\mathcal{B}), \langle\!\langle , \rangle\!\rangle_{H^{\mathbf{d}}}\big)$, respectively. Note that the completion of $(\mathfrak{X}_c(\mathcal{B}), \langle\!\langle , \rangle\!\rangle_{L^2})$ and $\big(\Gamma_c(S^2T\mathcal{B}), \langle\!\langle , \rangle\!\rangle_{L^2}\big)$ are $L^2\mathfrak{X}(\bar{\mathcal{B}})$ and $L^2\Gamma(S^2T\bar{\mathcal{B}})$, respectively. One can define closed extensions of $\mathbf{grad}^s_c$, $\mathbf{curl}\circ\mathbf{curl}_c$, and $\mathbf{div}_c$ using these Hilbert spaces. In particular, (3.28) and (3.30) imply that the closed operators $-\mathbf{div}_c$, $\mathbf{curl}\circ\mathbf{curl}_c$, and $-\mathbf{grad}^s_c$ are the adjoint operators of $\mathbf{grad}^s$, $\mathbf{curl}\circ\mathbf{curl}$, and $\mathbf{div}$, respectively. Thus, (3.27) admits the following dual complex

$$0 \longleftarrow L^2\mathfrak{X}(\bar{\mathcal{B}}) \xleftarrow{-\mathbf{div}_c} H^{\mathbf{d}}\Gamma_c(S^2T\mathcal{B}) \xleftarrow{\mathbf{curl}\circ\mathbf{curl}_c} H^{\mathbf{cc}}\Gamma_c(S^2T\mathcal{B}) \xleftarrow{-\mathbf{grad}^s_c} H^{\mathbf{g}}_s\mathfrak{X}_c(\mathcal{B}) \longleftarrow 0.$$

Theorem 3.5 of [13] suggests that the cohomology groups of (3.27) and their duals are isomorphic to their smooth subcomplexes and since the cohomologies of the smooth linear elasticity complex on $\bar{\mathcal{B}}$ are finite dimensional, one concludes that (3.27) and its dual are Fredholm. More specifically, let $\mathsf{H}^1_{E3}(\bar{\mathcal{B}}) := \ker \mathbf{curl}\circ\mathbf{curl}/\mathrm{im}\,\mathbf{grad}^s$, and $\mathsf{H}^2_{E3}(\bar{\mathcal{B}}) := \ker \mathbf{div}/\mathrm{im}\,\mathbf{curl}\circ\mathbf{curl}$, be the cohomology groups of (3.27). Also let $\mathcal{H}^1_{E3} := \ker \mathbf{curl}\circ\mathbf{curl} \cap \ker \mathbf{div}_c$, and $\mathcal{H}^2_{E3} := \ker \mathbf{div} \cap \ker \mathbf{curl}\circ\mathbf{curl}_c$. By using the cohomology groups of the linear elasticity complex derived by Calabi [14] and the Fredholm property, one obtains the following result.

Theorem 26. *The Hilbert complex (3.27) is Fredholm and*

$$\dim \mathsf{H}^i_{E3}(\bar{\mathcal{B}}) = \dim \mathcal{H}^i_{E3}(\bar{\mathcal{B}}) = 6 \dim \mathrm{H}^i_{dR}(\bar{\mathcal{B}}),\ \ i = 1, 2,$$

and hence, (3.27) induces the following L^2-orthogonal decompositions: The Hodge decompositions

$$L^2\Gamma(S^2T\bar{\mathcal{B}}) = \mathbf{grad}^s\big(H^{\mathbf{g}}_s\mathfrak{X}(\bar{\mathcal{B}})\big) \oplus \mathcal{H}^1_{E3}(\bar{\mathcal{B}}) \oplus \mathbf{curl}\circ\mathbf{curl}\big(H^{\mathbf{cc}}\Gamma_c(S^2T\mathcal{B})\big)\,,$$
$$L^2\Gamma(S^2T\bar{\mathcal{B}}) = \mathbf{grad}^s\big(H^{\mathbf{g}}_s\mathfrak{X}_c(\mathcal{B})\big) \oplus \mathcal{H}^2_{E3}(\bar{\mathcal{B}}) \oplus \mathbf{curl}\circ\mathbf{curl}\big(H^{\mathbf{cc}}\Gamma(S^2T\bar{\mathcal{B}})\big)\,,$$

and, equivalently, the Helmholtz decompositions

$$L^2\Gamma(S^2T\bar{\mathcal{B}}) = \mathbf{grad}^s\big(H^{\mathbf{g}}_s\mathfrak{X}(\bar{\mathcal{B}})\big) \oplus \ker \mathbf{div}_c \tag{3.32a}$$
$$= \ker \mathbf{curl}\circ\mathbf{curl} \oplus \mathbf{curl}\circ\mathbf{curl}\big(H^{\mathbf{cc}}\Gamma_c(S^2T\mathcal{B})\big) \tag{3.32b}$$
$$= \mathbf{grad}^s\big(H^{\mathbf{g}}_s\mathfrak{X}_c(\mathcal{B})\big) \oplus \ker \mathbf{div} \tag{3.32c}$$
$$= \ker \mathbf{curl}\circ\mathbf{curl}_c \oplus \mathbf{curl}\circ\mathbf{curl}\big(H^{\mathbf{cc}}\Gamma(S^2T\bar{\mathcal{B}})\big)\,, \tag{3.32d}$$

where

$$\ker \mathbf{div}_c = \mathbf{curl}\circ\mathbf{curl}\big(H^{\mathbf{cc}}\Gamma_c(S^2T\mathcal{B})\big) \oplus \mathcal{H}^1_{E3}(\bar{\mathcal{B}}), \tag{3.33a}$$
$$\ker \mathbf{curl}\circ\mathbf{curl} = \mathbf{grad}^s\big(H^{\mathbf{g}}_s\mathfrak{X}(\bar{\mathcal{B}})\big) \oplus \mathcal{H}^1_{E3}(\bar{\mathcal{B}}), \tag{3.33b}$$
$$\ker \mathbf{div} = \mathbf{curl}\circ\mathbf{curl}\big(H^{\mathbf{cc}}\Gamma(S^2T\bar{\mathcal{B}})\big) \oplus \mathcal{H}^2_{E3}(\bar{\mathcal{B}}), \tag{3.33c}$$
$$\ker \mathbf{curl}\circ\mathbf{curl}_c = \mathbf{grad}^s\big(H^{\mathbf{g}}_s\mathfrak{X}_c(\mathcal{B})\big) \oplus \mathcal{H}^2_{E3}(\bar{\mathcal{B}}). \tag{3.33d}$$

Remark 27. The Hodge decompositions in the above theorem are equivalent to the Hodge decompositions (2.21) and (4.1) of Geymonat and Krasucki [23]. The Helmholtz decomposition (3.32a) is equivalent to the orthogonal decomposition given by Ting [37, Theorem 3.1]. The decomposition for divergence-free symmetric tensors introduced by Gurtin [27, Theorem 4.4] is equivalent to (3.33c). As was mentioned earlier, the linear elasticity complex is a special case of the Calabi complex, which is valid on manifolds with constant curvature. Therefore, by following the above procedure for the Calabi complex, one can also obtain orthogonal decompositions for symmetric tensors on manifolds with constant curvature.

