

Homework
Due Friday, 31 October

26. Stress-strain relations under the plane strain conditions

Starting from a familiar form of Hooke's law in three dimensions, namely,

$$\varepsilon_x = (\sigma_x - \nu\sigma_y - \nu\sigma_z) / E, \quad \gamma_{xy} = 2(1 + \nu)\sigma_{xy} / E, \text{ etc.},$$

show that under the plane strain conditions the stress-strain relation takes the following form:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & 0.5 - \nu \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}.$$

27. Getting weak: derive weak statements from differential equations

Consider heat conduction in a three dimensional body. First recall the governing equations. Let $T(x, y, z, t)$ be the temperature field, and $\mathbf{J}(x, y, z, t)$ be the vector field of heat flux (i.e., energy across unit area per unit time).

Material model. Fourier's law relates the heat flux to the temperature gradient:

$$\begin{aligned} J_x &= -k \frac{\partial T}{\partial x} \\ J_y &= -k \frac{\partial T}{\partial y} \\ J_z &= -k \frac{\partial T}{\partial z} \end{aligned}$$

where k is the heat conductivity.

Conservation of Energy. Let ρ be the mass density, and c be the heat capacity (i.e., the energy needed to increase the temperature per unit mass per degree). For a unit volume of the body to change temperature by dT , the energy needed is $c\rho dT$. Energy balance requires that

$$c\rho \frac{\partial T}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

in the volume of the body.

The body exchanges heat with the ambient by convection. Let h be the convection coefficient (energy per area per time per degree). Conservation of energy requires that

$$n_x J_x + n_y J_y + n_z J_z = h(T - T_a)$$

on the surface of the body. Here (n_x, n_y, n_z) is the unit vector normal to the surface, and T_a is the ambient temperature.

Action items. (a) Start with a weighted residual and derive the equivalent statement. Let ϕ be an arbitrary scalar field (i.e., a test function). Show that the energy balance equations are equivalent to requiring that the equation

$$\int \left(c\rho \frac{\partial T}{\partial t} \phi - J_x \frac{\partial \phi}{\partial x} - J_y \frac{\partial \phi}{\partial y} - J_z \frac{\partial \phi}{\partial z} \right) dV + \int h(T - T_a) \phi dA = 0,$$

hold true for every field ϕ . The first integral is over the volume of the body. The second integral is over the surface of the body. (b) Outline a finite element method for heat conduction.

28. Potential energy and the Rayleigh-Ritz Method

Here is yet another way to do pretty much the same thing. Applying a constant external force to a body is equivalent to hanging a weight. For example, consider the structure in the figure. Assume the rope that hangs the weight is inextensible. When the node moves by displacement u , the weight P drops by the same distance, so that the potential energy of the weight changes by $-Pu$. Now regard the body and the weights as a single system. The potential energy of this system, Π , is the sum of the elastic energy in the body and the potential energy of the weights. We will call the potential energy due to the fixed load the **work potential**.

Consider a three-dimension elastic body, subject to the body force \mathbf{b} in the volume, and the traction \mathbf{t} on one part of the surface of the body, S_t . On the other part of the surface, S_u , the displacement is prescribed. The work potential is

$$WP = -\int b_i u_i dV - \int t_i u_i dA.$$

The first integral extends over the volume of the body. The second integral extends over the part of the surface S_t , where the traction is prescribed. Note that \mathbf{b} and \mathbf{t} are known external forces. The displacement field, \mathbf{u} , is unknown, except for its prescribed values on the part of the surface of the body, S_u . For *any* displacement field \mathbf{u} , one can calculate the work potential. A relation that maps a field to a number is known as a **functional**. The work potential is a functional of the displacement field.

The elastic energy in the body is

$$U = \frac{1}{2} \int C_{ijpq} \frac{\partial u_i}{\partial x_j} \frac{\partial u_p}{\partial x_q} dV.$$

The integral extends over the volume of the body. The elastic energy is also a functional of the displacement field.

