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Improved Muller approximate solution of the pull-off of a sphere from a viscoelastic substrate

M. Ciavarella

Politecnico di BARI. DMMM Department, Bari, Italy

ABSTRACT

The detachment of a sphere from a viscoelastic substrate is clearly a fundamental problem. In the case viscoelastic dissipation is concentrated at the contact edge, and the work of adhesion follows a quite popular simplified model, Muller has suggested an approximate solution, which however is based on an empirical observation. We revisit Muller's solution and show it leads to very poor fitting of the actual full numerical results, particularly for the radius of contact at pull-off, and we suggest an improved fitting of the pull-off which works extremely well over a very wide range of withdrawing speeds, and correctly converges to the JKR value at very low speeds.

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Viscoelasticity; adhesion; JKR theory; viscoelastic material

1. Introduction

The problem of viscoelastic dissipation during crack growth or contact peeling has attracted much interest due to its fundamental importance in many areas of science and technology. Many authors have applied fracture mechanics concepts and made extensive measurements [1–8] postulating peeling involves an effective work of adhesion w as the product of adiabatic value w_0 and a function of velocity of peeling of the contact/crack line and temperature, namely

$$w = w_0\left[1 + k\left(a_T v_p\right)^n\right] \tag{1}$$

where k, n are constants of the material, with n in the range 0.1–0.8 and a_T is the WLF factor [9] which permits to translate results at various temperatures T from measurement at a certain standard temperature. The details of the derivation from crack models involving cohesive Barenblatt zones or models 'truncating' or 'blunting' crack tip dissipation [2,6,8] vary, but the form (1) remains the most popular simple choice, and therefore a baseline for comprehension of possible mechanics of contact and crack problems.

In the case of adhesive contact of the fundamental spherical geometry, various authors [3,6,10] have attempted to apply the fracture mechanics formulation with the model (1), and some approximate results have been given in terms of explicit

CONTACT M. Ciavarella Mciava@poliba.it Politecnico di BARI. DMMM department, Bari 70016, Italy

dependences of the pull off force or work, contact radius and approach at pull-off see ref. [10], which we shall revisit here in comparison with full numerical simulation, finding very significant discrepancies, and suggesting some improved fitting of the numerical results, at least for the pull-off force which is the quantity of greater interest.

2. Spherical contact mechanics theory

The fracture mechanics formulation for the adhesive contact problem for a sphere is classic, and we shall revisit here only the essentials. We consider the stress intensity factor at the contact edge is due to the difference between P_1, the load required to maintain a contact radius a in the absence of adhesion

$$P_1(a) = \frac{4}{3}\frac{E^*}{R}a^3 \tag{2}$$

where $E^* = E/(1-\nu^2)$ is the plane strain elastic modulus (E being Young's modulus and ν Poisson's ratio) and P, the smaller load to maintain the same contact radius in the presence of adhesion. So we find the strain energy release rate as[1]

$$G(a,P) = \frac{K(a,P)^2}{2E^*} = \frac{(P_1(a)-P)^2}{8\pi E^* a^3} \tag{3}$$

In the adhesionless conditions, the remote approach is $\alpha_1(a) = \frac{a^2}{R}$, so in the adhesive condition we have to decrease this by an amount given by a flat punch displacement $\Delta\alpha = \frac{P_1 - P}{2E^* a}$ (since in moving from the adhesionless to the adhesive solution we keep the contact area constant) giving the general result for approach

$$\alpha(a,P) = \frac{a^2}{R} - \frac{P_1(a) - P}{2E^* a} \tag{4}$$

from which we can obtain $P(a,\alpha)$ using (2)

$$P(a,\alpha) = P_1(a) + 2E^* a\alpha(a,P) - 2E^*\frac{a^3}{R}$$

$$= \frac{2E^* a}{R}\left(R\alpha(a,P) - \frac{a^2}{3}\right) \tag{5}$$

which corresponds to Muller [10] Equation (10), whereas using (3)

$$G(a) = \frac{(P_1(a)-P)^2}{8\pi E^* a^3} = \frac{E^*}{2\pi a R^2}\left(R\alpha(a) - a^2\right)^2 \tag{6}$$

which corresponds to Muller [10] Equation 15 except for a factor 2 misprint. For the elastic case, JKR[11] theory is obtained by using (6) and (4)

$$P = \frac{4}{3}\frac{E^*}{R}a^3 - \sqrt{8\pi w_0 E^* a^3} \tag{7}$$

Putting

$$\zeta = \left(\frac{\pi w_0}{6RE^*}\right)^{1/3} \tag{8}$$

we have at $P = 0$ from (7) and (5)

$$a_0 = \left(\frac{9}{2}\pi R^2 \frac{w_0}{E^*}\right)^{1/3} = 3R\zeta \tag{9}$$

$$\alpha_0 = \frac{a_0^2}{3R} = 3R\zeta^2 \tag{10}$$

where there is a factor 3 misprint in Muller [10] Equation 19.