Corollary 28. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary. Then, $\boldsymbol{T} \in L^2\Gamma(S^2T\bar{\mathcal{B}})$ admits a $\mathbf{grad}^s$-potential if and only if*

$$\mathbf{curl} \circ \mathbf{curl}\, \boldsymbol{T} = 0,\ \ \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \ \forall \boldsymbol{Q} \in \mathcal{H}^1_{E3}(\bar{\mathcal{B}}). \tag{3.34}$$

Similarly, $\boldsymbol{T}$ admits a $\mathbf{curl} \circ \mathbf{curl}$-potential if and only if

$$\mathbf{div}\, \boldsymbol{T} = 0,\ \ \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \ \forall \boldsymbol{Q} \in \mathcal{H}^2_{E3}(\bar{\mathcal{B}}). \tag{3.35}$$

Remark 29. For sufficiently smooth symmetric tensor fields, the condition (3.34) is equivalent to those given by Georgescu [22, Theorem 5.3] and Yavari [40, Proposition 2.8]. Similarly, (3.35) is equivalent to the conditions in Gurtin [27, Theorem 3.4] and Georgescu [22, Theorem 5.2].

Next, we derive orthogonal decompositions for symmetric tensors by imposing proper boundary conditions on $\partial_j \bar{\mathcal{B}}$, $j = 1, 2$. To this end, we define the following linear subspaces of $\Gamma(S^2T\bar{\mathcal{B}})$:

$$\begin{aligned}
\Gamma_j(S^2T\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{T} \in \Gamma(S^2T\bar{\mathcal{B}}) : \langle \boldsymbol{T}, \boldsymbol{N} \rangle|_{\partial_j \bar{\mathcal{B}}} = 0 \right\}, \\
\Gamma_{\mathrm{n}_j}(S^2T\bar{\mathcal{B}}) &:= \left\{ \boldsymbol{T} \in \Gamma(S^2T\bar{\mathcal{B}}) : \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I}, \overrightarrow{\mathbf{curl}\, \boldsymbol{T}}_{\mathbf{E}_I} \perp \partial_j \bar{\mathcal{B}},\ I = 1, 2, 3 \right\}.
\end{aligned}$$

If $\boldsymbol{T}$ represents a Cauchy stress, then $\boldsymbol{T} \in \Gamma_j(S^2T\bar{\mathcal{B}})$ if and only if the traction of $\boldsymbol{T}$ on $\partial_j \bar{\mathcal{B}}$ vanishes. Let $H^{\mathbf{g}}_s\mathfrak{X}_j(\bar{\mathcal{B}})$, $H^{\mathbf{cc}}\Gamma_{\mathrm{n}_j}(S^2T\bar{\mathcal{B}})$, and $H^{\mathbf{d}}\Gamma_j(S^2T\bar{\mathcal{B}})$ be the completions of $\big(\mathfrak{X}_j(\bar{\mathcal{B}}), \langle\!\langle, \rangle\!\rangle_{H^{\mathbf{g}}_s}\big)$, $\big(\Gamma_{\mathrm{n}_j}(S^2T\bar{\mathcal{B}}), \langle\!\langle, \rangle\!\rangle_{H^{\mathbf{cc}}}\big)$, and $\big(\Gamma_j(S^2T\bar{\mathcal{B}}), \langle\!\langle, \rangle\!\rangle_{H^{\mathbf{d}}}\big)$, respectively. One can write the following unbounded, densely-defined, closed operators:

$$\begin{aligned}
\mathbf{grad}^s_j &: L^2\mathfrak{X}(\bar{\mathcal{B}}) \to L^2\Gamma(S^2T\bar{\mathcal{B}}), & \mathrm{D}(\mathbf{grad}^s_j) &= H^{\mathbf{g}}_s\mathfrak{X}_j(\bar{\mathcal{B}}), \\
\mathbf{curl}\circ\mathbf{curl}_j &: L^2\Gamma(S^2T\bar{\mathcal{B}}) \to L^2\Gamma(S^2T\bar{\mathcal{B}}), & \mathrm{D}(\mathbf{curl}\circ\mathbf{curl}_j) &= H^{\mathbf{cc}}\Gamma_{\mathrm{n}_j}(S^2T\bar{\mathcal{B}}), \\
\mathbf{div}_j &: L^2\Gamma(S^2T\bar{\mathcal{B}}) \to L^2\mathfrak{X}(\bar{\mathcal{B}}), & \mathrm{D}(\mathbf{div}_j) &= H^{\mathbf{d}}\Gamma_j(S^2T\bar{\mathcal{B}}).
\end{aligned}$$

Green's formula (3.28) suggests that $-\mathbf{div}_2$ is the adjoint operator of $\mathbf{grad}^s_1$. For obtaining the adjoint operator of $\mathbf{curl} \circ \mathbf{curl}_1$, note that for $\boldsymbol{T} \in \Gamma(S^2T\bar{\mathcal{B}})$, one has $\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} = \overleftarrow{\boldsymbol{T}}_{\mathbf{E}_I}$, and $\overrightarrow{\mathbf{curl}\, \boldsymbol{T}}_{\mathbf{E}_I} = \overleftarrow{\mathbf{f}(\boldsymbol{T})}_{\mathbf{E}_I}$, $I = 1, 2, 3$, where $\mathbf{f}$ is defined in (3.29). These facts together with (3.30) imply that $\mathbf{curl} \circ \mathbf{curl}_2$ is the adjoint operator of $\mathbf{curl} \circ \mathbf{curl}_1$.