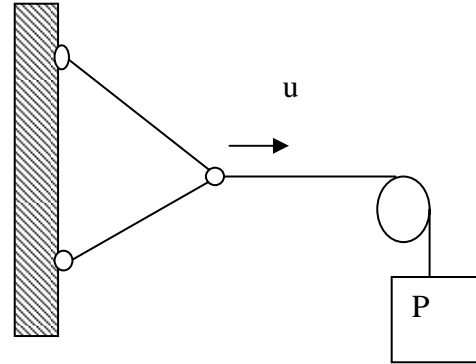
Potential energy is a functional of displacement. By definition, the potential energy is the sum of the elastic energy and the work potential:

$$\Pi(\mathbf{u}) = \frac{1}{2} \int C_{ijpq} \frac{\partial u_i}{\partial x_j} \frac{\partial u_p}{\partial x_q} dV - \int b_i u_i dV - \int t_i u_i dA.$$

The potential energy is a functional of the displacement field. The body force is prescribed over the volume of the body, and the traction is prescribed on the surface S_t . The displacement need not be the actual displacement field occurring in the body, but can be any field that satisfies the prescribed displacement on the surface S_u . The first two integral extends over the volume of the body. The third integral extends over the area S_t .

Principle of minimum potential energy. The principle of minimum potential energy states that, *of all displacement fields that satisfy the prescribed values on the surface of a body, the displacement field corresponding to equilibrium minimizes the potential energy.*

Proof. Let \mathbf{u} be the displacement field corresponding to equilibrium, and \mathbf{h} be a variation in the displacement field. The variation is arbitrary, except that $\mathbf{h} = \mathbf{0}$ on S_u . For the virtual displacement $\mathbf{u} + \mathbf{h}$, the strain is calculated by inserting $\mathbf{u} + \mathbf{h}$ in the usual displacement relation. A direct calculation shows that



$$\begin{aligned}\Pi(\mathbf{u} + \mathbf{h}) - \Pi(\mathbf{u}) &= \frac{1}{2} \int \left\{ c_{ijpq} \frac{\partial(\mathbf{u}_i + \mathbf{h}_i)}{\partial x_j} \frac{\partial(\mathbf{u}_p + \mathbf{h}_p)}{\partial x_q} - c_{ijpq} \frac{\partial \mathbf{u}_i}{\partial x_j} \frac{\partial \mathbf{u}_p}{\partial x_q} \right\} dV - \int b_i h_i dV - \int t_i h_i dA \\ &= \frac{1}{2} \int c_{ijpq} \frac{\partial h_i}{\partial x_j} \frac{\partial h_p}{\partial x_q} dV + \int c_{ijpq} \frac{\partial \mathbf{u}_p}{\partial x_q} \frac{\partial h_i}{\partial x_j} dV - \int b_i h_i dV - \int t_i h_i dA\end{aligned}$$

Because \mathbf{u} is the equilibrium displacement, $\sigma_{ij} = c_{ijpq} \partial u_p / \partial x_q$. According to the weak statement, the last three terms cancel each other. Consequently,

$$\Pi(\mathbf{u} + \mathbf{h}) - \Pi(\mathbf{u}) = \frac{1}{2} \int c_{ijpq} \frac{\partial h_i}{\partial x_j} \frac{\partial h_p}{\partial x_q} dV.$$

This difference is always positive unless \mathbf{h} vanishes everywhere in the body.

Rayleigh-Ritz Method. The principle of minimum potential energy calls for a search of the winner among *all* displacement fields that satisfy the displacement boundary conditions. Now if we limit the scope of the search to a *subset* of the admissible displacement fields, we will not find the real winner, i.e., the actual displacement field, but an approximate. Of the displacement fields in the subset, the displacement field that we select minimizes the potential energy.