3. Viscoelasticity

Now, for a viscoelastic material, the material dissipation at the crack tip/contact edge requires that energy balance imposes the velocity of crack according to (1). Further, we can write the velocity of the contact edge as

$$v_p = -\frac{da}{dt} = v\frac{da}{d\alpha} \tag{11}$$

where v is the remote pull-off rate imposed by the loading equipment. The condition $G(a) = w$ therefore defines a differential equation for $a = a(\alpha)$ obtained using (6, 11)

$$\frac{1}{k^{1/n}a_T v}\left[\frac{E^*}{2\pi a R^2 w_0}\left(R\alpha(a) - a^2\right)^2 - 1\right]^{1/n} = \frac{da}{d\alpha} \tag{12}$$

By using the JKR values at zero load (9, 10) and the JKR values for pull-off for $P_0 = \frac{3}{2}\pi R w_0$, and finally the adiabatic work of adhesion for G, we obtain the dimensionless variables

$$G' = \frac{G}{w_0};\; P' = \frac{P}{P_0};\; a' = \frac{a}{a_0};\; \alpha' = \frac{\alpha}{\alpha_0} \tag{13}$$

If we now remove the (') for simplicity in the following equations, we rewrite (12) as

$$\frac{da}{d\alpha} = \beta^{-1}\left[a^3\left(\frac{\alpha}{3a^2} - 1\right)^2 - \frac{4}{9}\right]^{1/n} \tag{14}$$

where we have introduced the only dimensionless factor in the problem, apart from n, namely

$$\beta = \left(\frac{6RE^*}{\pi w_0}\right)^{1/3} \left(\frac{4k}{9}\right)^{1/n} a_T v \tag{15}$$

The latter two equation s correspond to Muller [10] Equation (23,24). The differential Equation (14) can be solved for initial conditions starting from a point on the loading curve[2], which is the JKR curve which in this dimensionless notation and in parametric form is

$$P(a) = 4\left(a^3 - a^{3/2}\right) \tag{16}$$

and

$$\alpha(a) = 3a^2 - 2a^{1/2} \tag{17}$$

After $a(\alpha)$ is obtained, we can compute the load which in dimensionless form is obtained from

$$P(a, \alpha) = 2a\left(\alpha - a^2\right) \tag{18}$$

Notice that the strain energy release rate in dimensionless form is

$$G = \frac{9}{4} a^3 \left(\frac{\alpha}{3a^2} - 1\right)^2 \tag{19}$$

3.1. Muller's approximate solution

Muller [10] in searching for the pull-off as the minimum of the $P(\alpha)$ curve, postulates that this is close to the minimum of $P(\alpha) + G(\alpha)$ which is also 0 in the minimum. There is no fundamental reason for this mix of the dimensionless load with the dimensionless strain energy release rate to have any special property, and indeed we found the two minima are not necessarily very close. Muller's postulate anyway leads to radius of contact, approach and load at pull-off,

$$a_m = \kappa \beta^q \tag{20}$$

$$\alpha_m = -\kappa^2 \beta^{2q} \tag{21}$$

$$P_m = |P_{\min}| = 4\kappa^3 \beta^{3q} \tag{22}$$

where $q = n/(n+3)$ and $\kappa = \left(\frac{9/16}{4^n}\right)^{1/(n+3)}$. Notice obviously that this result at zero velocity would give *incorrect results* as all values go to zero, rather than the asymptotic values of JKR theory for thermodynamic equilibrium.

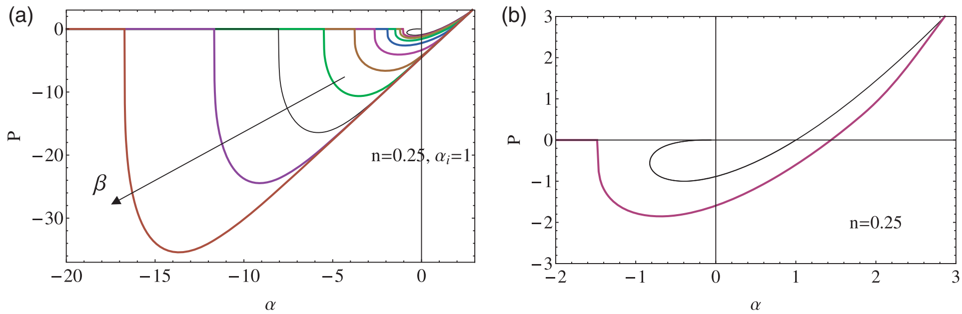

Figure 1. Dimensionless load P dimensionless approach α (a) for various $\beta = 2 \times 10^{-5} * 15^i$, $(i = 1, 10)$ and for $n = 0.25$. The inner black curve is the adiabatic JKR curve. (b) very weak dependence of pull-off on initial conditions (initial load $P = 0, 5$) for an example case $\beta = 0.0675$.