Imposing the above boundary conditions on the linear elasticity complex does not result in a complex, i.e. $\mathbf{grad}^s_j(\mathfrak{X}_j(\bar{\mathcal{B}})) \not\subset \Gamma_{\mathrm{n}_j}(S^2T\bar{\mathcal{B}})$, etc. Nevertheless, it is still possible to obtain Helmholtz-type orthogonal decompositions as follows. Recall that if D^a is the adjoint of a closed operator $D : H_1 \to H_2$, then $\ker D$ is closed in H_1 and $(\ker D)^\perp = \overline{\operatorname{im} D^a}$, where $\overline{\operatorname{im} D^a}$ is the closure of the image of D^a in H_1. Also since $H_1 = \ker D \oplus (\ker D)^\perp$, one concludes that:

Theorem 30. *The space $L^2\Gamma(S^2T\bar{\mathcal{B}})$ on a compact 3-manifold with boundary $\bar{\mathcal{B}} \subset \mathbb{R}^3$ admits the*

Helmholtz decompositions

$$L^2\Gamma(S^2T\bar{\mathcal{B}}) = \overline{\operatorname{im}\mathbf{grad}_1^s} \oplus \ker \mathbf{div}_2 = \ker \mathbf{curl}\circ\mathbf{curl}_1 \oplus \overline{\operatorname{im}\mathbf{curl}\circ\mathbf{curl}_2},$$

where $\overline{\operatorname{im}\mathbf{grad}_1^s}$ *and* $\overline{\operatorname{im}\mathbf{curl}\circ\mathbf{curl}_2}$ *are the closures of the images of* $\mathbf{grad}_1^s$ *and* $\mathbf{curl}\circ\mathbf{curl}_2$ *in* $L^2\Gamma(S^2T\bar{\mathcal{B}})$.

Remark 31. The first decomposition in the above theorem induced by $\mathbf{grad}_1^s$ and $\mathbf{div}_2$ is equivalent to the decomposition derived in [34, Theorem 3.1]. One does not need the complex structure for deriving the above theorem, which generalizes the Helmholtz decompositions of Theorem 26. Cantor [15] discussed a general framework for writing such decompositions by using elliptic operators. Berger and Ebin [10] used a similar approach for deriving Helmholtz-type decompositions for symmetric tensors on Riemannian manifolds with constant curvature.

Finally, we study decompositions for symmetric tensors on a compact 2-manifold $\bar{\mathcal{B}} \subset \mathbb{R}^2$. The 2D linear elasticity complex reads

$$0 \longrightarrow \mathfrak{X}(\bar{\mathcal{B}}) \xrightarrow{\mathbf{grad}^s} \Gamma(S^2T\bar{\mathcal{B}}) \xrightarrow{\mathrm{D_c}} C^\infty(\bar{\mathcal{B}}) \longrightarrow 0, \tag{3.36}$$

where in the Cartesian coordinates $\{X^I\}$, $\mathrm{D_c}\boldsymbol{T} := T^{11}{}_{,22} - 2T^{12}{}_{,12} + T^{22}{}_{,11}$. One can also write the complex

$$0 \longrightarrow C^\infty(\bar{\mathcal{B}}) \xrightarrow{\mathrm{D_s}} \Gamma(S^2T\bar{\mathcal{B}}) \xrightarrow{\mathbf{div}} \mathfrak{X}(\bar{\mathcal{B}}) \longrightarrow 0, \tag{3.37}$$

with $(\mathrm{D_s}f)^{11} = f_{,22}$, $(\mathrm{D_s}f)^{12} = -f_{,12}$, and $(\mathrm{D_s}f)^{22} = f_{,11}$. By restricting the above complexes to compactly supported sections on $\mathcal{B}$, one obtains the following complexes

$$0 \longrightarrow \mathfrak{X}_c(\mathcal{B}) \xrightarrow{\mathbf{grad}_c^s} \Gamma_c(S^2T\mathcal{B}) \xrightarrow{\mathrm{D_{cc}}} C_c^\infty(\mathcal{B}) \longrightarrow 0, \tag{3.38}$$

$$0 \longrightarrow C_c^\infty(\mathcal{B}) \xrightarrow{\mathrm{D_{sc}}} \Gamma_c(S^2T\mathcal{B}) \xrightarrow{\mathbf{div}_c} \mathfrak{X}_c(\mathcal{B}) \longrightarrow 0. \tag{3.39}$$

It is straightforward to show that $\mathrm{D_c}\boldsymbol{T} = *d\left(\mathbf{c}(\boldsymbol{T})\right)^\flat$, and $\mathrm{D_s}f = \mathbf{s}\big(\left(\delta(*f)\right)^\sharp\big)$, where $\sharp : \Omega^1(\bar{\mathcal{B}}) \to \mathfrak{X}(\bar{\mathcal{B}})$ is the inverse of $\flat$, and $\mathbf{c}$ and $\mathbf{s}$ were introduced in (3.15). By using these relations and following the approach of Lemma 25, one concludes that

$$\langle\!\langle \mathrm{D_c}\boldsymbol{T}, f\rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{T}, \mathrm{D_s}f\rangle\!\rangle_{L^2} + \widetilde{\mathrm{BC}}_1 + \widetilde{\mathrm{BC}}_2, \tag{3.40}$$

where

$$\widetilde{\mathrm{BC}}_1 = \int_{\partial\bar{\mathcal{B}}} f\, \mathrm{i}^*(\mathrm{t}\,\mathbf{c}(\boldsymbol{T}))^\flat,$$
$$\widetilde{\mathrm{BC}}_2 = \int_{\partial\bar{\mathcal{B}}} f_{,2}\, \mathrm{i}^*\Big(\mathrm{t}\,\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_1}\Big)^\flat - \int_{\partial\bar{\mathcal{B}}} f_{,1}\, \mathrm{i}^*\Big(\mathrm{t}\,\overrightarrow{\boldsymbol{T}}_{\mathbf{E}_2}\Big)^\flat.$$

