

Homework
Due Friday, 26 September

1. Nothing Is Continuum, but...

A pencil is not a continuum: it is a composite of wood, graphite, metal and rubber. Even a part, the metal for example, has its own structure when you look closely. It consists of grains, i.e., tiny crystals in different orientations, each being perhaps about 10^{-5} meter across. Further down, each tiny crystal has many levels of structures—dislocations, atoms, electrons, nucleus...

Every single thing has many levels of *structures*. Why should we even bother with the concept of a continuum? Because it is a useful *idealization*. When Newton analyzed the motion of the earth around the sun, he idealized the earth as a rigid body, and ignored the violent motion inside the earth, because the latter has little effect on the motion of the earth around the sun. Of course, this idealization becomes useless in thinking about tectonic plate motion.

A set of quantities is used to describe the structure at each level. Quantities of a low-level structure relate to quantities of a high-level structure. Consider, for example, the elasticity of a metal. We can measure Young's modulus by bending a rod. Elasticity is clearly related to how much the electron cloud changes shape in response to the load. If we think of the electron cloud, i.e., the interatomic bond, as a spring, Young's modulus must relate to the "atomic spring" constant.

Actions:

- a) Make the connection between Young's modulus and the atomic spring constant. Assume that the atoms crystallize to a cubic lattice. Consider the following quantities are known: the mass density and interatomic spacing.
- b) Estimate the frequency of atomic vibration using representative orders of magnitude for Young's modulus, density, and interatomic spacing.

2. Hooke's law in various forms

Start with one form of Hooke's law

$$\varepsilon_{11} = \frac{1}{E} (\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}), \dots, \varepsilon_{12} = \frac{\sigma_{12}}{2\mu}, \dots$$

(a) Prove that the shear modulus relates to Young's modulus and Poisson's ratio as

$$\mu = \frac{E}{2(1 + \nu)}.$$

Thus, Hooke's law can be expressed as

$$\varepsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$$

(b) Show the inverse relations, i.e., the stresses in terms of strains, are

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}.$$

where μ and λ are called Lamé's constants. Confirm that

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

(c) Bulk modulus B is defined as the hydrostatic pressure needed to cause per unit volume strain, i.e.,

$$p = -B \frac{\Delta V}{V}.$$

Show that the bulk modulus relates to Young's modulus and Poisson's ratio as

$$B = \frac{E}{3(1-2\nu)}.$$

3. Compatibility: the strain-displacement relations

When you are given the displacement field, you can always find the strain field by taking derivatives. Here is a problem in another direction. The strain field is known as follows.

$$\varepsilon_{xz} = Ay, \quad \varepsilon_{yz} = Bx, \quad \varepsilon_{zz} = C$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = 0$$

where A , B and C are known constants. Determine all possible displacement field compatible with the strain field.

4. Traction vector on a plane

The stress state at a material particle is described by the stress components

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -3 \\ 5 & -3 & 4 \end{bmatrix}.$$

A plane passes three points $(0,0,0)$, $(1,1,1)$ and $(1,2,3)$. This plane also passes the material particle under consideration.

- Compute the traction vector on the plane.
- Compute the magnitude of the normal stress on the plane.
- Compute the *magnitude* of the shearing stress on the plane.
- Compute the *direction* of the shearing stress on the plane.

5. Turbine blade: centrifugal force and creep

Turbine blades in aircraft engines offer a good story of advanced technologies. Here we focus on a key mechanics-materials issue. The blades rotate as a high speed at a high temperature. The centrifugal force causes tensile stress in the blades. All metals, including the nickel-based superalloys used today, creep under stress at high temperature. The creep strain rate of metals is a nonlinear function of stress, often fit to a power-law:

$$\dot{\varepsilon} = B\sigma^n$$

where B and n are constants best fit the experimental data. A main concern is that the tensile stress in a blade will cause it to elongate to an unacceptable length.

- Model a blade as a rod, fixed at one end, and rotating at a uniform speed. Derive the differential equation that governs the stress field in the rod.
- Derive the expression of the stress in the rod.
- Derive an expression of the elongation of the rod as a function of time.