Remark that the actual velocity of the crack line (recall a and α are dimensionless here, not to be confused with Equation (11))

$$\frac{v_p}{v} = \frac{1}{\zeta}\left(\frac{da}{d\alpha}\right)_m = \frac{1}{\zeta}\frac{1}{4a_m} \tag{23}$$

and given $a_m \sim 1$ while $\zeta \ll 1$, it is clear that $\frac{v_p}{v} \gg 1$ so that the velocity at the contact line can be much greater than the cross-head remote velocity, which permits to make the approximation that the bulk may be essentially in a relaxed elastic state. Notice however that, in concentrating the effect of dissipation at the crack tip, despite the dissipation can occur very far from it, there is another possible approximation: indeed, the form of solution we are using is unlikely to be reliable at extremely high speeds anyway, also for thermal effects and other possible physical factors.

4. Numerical results and fittings

Here we report some results of the numerical solution of the differential equation, comparison with Muller's approximate solution, and some improved fitting results for the pull-off, which is (perhaps) the most important quantity.

From Figure 1 we see the withdrawing curves for an example case of low $n = 0.25$, and (b) an example showing that initial conditions seem to very weakly affect the actual pull-off, as Muller had remarked. From Figure 2 we see that the contact radius at pull-off is very poorly predicted by Muller's approximate solution (20), and it is much more weakly dependent on β. In particular, at high β, Muller's solution predicts very large a_m which do not make much sense. Indeed, as we have seen there is not much dependence on the initial condition, we expect $a_m < 1$ as when we are unloading from equilibrium condition at zero load, and since we expect the radius to further decrease, a fortiori we obviously end up with a smaller radius that at zero load, which is $a_\mathrm{i} = 1$. An exception, where we see $a_m > 1$ but not by a large factor, is when there is some weak dependence on initial conditions and we start from very high loads (see example of $P_i = 5$ of Figure 2(a,c)). At low β, Muller's prediction underestimates the radius at pull-off, particularly at high β.

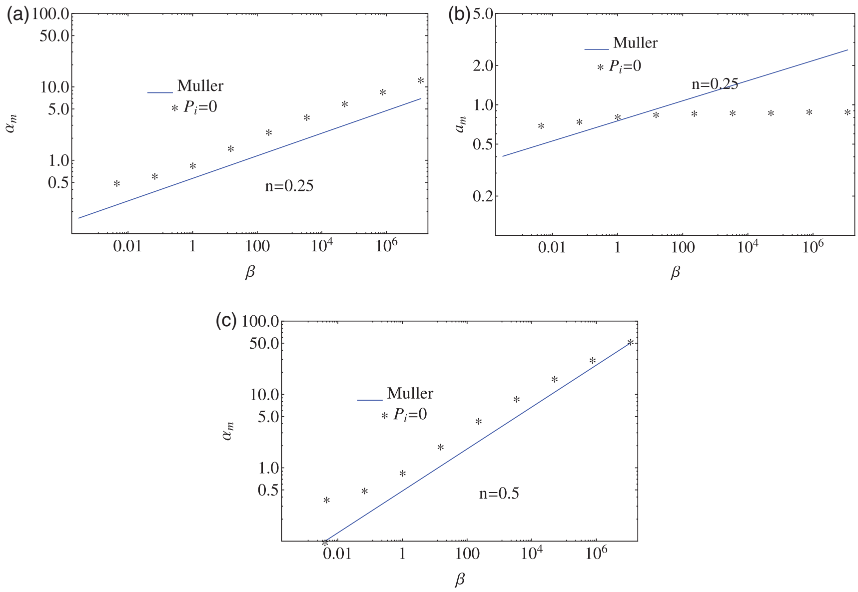

Figure 2. Dimensionless contact radius at pull-off a_m for $n = 0.25$ (a) $n = 0.5$ (b), $n = 0.75$ (c) as a function of the dimensionless speed factor β. (initial load in the figure $P = 0$ or 5).