Let $H^{\mathrm{D_c}}\Gamma(S^2T\bar{\mathcal{B}})$ and $H^{\mathrm{D_s}}(\bar{\mathcal{B}})$ be the completions of $\big(\Gamma(S^2T\bar{\mathcal{B}}), \langle\!\langle,\rangle\!\rangle_{H^{\mathrm{D_c}}}\big)$ and $\big(C^\infty(\bar{\mathcal{B}}), \langle\!\langle,\rangle\!\rangle_{H^{\mathrm{D_s}}}\big)$, respectively, where

$$\langle\!\langle \boldsymbol{S}, \boldsymbol{T}\rangle\!\rangle_{H^{\mathrm{D_c}}} := \langle\!\langle \boldsymbol{S}, \boldsymbol{T}\rangle\!\rangle_{L^2} + \langle\!\langle \mathrm{D_c}\boldsymbol{S}, \mathrm{D_c}\boldsymbol{T}\rangle\!\rangle_{L^2},$$
$$\langle\!\langle f, h\rangle\!\rangle_{H^{\mathrm{D_s}}} := \langle\!\langle f, h\rangle\!\rangle_{L^2} + \langle\!\langle \mathrm{D_s}f, \mathrm{D_s}h\rangle\!\rangle_{L^2},$$

with $\langle\!\langle f, h\rangle\!\rangle_{L^2} := \int_{\bar{\mathcal{B}}} fh\boldsymbol{\mu_G}$. Similarly, we define the Hilbert spaces of compactly supported sections

$H^{\mathrm{D_c}}\Gamma_c(S^2T\mathcal{B})$ and $H_c^{\mathrm{D_s}}(\mathcal{B})$. Suppose $L^2(\bar{\mathcal{B}})$ is the space of L^2 real-valued functions on $\bar{\mathcal{B}}$. Then, the Hilbert complexes associated to the smooth complexes (3.36) and (3.37) are

$$\begin{aligned} &0 \longrightarrow H_s^{\mathbf{g}}\mathfrak{X}(\bar{\mathcal{B}}) \xrightarrow{\mathbf{grad}^s} H^{\mathrm{D_c}}\Gamma(S^2T\bar{\mathcal{B}}) \xrightarrow{\mathrm{D_c}} L^2(\bar{\mathcal{B}}) \longrightarrow 0, \\ &0 \longrightarrow H^{\mathrm{D_s}}(\bar{\mathcal{B}}) \xrightarrow{\mathrm{D_s}} H^{\mathbf{d}}\Gamma(S^2T\bar{\mathcal{B}}) \xrightarrow{\mathbf{div}} L^2\mathfrak{X}(\bar{\mathcal{B}}) \longrightarrow 0. \end{aligned} \tag{3.41}$$

Remark 32. As was discussed in [6, 7], the Hilbert complexes (3.41) and (3.27) can be used for developing stable mixed finite element formulations for 2D and 3D linear elasticity.

By using Green's formulae (3.28) and (3.40), one concludes that the dual complexes of the above Hilbert complexes are

$$0 \longleftarrow L^2\mathfrak{X}(\bar{\mathcal{B}}) \xleftarrow{-\mathbf{div}_c} H^{\mathbf{d}}\Gamma_c(S^2T\mathcal{B}) \xleftarrow{\mathrm{D_{sc}}} H_c^{\mathrm{D_s}}(\mathcal{B}) \longleftarrow 0, \tag{3.42}$$

$$0 \longleftarrow L^2(\bar{\mathcal{B}}) \xleftarrow{\mathrm{D_{cc}}} H^{\mathrm{D_c}}\Gamma_c(S^2T\mathcal{B}) \xleftarrow{-\mathbf{grad}_c^s} H_s^{\mathbf{g}}\mathfrak{X}_c(\mathcal{B}) \longleftarrow 0. \tag{3.43}$$

Let $\mathsf{H}^1_{E2}(\bar{\mathcal{B}}) := \ker \mathrm{D_c}/\operatorname{im}\mathbf{grad}^s$, $\mathsf{H}^1_{E2'}(\bar{\mathcal{B}}) := \ker \mathbf{div}/\operatorname{im}\mathrm{D_s}$, $\mathcal{H}^1_{E2}(\bar{\mathcal{B}}) := \ker \mathrm{D_c} \cap \ker \mathbf{div}_c$, and $\mathcal{H}^1_{E2'}(\bar{\mathcal{B}}) := \ker \mathrm{D_{cc}} \cap \ker \mathbf{div}$. Using the Poincaré duality $\mathrm{H}^1_{dR}(\mathcal{B}) \approx \mathrm{H}^1_{dR_c}(\mathcal{B})$ for the 2-manifold $\mathcal{B}$, we can write the following 2D analogue of Theorem 26.

Theorem 33. *The Hilbert complexes (3.41) are Fredholm and*

$$\dim \mathsf{H}^1_{E2}(\bar{\mathcal{B}}) = \dim \mathsf{H}^1_{E2'}(\bar{\mathcal{B}}) = \dim \mathcal{H}^1_{E2}(\bar{\mathcal{B}}) = \dim \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}) = 3 \dim \mathrm{H}^1_{dR}(\bar{\mathcal{B}}).$$

These Hilbert complexes induce the following L^2-orthogonal decompositions: The Hodge decompositions

$$\begin{aligned} L^2\Gamma(S^2T\bar{\mathcal{B}}) &= \mathbf{grad}^s\left(H_s^{\mathbf{g}}\mathfrak{X}(\bar{\mathcal{B}})\right) \oplus \mathcal{H}^1_{E2}(\bar{\mathcal{B}}) \oplus \mathrm{D_s}\left(H_c^{\mathrm{D_s}}(\mathcal{B})\right), \\ L^2\Gamma(S^2T\bar{\mathcal{B}}) &= \mathbf{grad}^s\left(H_s^{\mathbf{g}}\mathfrak{X}_c(\mathcal{B})\right) \oplus \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}) \oplus \mathrm{D_s}\left(H^{\mathrm{D_s}}(\bar{\mathcal{B}})\right), \end{aligned}$$

and, equivalently, the Helmholtz decompositions

$$\begin{aligned} L^2\Gamma(S^2T\bar{\mathcal{B}}) &= \mathbf{grad}^s\big(H_s^{\mathbf{g}}\mathfrak{X}(\bar{\mathcal{B}})\big) \oplus \ker \mathbf{div}_c = \ker \mathrm{D_c} \oplus \mathrm{D_s}\big(H_c^{\mathrm{D_s}}(\mathcal{B})\big) \\ &= \mathbf{grad}^s\left(H_s^{\mathbf{g}}\mathfrak{X}_c(\mathcal{B})\right) \oplus \ker \mathbf{div} = \ker \mathrm{D_{cc}} \oplus \mathrm{D}_s\big(H^{\mathrm{D_s}}(\mathcal{B})\big), \end{aligned}$$

