

### Homework Due Friday, 10 October

This set of homework relies on a few elementary facts of the algebra of vectors and tensors. If you are vague about these facts, see some old notes I wrote when I taught ES 240 in 2006: <http://www.imechanica.org/node/205/revisions/1385/view>

#### 11. Positive-definite elastic energy density

For many materials the elastic energy density may be taken as a positive-definite quadratic form of the strain tensor. That is,  $w > 0$  for any state of strain, except that  $w = 0$  when the strain tensor vanishes. For an isotropic, linearly elastic solid, confirm that this positive-definite requirement is equivalent to require that  $E > 0$  and  $-1 < \nu < 1/2$ .

#### 12. The coefficient of thermal expansion (CTE) is a second-rank tensor.

When an anisotropic solid is subject to a change in temperature, all components of the strain tensor can change. Consequently, the coefficient of thermal expansion (CTE) is a second-rank tensor. For a crystal of cubic symmetry, show that the CTE is the same in all directions.

#### 13. Hooke's law for anisotropic, linearly elastic solids

Hooke's law connecting stresses  $\sigma_{ij}$  and strains  $\varepsilon_{ij}$  for a generally anisotropic solid can be written in one of the following forms

$$\begin{aligned}\varepsilon_{ij} &= S_{ijpq} \sigma_{pq}, & \sigma_{ij} &= C_{ijpq} \varepsilon_{pq} \\ \varepsilon_i &= S_{ip} \sigma_p, & \sigma_i &= c_{ip} \varepsilon_p\end{aligned}$$

The usual correspondence is adopted, i.e.

$$\begin{aligned}\varepsilon_1 &= \varepsilon_{xx}, \varepsilon_2 = \varepsilon_{yy}, \varepsilon_3 = \varepsilon_{zz}, \varepsilon_4 = 2\varepsilon_{yz}, \varepsilon_5 = 2\varepsilon_{zx}, \varepsilon_6 = 2\varepsilon_{xy} \\ \sigma_1 &= \sigma_{xx}, \sigma_2 = \sigma_{yy}, \sigma_3 = \sigma_{zz}, \sigma_4 = \sigma_{yz}, \sigma_5 = \sigma_{zx}, \sigma_6 = \sigma_{xy}\end{aligned}$$

The fourth-order tensors  $\mathbf{S}$  and  $\mathbf{C}$  are the compliance and stiffness tensors, respectively. The 6 by 6 matrices  $\mathbf{s}$  and  $\mathbf{c}$  ( $\mathbf{s} = \mathbf{c}^{-1}$ ) are conventional compliance and stiffness matrices.

i) Confirm that

$$\begin{aligned}c_{11} &= C_{1111}, & c_{14} &= C_{1123}, & c_{44} &= C_{2323} \\ s_{11} &= S_{1111}, & s_{14} &= 2S_{1123}, & s_{44} &= 4S_{2323}\end{aligned}$$

ii) Let  $\mathbf{m} = m_i \mathbf{e}_i$  be a unit vector. Show that when the solid is under a uniaxial stress in the direction  $\mathbf{m}$ , Young's modulus in this direction is given by

$$\frac{1}{E(\mathbf{m})} = S_{ijpq} m_i m_j m_p m_q$$

iii) For a cubic solid show the Young's modulus in direction  $\mathbf{m}$  is given by

$$\frac{1}{E(\mathbf{m})} = (m_1^4 + m_2^4 + m_3^4) \left( s_{11} - s_{12} - \frac{1}{2} s_{44} \right) + \left( s_{12} + \frac{1}{2} s_{44} \right)$$

iv) Find the orientations for a cubic solid which attain minimum or maximum Young's modulus.

#### 14. Invariants of a tensor

When the basis changes, the components of a vector change, but the length of the vector is invariant. Let  $\mathbf{f}$  be a vector, and  $f_i$  be the components of the vector for a given basis. The length of the vector is the square root of

$$f_i f_i.$$

The index  $i$  is dummy. Thus, this combination of the components of a vector is a scalar, which is invariant under any change of basis. For a vector, there is only one independent invariant. Any other invariant of the vector is a function of the length of the vector.

This observation can be extended to high-order tensors. By definition, **an invariant of a tensor is a scalar formed by a combination of the components of the tensor**. For example, for a symmetric second-rank tensor  $\sigma_{ij}$ , we can form three independent invariants:

$$\sigma_{ii}, \quad \sigma_{ij}\sigma_{ij}, \quad \sigma_{ij}\sigma_{jk}\sigma_{ki}.$$

In each case, all indices are dummy, resulting in a scalar. Any other invariant of the tensor is a function of the above three invariants.

Actions:

- (a) For a nonsymmetric second-rank tensor, give all the independent invariants. Write each invariant using the summation convention, and then write it explicitly in all its terms.
- (b) Give all the independent invariants of a third-rank tensor. Use the summation convention.

#### 15. A “derivation” of the Mises (1913) yield criterion

Given a metal, we determine its yield strength,  $Y$ , by subjecting the metal to a tensile force, and recording the tensile stress at which the material starts to deform plastically. We can also subject the same material to a shear stress, and record the shear stress at which the material starts to deform plastically. In this experiment, we measure the yield strength in shear. Of course, we can subject the metal to any other types of stress state, say biaxial tension, and record yet another yield strength. Clearly, for the given metal, the yield strength is specific to the type of the stress state. It is impractical to determine the yield condition experimentally for all types of stress state. How do we obtain the yield condition under a given stress state?

Mises provided the following answer. The Mises criterion is not an exact law of nature. Rather, it is a compromise between obtaining the truth and the amount of work you need to do. It is comforting that the Mises criterion can be derived from a few reasonable assumptions.

*Isotropy.* For an isotropic material, the stresses can affect the yield condition only through the invariants:

$$\begin{aligned} A &= \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \\ B &= \sigma_{ij}\sigma_{ij} = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}\sigma_{12} + 2\sigma_{13}\sigma_{13} + 2\sigma_{23}\sigma_{23} \\ C &= \sigma_{ij}\sigma_{jk}\sigma_{ki}. \end{aligned}$$

*No Bauschinger effect.* If the stress state  $\sigma_{ij}$  causes yielding, so does  $-\sigma_{ij}$ . We would like our yield condition to be quadratic in stress components. This is not necessary in order to avoid the Bauschinger effect, but is a simple way to do so. Any quadratic invariant can be expressed as a linear combination of  $A^2$  and  $B$ . Thus, the yield condition is

$$\sigma_{ij}\sigma_{ij} + \alpha(\sigma_{pp})^2 = \text{constant}$$

where  $\alpha$  is a number.

*Hydrostatic stress does not affect yielding.* Experiments by Percy W. Bridgeman and others have long shown that superimposing any hydrostatic stress  $q$  does not affect yielding. Thus, if the stress state  $\sigma_{ij}$  causes yield, so does  $\sigma_{ij} + q\delta_{ij}$ .

Actions:

(a) Show that  $\alpha = -1/3$ . Consequently, the yield condition is

$$\sigma_{ij}\sigma_{ij} - \frac{1}{3}(\sigma_{pp})^2 = \text{constant}.$$

This is the same as the Mises condition.

(b) Determine the constant in terms of the yield strength  $Y$  measured under the uniaxial tension.