Here is the Rayleigh-Ritz method that implements the idea. For simplicity, we assume that the boundary condition that the displacements vanish on S_u . Let $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n$ be a set of known virtual displacement fields. A linear combination is also a virtual displacement field:

$$\mathbf{u} = a_1 \mathbf{h}_1 + a_2 \mathbf{h}_2 + \dots + a_n \mathbf{h}_n,$$

where a_1, a_2, \dots, a_n are arbitrary numbers. This gives us a family of functions. In the jargon of linear algebra, we say that $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n$ is a set of bases, and all their linear combinations form a space. This is a subspace in that there are other functions that cannot be represented this way. Inserting this expression of \mathbf{u} into the potential energy, we express the potential energy as a function of a_1, a_2, \dots, a_n . We will select the coefficients a_1, a_2, \dots, a_n to minimize the potential energy.

Action items. An infinite body has a spherical cavity, radius R and subject to an internal pressure p .

(a) Express the potential energy of the body as a functional of the radial displacement field $u(r)$. (b) Functions like r^{-1}, r^{-2}, \dots are all virtual displacement fields. Assume a virtual displacement field

$$u = \frac{A}{r^2}.$$

Use the Rayleigh-Ritz method to determine A .

(c) Calculate the corresponding stress field. Compare with the exact solution to the problem. Explain your findings.

29. Constant strain triangle

We want to formulate the finite element method to solve a plane elasticity problem. Divide the body in the (x, y) plane into triangular elements.

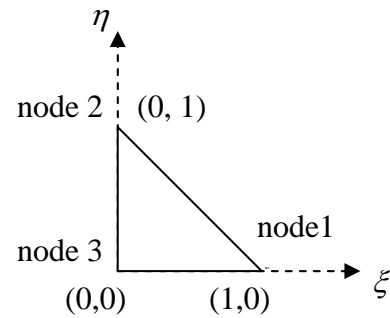
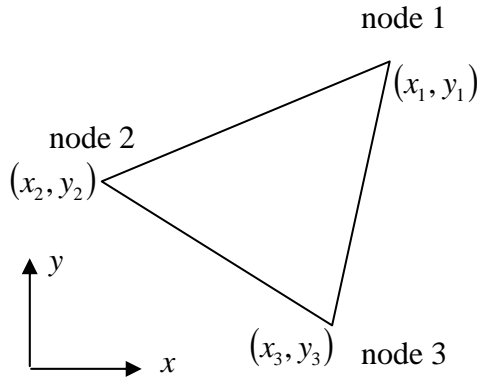
(a) Shape functions. Consider one triangle. Label its three nodes, *counterclockwise*, as 1, 2, 3. The three nodes have coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Map a triangle on a different plane, (ξ, η) , to the triangle in the (x, y) plane. The following functions map a point in the (ξ, η) plane to a point in the (x, y) plane:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

Show that the shape functions are

$$N_1 = \xi, \quad N_2 = \eta, \quad N_3 = 1 - \xi - \eta.$$



(b) The Jacobian matrix. Calculate the Jacobian matrix of the transformation, namely,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}.$$

Also calculate $\det \mathbf{J}$ and \mathbf{J}^{-1} .

(c) Displacement field and its gradient. Let the displacements at the three nodes be (u_1, v_1) , (u_2, v_2) , (u_3, v_3) . Interpolate the displacement (u, v) of a point inside the triangle as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Calculate the displacement gradients in the (x, y) plane: $\partial u / \partial x, \partial u / \partial y, \partial v / \partial x, \partial v / \partial y$.

(d) Strain field. The strain column of a point in the element is linear in the nodal displacement column. We write

$$\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{q}.$$

Work out the entries to the matrix \mathbf{B} . You will find the strain field in this element is constant. This element is known as the constant-strain triangle.

(e) Stress field. The stress column is also linear in the displacement column:

$$\boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon}.$$

Give the stiffness matrix \mathbf{D} .

(f) Stiffness matrix. Calculate the stiffness matrix of the element.

(g) Force column. Calculate the force column of the element.

30. Gaussian quadrature

- (a) Determine the points and weights of the three-point Gaussian quadrature. Compare your results with the values listed in a reference. Please list your reference.
- (b) Use the three-point Gaussian quadrature to evaluate the following integral

$$\int_0^{\pi} \sin x dx.$$

Compare with the exact result.