where

$$\begin{aligned} \ker \mathbf{div}_c &= \mathrm{D_s}\big(H_c^{\mathrm{D_s}}(\mathcal{B})\big) \oplus \mathcal{H}^1_{E2}(\bar{\mathcal{B}}), & \ker \mathrm{D_c} &= \mathbf{grad}^s\big(H_s^{\mathbf{g}}\mathfrak{X}(\bar{\mathcal{B}})\big) \oplus \mathcal{H}^1_{E2}(\bar{\mathcal{B}}), \\ \ker \mathbf{div} &= \mathrm{D_s}\big(H^{\mathrm{D_s}}(\bar{\mathcal{B}})\big) \oplus \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}), & \ker \mathrm{D_{cc}} &= \mathbf{grad}^s\left(H_s^{\mathbf{g}}\mathfrak{X}_c(\mathcal{B})\right) \oplus \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}). \end{aligned}$$

Corollary 34. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^2$ be a smooth, compact 2-manifold with boundary. Then, $\boldsymbol{T} \in L^2\Gamma(S^2T\bar{\mathcal{B}})$ admits a $\mathbf{grad}^s$-potential if and only if*

$$\mathrm{D_c}\boldsymbol{T} = 0, \;\; \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0, \;\; \forall \boldsymbol{Q} \in \mathcal{H}^1_{E2}(\bar{\mathcal{B}}).$$

Moreover, $\boldsymbol{T}$ admits a $\mathrm{D_s}$-potential if and only if

$$\mathbf{div}\,\boldsymbol{T} = 0, \;\; \langle\!\langle \boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0, \;\; \forall \boldsymbol{Q} \in \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}).$$

Let

$$C_j^\infty(\bar{\mathcal{B}}) := \left\{ f \in C^\infty(\bar{\mathcal{B}}) : f \text{ and } \operatorname{grad} f \text{ vanish on } \partial_j \bar{\mathcal{B}} \right\},$$
$$\widetilde{\Gamma}_{\mathrm{n}_j}(S^2 T\bar{\mathcal{B}}) := \left\{ \boldsymbol{T} \in \Gamma(S^2 T\bar{\mathcal{B}}) : \mathbf{c}(\boldsymbol{T}), \overrightarrow{\boldsymbol{T}}_{\mathbf{E}_I} \perp \partial_j \bar{\mathcal{B}},\ I = 1, 2 \right\},$$

and suppose that the Hilbert spaces $H_j^{\mathrm{D_s}}(\bar{\mathcal{B}})$, and $H^{\mathrm{D_c}}\Gamma_{\mathrm{n}_j}(S^2 T\bar{\mathcal{B}})$ are the completions of $\left(C_j^\infty(\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathrm{D_s}}} \right)$ and $\left(\widetilde{\Gamma}_{\mathrm{n}_j}(S^2 T\bar{\mathcal{B}}), \langle\!\langle , \rangle\!\rangle_{H^{\mathrm{D_c}}} \right)$, respectively. The relation (3.40) suggests that the closed operator $\mathrm{D}_{\mathrm{s}_2}$ is the adjoint operator of $\mathrm{D}_{\mathrm{c}_1}$, where

$$\mathrm{D}_{\mathrm{c}_1} : L^2\Gamma(S^2 T\bar{\mathcal{B}}) \to L^2(\bar{\mathcal{B}}), \qquad \mathrm{D}(\mathrm{D}_{\mathrm{c}_1}) = H^{\mathrm{D_c}}\Gamma_{\mathrm{n}_j}(S^2 T\bar{\mathcal{B}}),$$
$$\mathrm{D}_{\mathrm{s}_2} : L^2(\bar{\mathcal{B}}) \to L^2\Gamma(S^2 T\bar{\mathcal{B}}), \qquad \mathrm{D}(\mathrm{D}_{\mathrm{s}_2}) = H_2^{\mathrm{D_s}}(\bar{\mathcal{B}}).$$

Thus, one concludes that:

Theorem 35. *The space $L^2\Gamma(S^2 T\bar{\mathcal{B}})$ on a compact 2-manifold with boundary $\bar{\mathcal{B}} \subset \mathbb{R}^2$ admits the following Helmholtz decompositions*

$$L^2\Gamma(S^2 T\bar{\mathcal{B}}) = \overline{\operatorname{im} \mathbf{grad}_1^s} \oplus \ker \mathbf{div}_2 = \ker \mathrm{D}_{\mathrm{c}_1} \oplus \overline{\operatorname{im} \mathrm{D}_{\mathrm{s}_2}},$$

where $\overline{\operatorname{im} \mathbf{grad}_1^s}$ and $\overline{\operatorname{im} \mathrm{D}_{\mathrm{s}_2}}$ are the closures of the images of $\mathbf{grad}_1^s$ and $\mathrm{D}_{\mathrm{s}_2}$ in $L^2\Gamma(S^2 T\bar{\mathcal{B}})$, respectively.

4 Applications in Nonlinear Elasticity

Second-order tensors have various applications in continuum mechanics. Let $\boldsymbol{U} \in \Gamma(T\varphi(\bar{\mathcal{B}}))$ be a displacement field on $\bar{\mathcal{B}}$. The two-point tensor $\boldsymbol{K} = \mathbf{Grad}\, \boldsymbol{U}$ is the displacement gradient associated to $\boldsymbol{U}$ and $\mathbf{Curl}^{\mathsf{T}} \boldsymbol{K} = 0$ is the necessary condition for compatibility of $\boldsymbol{K}$, i.e. the existence of a displacement field with displacement gradient $\boldsymbol{K}$. On the other hand, $\boldsymbol{P} \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$ can represent a first Piola-Kirchhoff stress tensor and consequently, $\mathbf{Div}\, \boldsymbol{P} = 0$, expresses the equilibrium equation and also the necessary condition for the existence of a stress function $\boldsymbol{\Psi} \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$ such that $\boldsymbol{P} = \mathbf{Curl}^{\mathsf{T}} \boldsymbol{\Psi}$. Thus, the Hilbert complex (3.22) describes both the kinematics and the kinetics of a motion φ (also see the discussions in [2, §3]). Similarly, the Hilbert complex (3.27) describes both the kinematics and the kinetics of a linearly elastic body.