Also not very good predictions, but perhaps better than for contact radius, are those for the approach at pull-off (Figure 3). Here, the actual results tend to be higher than Muller's prediction (21), at all speeds, and start off with a value near $\alpha_m = -0.5$ rather than from 0.

Considering these poor performances on a_m and α_m, the results for the pull-off load vs Muller's prediction (see Figure 4) are relatively good (blue line vs the markers of the numerical simulations), which is probably why he was satisfied in his paragraph 'comparison with exact calculation' where he has only comparison with pull-off load or work for pull-off, but still we find them only rough 'estimates'. It is easy to obtain much better fit of the results, considering we have only two independent dimensionless parameters, n and β of course, so we improve Muller's prediction in two respects:

1. We add a crossover towards the JKR value $P = 1$, by adding '1' to Muller's Equation (22) the JKR load;
2. We improve the power law exponent at large β with a corrective factor to Muller's Equation (22) in the form

$$P_m = |P_{\min}| = 1 + 4\kappa^3 \beta^{3q/c(n)} \tag{24}$$

where

$$c(n) = 1.1 + n/1.65 \tag{25}$$

This improvement shows clearly a much better fit with respect to detailed numerical calculations in the entire range of realistic values for n and of β covering 10 orders of

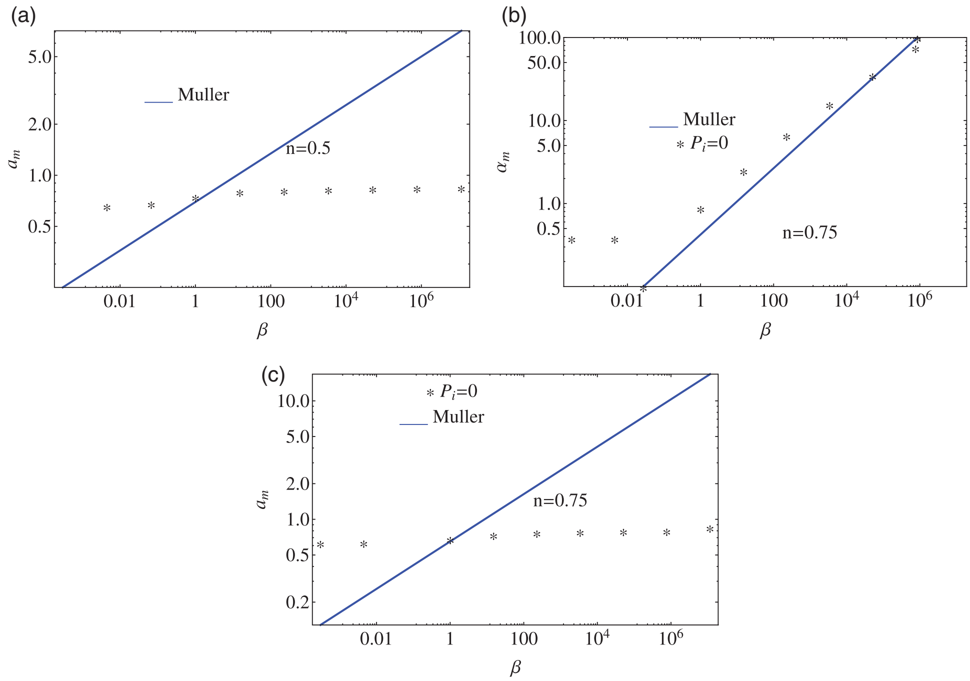

Figure 3. Dimensionless absolute value of approach at pull-off $|\alpha_m|$ for $n = 0.25$ (a) $n = 0.5$ (b), $n = 0.75$ (c) as a function of the dimensionless speed factor β. (initial load in the figure $P = 0$ or 5).