In this section, we use the orthogonal decompositions introduced in the previous sections for formulating the compatibility equations (for linear and nonlinear strains) and for deriving the necessary and sufficient conditions for the existence of stress functions on non-contractible bodies. In particular, these decompositions allow one to study the effect of Dirichlet boundary conditions and their topological properties (i.e. topological properties of regions on which these boundary conditions are imposed) on the compatibility equations. Moreover, the compatibility equations written using the orthogonal decompositions are also valid for non-smooth L^2 strains such as those associated to deformations of multiphase materials.

4.1 The Compatibility Problems with Dirichlet Boundary Conditions

The Hodge decomposition for $\mathbb{R}^n$-valued one-forms can be used for writing the nonlinear compatibility equations in the presence of Dirichlet boundary conditions as follows. Any $C^{r,\mu}$-mapping $\varphi : \bar{\mathcal{B}} \to \mathbb{R}^n$ with $r \geq 0$ and $0 < \mu < 1$ on a body $\bar{\mathcal{B}}$ induces a $C^{r,\mu}$-displacement field $\boldsymbol{U}(X) := \varphi(X) - X$, $\forall X \in \bar{\mathcal{B}}$. Let $\boldsymbol{\Upsilon}$ and $\boldsymbol{\kappa}$ be $\mathbb{R}^3$-valued zero- and one-forms of classes $C^{r+1,\mu}$ and $C^{r,\mu}$, respectively, such that $\boldsymbol{\kappa} = \boldsymbol{d}\boldsymbol{\Upsilon}$. Then, $\boldsymbol{\Upsilon}$ induces a $C^{r+1,\mu}$-mapping $\varphi(X) = X + \boldsymbol{\Upsilon}(X)$, and $C^{r+1,\mu}$-displacement $\boldsymbol{U} := \boldsymbol{I}_0^{-1}(\boldsymbol{\Upsilon})$, with the displacement gradient $\mathbf{Grad}\,\boldsymbol{U} = \boldsymbol{I}_1^{-1}(\boldsymbol{\kappa})$. Clearly $\mathsf{t}\boldsymbol{\Upsilon}|_{\partial_1\bar{\mathcal{B}}} = 0$, if and only if $\boldsymbol{U}|_{\partial_1\bar{\mathcal{B}}} = 0$. Thus, by using Remarks 6 and 3, one obtains the following theorem.

Theorem 36. *Given an $\mathbb{R}^n$-valued one-form $\boldsymbol{\kappa}$ of class $C^{r,\mu}$ with $r \geq 0$ and $0 < \mu < 1$ on $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, there exists a $C^{r+1,\mu}$-displacement $\boldsymbol{U}$ with $\boldsymbol{U}|_{\partial_1\bar{\mathcal{B}}} = 0$, such that $\mathbf{Grad}\,\boldsymbol{U} = \boldsymbol{I}_1^{-1}(\boldsymbol{\kappa})$ (or $\mathbf{Grad}\,\boldsymbol{U} = \boldsymbol{J}_1^{-1}(\boldsymbol{\kappa})$, if $n = 2$) if and only if*

$$\mathsf{t}\boldsymbol{\kappa}|_{\partial_1\bar{\mathcal{B}}} = 0,\ \boldsymbol{d}\boldsymbol{\kappa} = 0,\ \langle\!\langle \boldsymbol{\kappa}, \boldsymbol{\chi} \rangle\!\rangle_{\boldsymbol{L}^2} = 0,\ \forall \boldsymbol{\chi} \in \boldsymbol{\mathcal{H}}^1_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}), \tag{4.1}$$

where $\dim \boldsymbol{\mathcal{H}}^1_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) = n \dim \mathrm{H}^1_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$.

Remark 37. By choosing $\partial_1\bar{\mathcal{B}} = \varnothing$, one obtains the nonlinear compatibility equations without boundary conditions. In this case and for sufficiently smooth strains, (4.1) is equivalent to the necessary and sufficient conditions discussed in [40, 2], also see Remark 7. However, unlike the integral conditions of these references, the inner-product condition $\langle\!\langle \boldsymbol{\kappa}, \boldsymbol{\chi} \rangle\!\rangle_{\boldsymbol{L}^2} = 0$ still makes sense for L^2-strains. Thus, the condition (4.1) is also useful for studying the compatibility of non-smooth strains such as those associated with multiphase materials. Also note that the above theorem does not guarantee the mapping φ associated to $\boldsymbol{\kappa}$ to be an embedding, cf. [2, Remark 15].

Remark 38. In [3, Theorem 7], the above theorem is extended to non-homogeneous boundary conditions. In [4, Theorem 4], the analogue of this theorem for multiphase bodies is derived. The main tools for obtaining these extensions are the standard Hodge-Morrey decomposition and an appropriate extension of the classical Friedrichs decomposition for harmonic fields, see [3, Theorem 1].

Example 39. Theorem 36 allows one to study the effect of topological properties of both $\bar{\mathcal{B}}$ and $\partial_1\bar{\mathcal{B}}$ on the nonlinear compatibility equations. For example, consider the body $\bar{\mathcal{B}}$ depicted in Fig. 1. Since $\bar{\mathcal{B}}$ is simply-connected, we have $\boldsymbol{\mathcal{H}}^1_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) = 0$, for $\partial_1\bar{\mathcal{B}} = \varnothing$, and (4.1) without boundary conditions simply reads $\boldsymbol{d}\boldsymbol{\kappa} = 0$. If we impose the Dirichlet boundary condition on $\partial_1\bar{\mathcal{B}} = \mathcal{C}_1$, then $\boldsymbol{\mathcal{H}}^1_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}}) = 0$, and (4.1) becomes $\mathsf{t}\boldsymbol{\kappa}|_{\partial_1\bar{\mathcal{B}}} = 0$, and $\boldsymbol{d}\boldsymbol{\kappa} = 0$. However, if $\partial_1\bar{\mathcal{B}} = \mathcal{C}_1 \cup \mathcal{C}_2$, or $\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$, then $\dim \boldsymbol{\mathcal{H}}^1_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$ will be 3 and 6, respectively. Thus, although $\bar{\mathcal{B}}$ is simply-connected, $\boldsymbol{d}\boldsymbol{\kappa} = 0$, is no longer a sufficient condition for the compatibility of $\boldsymbol{\kappa}$ subject to the given Dirichlet boundary conditions. For all the above cases, we have $\mathrm{H}^0_{dR}(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}}) = 0$, which implies that the displacement $\boldsymbol{U}$ associated to $\boldsymbol{\kappa}$ is unique.

The linear compatibility equations on non-simply-connected bodies are stated in Georgescu [22, Theorem 5.3] and Yavari [40, Proposition 2.8]. Alternatively, by using Corollaries 28 and 34, one can write the linear compatibility equations as follows.

Theorem 40. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, be a smooth, compact n-manifold with boundary. Let $n = 3$. A symmetric second-order L^2-tensor $\boldsymbol{e}$ is the linear strain induced by a displacement field $\boldsymbol{U}$ if and only if*

$$\mathbf{curl} \circ \mathbf{curl}\,\boldsymbol{e} = 0,\ \langle\!\langle \boldsymbol{e}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{Q} \in \mathcal{H}^1_{E3}(\bar{\mathcal{B}}),$$

where $\dim \mathcal{H}^1_{E3}(\bar{\mathcal{B}}) = 6 \dim \mathrm{H}^1_{dR}(\bar{\mathcal{B}})$. *For $n = 2$, the linear compatibility equations read*

$$\mathrm{D_c}\boldsymbol{e} = 0,\ \langle\!\langle \boldsymbol{e}, \boldsymbol{Q} \rangle\!\rangle_{L^2} = 0,\ \forall \boldsymbol{Q} \in \mathcal{H}^1_{E2}(\bar{\mathcal{B}}),$$

where $\dim \mathcal{H}^1_{E2}(\bar{\mathcal{B}}) = 3 \dim \mathrm{H}^1_{dR}(\bar{\mathcal{B}})$.

Remark 41. Theorem 5.3 of Georgescu [22] is equivalent to the above theorem. There, he gives alternative representations of the elements of $\mathcal{H}^1_{E2}(\bar{\mathcal{B}})$ and $\mathcal{H}^1_{E3}(\bar{\mathcal{B}})$ as the tensor product of Killing vector fields and harmonic vector fields that satisfy the appropriate boundary conditions. Note that the inner product conditions in the above theorem are also valid for L^2-strains.

By using Theorems 30 and 35, one obtains the linear compatibility equation with Dirichlet boundary condition $\boldsymbol{U}|_{\partial_1\bar{\mathcal{B}}} = 0$, in the following sense.

Theorem 42. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, be a smooth, compact n-manifold with boundary. Then, a symmetric second-order L^2-tensor $\boldsymbol{e}$ belongs to $\overline{\operatorname{im}\mathbf{grad}_1^s}$, i.e. the closure of $\operatorname{im}\mathbf{grad}_1^s$ in $L^2\Gamma(S^2T\bar{\mathcal{B}})$, if and only if*

$$\langle\!\langle \boldsymbol{e}, \boldsymbol{Q}\rangle\!\rangle_{L^2} = 0, \ \forall \boldsymbol{Q} \in \ker \mathbf{div}_2. \tag{4.2}$$

Remark 43. Roughly speaking, Theorem 42 says that a symmetric tensor $\boldsymbol{e}$ is the linear strain induced by a displacement $\boldsymbol{U}$ with $\boldsymbol{U}|_{\partial_1\bar{\mathcal{B}}} = 0$, if and only if the work of $\boldsymbol{e}$ and any virtual stress that is equilibrated and has zero traction on $\partial_2\bar{\mathcal{B}}$ vanishes. A similar result is proved by Dorn and Schild [18] on simply-connected bodies. They obtained a sufficient condition for the existence of a displacement field satisfying arbitrary boundary conditions on $\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$, that induces a given linear strain $\boldsymbol{e}$.

Remark 44. The linear compatibility equations derived by Ting [37, Theorem 3.1] correspond to the case $\partial_1\bar{\mathcal{B}} = \varnothing$ in the above theorem. Note that the condition (4.2) is not useful in practice as the space $\ker \mathbf{div}_2$ is infinite-dimensional. The generalization of Theorem 40 with non-homogenous boundary conditions is given in [3, Theorem 15].

4.2 Stress Functions

Next, we study the existence of stress functions for the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$, i.e. the existence of $\mathbf{S}$ and $\mathbf{Curl}^{\mathsf{T}}$-potentials for $\boldsymbol{P}$. Let $\partial_1\bar{\mathcal{B}} = \partial\bar{\mathcal{B}}$. For $n = 3$, let $\mathcal{H}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}}) := \mathcal{H}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$, and $H^{\mathbf{C}}\Gamma_{\mathrm{n}}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) := H^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$. Similarly, for $n = 2$, let $\bar{\mathcal{H}}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}}) := \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}_1,\mathrm{t}_2}(\bar{\mathcal{B}})$, and $\bar{H}^{\mathbf{C}}\Gamma_{\mathrm{n}}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) := \bar{H}^{\mathbf{C}}\Gamma_{\mathrm{n}_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$. Then, by using Corollaries 20 and 23, Remark 5, and the decompositions for $\ker \mathbf{Div}$ given in Theorems 19 and 22, one obtains the following results.

Theorem 45. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, be a smooth, compact n-manifold with boundary. A first Piola-Kirchhoff stress $\boldsymbol{P} \in L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ can be written as $\boldsymbol{P} = \mathbf{Curl}^{\mathsf{T}}\boldsymbol{\Psi}$ ($\boldsymbol{P} = \mathbf{S}(\boldsymbol{W})$, if $n = 2$), if and only if*

$$\mathbf{Div}\,\boldsymbol{P} = 0, \ \langle\!\langle \boldsymbol{P}, \boldsymbol{K}\rangle\!\rangle_{L^2} = 0, \ \forall \boldsymbol{K} \in \mathcal{H}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}}) \ (\forall \boldsymbol{K} \in \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}}), \ \textit{if } n = 2),$$

where $\dim \mathcal{H}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}}) = 3 \dim \mathrm{H}^2_{dR}(\bar{\mathcal{B}})$, and $\dim \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}}) = 2 \dim \mathrm{H}^1_{dR}(\bar{\mathcal{B}})$. Moreover, potential $\boldsymbol{\Psi}$ ($\boldsymbol{W}$) can be uniquely chosen if we further assume that

$$\boldsymbol{\Psi} \in \mathbf{Curl}^{\mathsf{T}}\left(H^{\mathbf{C}}\Gamma_{\mathrm{n}}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})\right) \ \left(\boldsymbol{W} \in \mathbf{C}\left(\bar{H}^{\mathbf{C}}\Gamma_{\mathrm{n}}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})\right)\right).$$