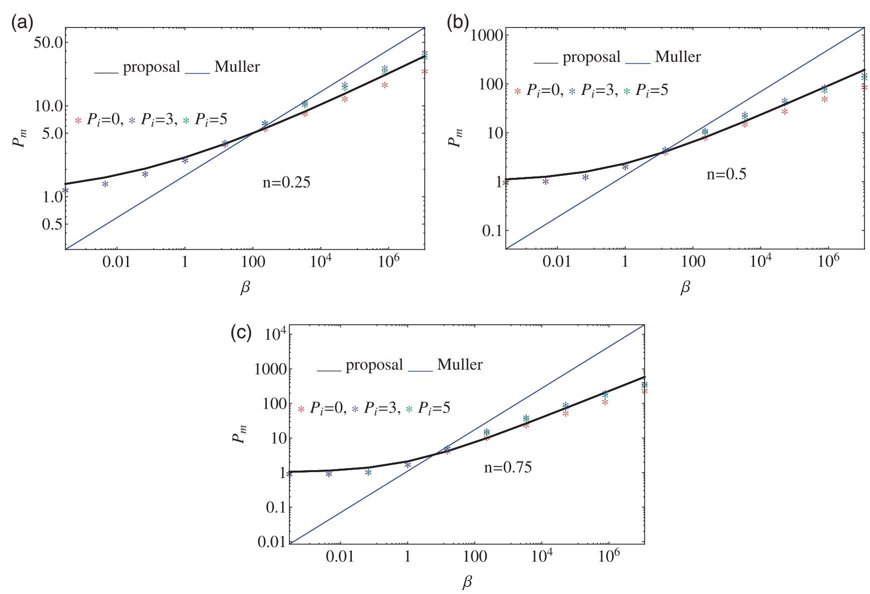

Figure 4. Absolute value of the dimensionless load at pull off P_m for $n = 0.25$ (a) $n = 0.5$ (b), $n = 0.75$ (c) as a function of the dimensionless speed factor β. (initial load as indicated by different colors in the markers in the figure $P_i = 0, 3, 5$). Blue power law curve is the Muller [10] prediction (22), while the thick black solid line is our proposal (24).

magnitude in β which is probably more than enough considering the other approximations made in the model, namely the form of the work of adhesion, that there is no viscoelasticity in the bulk, no thermal effects, and so on.

Notice that Violano and Afferrante [12] have numerically solved the Muller equation s, and found good correlation with experimental results. This suggests that our solution would be very valuable for an analytical fitting of experiments such as those of Violano and Afferrante [12].

5. Conclusions

We have revisited the Muller approximate solution for the pull-off of sphere from a flat viscoelastic material, finding significant errors in the approximate solution, which stem from the rather arbitrary assumption that the pull-off condition occurs when the sum of a dimensionless load and a dimensionless strain energy release rate has a minimum. We have added a 'cross-over' towards the JKR solution for very low velocities, and corrected the power law enhancement of pull-off with velocity of withdrawal. The solution can be useful for quick estimates of the effect of viscoelasticity on the increase of adhesion in spherical geometries.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes

1. The factor 2 which is missing in Muller [10] comes from the fact that strain energy exists only in one material, assuming the other is rigid. For two identical materials, $\frac{1}{E^*} = \frac{2}{E_1^*}$ and we return to the standard LEFM case with $G(a) = \frac{K(a)^2}{E_1^*}$.
2. Strictly speaking, during loading adhesion is reduced with respect to the adiabatic value at zero speed, but we neglect this effect, or else we consider that loading occurs near thermodynamic equilibrium.

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References

[1] Andrews EH, Kinloch AJ. Mechanics of elastomeric adhesion. In: Journal of polymer science: polymer symposia (Vol. 46, No. 1). New York: Wiley Subscription Services, Inc., A Wiley Company; 1974. pp. 1–14.
[2] Barber M, Donley J, Langer JS. Steady-state propagation of a crack in a viscoelastic strip. Phys Rev A. 1989;40(1):366–376.
[3] Barquins M, Maugis D. Tackiness of elastomers. J Adhes. 1981;13(1):53–65.
[4] Gent AN, Petrich RP. Adhesion of viscoelastic materials to rigid substrates. Proc Royal Soc London. A. Math Phys Sci. 1969;310(1502):433–448.

[5] Gent AN, Schultz J. Effect of wetting liquids on the strength of adhesion of viscoelastic material. J Adhes. 1972;3(4):281–294.
[6] Greenwood JA, Johnson KL. The mechanics of adhesion of viscoelastic solids. Philos Mag A. 1981;43(3):697–711.
[7] Maugis D, Barquins M. Fracture mechanics and adherence of viscoelastic solids. In: Adhesion and adsorption of polymers. Boston (MA): Springer; 1980. p. 203–277.
[8] Persson BNJ, Brener EA. Crack propagation in viscoelastic solids. Phys Rev E. 2005; 71(3):036123.
[9] Williams ML, Landel RF, Ferry JD. The temperature dependence of relaxation mechanisms in amorphous polymers and other glass-forming liquids. J Am Chem Soc. 1955; 77(14):3701–3707.
[10] Muller VM. On the theory of pull-off of a viscoelastic sphere from a flat surface. J Adhes Sci Technol. 1999;13(9):999–1016.
[11] Johnson KL, Kendall K, Roberts AD. Surface energy and the contact of elastic solids. Proc R Soc Lond. 1971;A324:301–313.
[12] Violano G, Afferrante L. Adhesion of compliant spheres: an experimental investigation. Procedia Struct Integr. 2019;24:251–258.