In general, any divergence-free first Piola-Kirchhoff stress $\boldsymbol{P}$ can be uniquely decomposed as $\boldsymbol{P} = \widetilde{\boldsymbol{P}} + \boldsymbol{P}_{\mathcal{H}}$, where $\widetilde{\boldsymbol{P}}$ admits a $\mathbf{Curl}^{\mathsf{T}}$-potential ($\mathbf{S}$-potential if $n = 2$) and $\boldsymbol{P}_{\mathcal{H}} \in \mathcal{H}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}})$ ($\boldsymbol{P}_{\mathcal{H}} \in \bar{\mathcal{H}}^{\varphi}_{\mathrm{n}}(\bar{\mathcal{B}})$, if $n = 2$).

By using operators $\mathbf{curl}^{\mathsf{T}}$ and $\mathbf{s}$, one can write the analogue of the above theorem for the Cauchy stress tensor $\boldsymbol{\sigma}$ as well. Since $\boldsymbol{\sigma}$ is usually symmetric, one can also define $\mathbf{curl} \circ \mathbf{curl}$ and $\mathrm{D_s}$-potentials for it, which are called Beltrami and Airy stress functions, respectively [38, 27]. In particular, Corollaries 28 and 34, and Theorems 26 and 33 allow us to write the following theorem.

Theorem 46. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^n$, $n = 2, 3$, be a smooth, compact n-manifold with boundary. A Cauchy stress $\boldsymbol{\sigma} \in L^2\Gamma(S^2T\bar{\mathcal{B}})$ can be written as $\boldsymbol{\sigma} = \mathbf{curl}\circ\mathbf{curl}\,\boldsymbol{\Phi}$ ($\boldsymbol{\sigma} = \mathrm{D_s}(f)$, if $n = 2$), if and only if*

$$\mathbf{div}\,\boldsymbol{\sigma} = 0, \ \langle\!\langle \boldsymbol{\sigma}, \boldsymbol{Q}\rangle\!\rangle_{L^2} = 0, \ \forall \boldsymbol{Q} \in \mathcal{H}^2_{E3}(\bar{\mathcal{B}}) \ (\forall \boldsymbol{Q} \in \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}), \ \textit{if } n = 2),$$

where $\dim \mathcal{H}^2_{E3}(\bar{\mathcal{B}}) = 6 \dim \mathrm{H}^2_{dR}(\bar{\mathcal{B}})$*, and* $\dim \mathcal{H}^1_{E2'}(\bar{\mathcal{B}}) = 3 \dim \mathrm{H}^1_{dR}(\bar{\mathcal{B}})$*. A Beltrami stress function* $\boldsymbol{\Phi}$ *(Airy stress function* f*) can be uniquely chosen if we also assume that* $\boldsymbol{\Phi} \in \mathbf{curl} \circ \mathbf{curl}\left(H^{\mathbf{cc}}\Gamma_c(S^2T\mathcal{B})\right)$ $\left(f \in \mathrm{D_c}\left(H^{\mathrm{D_c}}\Gamma_c(S^2T\mathcal{B})\right)\right)$*. In general, any divergence-free Cauchy stress* $\boldsymbol{\sigma}$ *can be uniquely decomposed as* $\boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_{\mathcal{H}}$*, where* $\widetilde{\boldsymbol{\sigma}}$ *admits a Beltrami stress function (Airy stress function if* $n = 2$*) and* $\boldsymbol{\sigma}_{\mathcal{H}} \in \mathcal{H}^2_{E3}(\bar{\mathcal{B}})$ *(*$\boldsymbol{\sigma}_{\mathcal{H}} \in \mathcal{H}^1_{E2'}(\bar{\mathcal{B}})$*, if* $n = 2$*).*

Remark 47. The decomposition for divergence-free Cauchy stresses in the above theorem is equivalent to the decomposition introduced by Gurtin [27, Theorem 4.4], see Remark 27. The necessary and sufficient condition for the existence of stress functions given in Georgescu [22, Theorem 5.2] is equivalent to the condition of the above theorem. He gives an alternative representation of the elements of $\mathcal{H}^1_{E2'}(\bar{\mathcal{B}})$ and $\mathcal{H}^2_{E3}(\bar{\mathcal{B}})$ as the tensor product of Killing vector fields and harmonic vector fields.

Remark 48. Airy and Beltrami stress functions are not unique, in general. Wang and Rutqvist [39] derived an expression for Beltrami stress functions that explains their degree of non-uniqueness. In the above theorem, we give additional conditions that allow one to uniquely choose Airy and Beltrami stress functions.

5 Applications in Computational Mechanics

In this final section, we briefly discuss an important application of the Hilbert complexes obtained in this paper for deriving a new class of numerical schemes suitable for large-strain deformations of solids. These numerical schemes are called *compatible-strain mixed finite element methods* (CSFEM) and were first introduced in [5].

The main tool for deriving CSFEMs is the finite element exterior calculus (FEEC) discussed in [6, 8]. FEEC allows one to discretize the Hilbert complex associated to the de Rham complex. These discrete complexes have the same cohomology groups as the de Rham complex and their underlying spaces are finite element spaces that can be efficiently implemented in numerical schemes. The commutative diagrams (3.3), (3.13), and (3.14) enable one to discretize the Hilbert complexes (3.7), (3.16), and (3.17) by using FEEC. As we mentioned earlier, these Hilbert complexes describe the kinematics and the kinetics of large deformations of solids. Thus, we can use FEEC to obtain trial spaces for strain and stress.

CSFEMs are derived by considering a mixed formulation for nonlinear elasticity in terms of the displacement, the displacement gradient, and the first Piola-Kirchhoff stress. This mixed formulation is then discretized using the above finite element spaces for strain and stress. The main feature of CSFEMs is that by construction, the trial spaces for strain satisfy the compatibility condition. In [5], by considering several benchmark problems, it is shown that CSFEMs have optimal convergence rates and have good convergence on domains with complex geometries. Moreover, CSFEMs can accurately approximate stress and do not suffer from numerical instabilities such as locking and hourglass-type instabilities.

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