THE *J* INTEGRAL

Fracture Mechanics, with or without Field Theory

For a crack in an elastic body subject to a load, the elastic energy stored in the body is a function of two independent variables: the displacement of the load, and the area of the crack, $U(\Delta, A)$. This function of two independent variables can be determined by alternative methods:

1. *Experimental method*. Measure the load-displacement curve for a body containing a pre-cut crack. During the measurement, the crack does not extend. Integrate the load-displacement curve to obtain the elastic energy $U(\Delta, A)$ for the body with the fixed area of crack. Repeat the procedure for the body with a pre-cut crack of a different area. This method invokes no field theory of elasticity.
2. *Computational method*. Solve the boundary-value problem for a body containing a pre-cut crack of a given area. Calculate the energy density at every material particle in the body. Integrate the energy density overall all material particles to obtain the elastic energy for the body with the specific area of crack. Repeat the procedure for the body with a pre-cut crack of a different length. This method requires that the body be modeled by a field theory of elasticity.

The energy release rate is defined as

$$G = -\frac{\partial U(\Delta, A)}{\partial A}.$$

This definition of the energy release rate assumes that the body is elastic, but invokes no field theory of elasticity. Indeed, the energy release rate can be determined experimentally by measuring the load-displacement curves of identically loaded bodies with cracks of different areas. No field need be measured.

The J integral. Many materials, however, can be modeled with a field theory of elasticity. When a material is modeled by such a field theory, we can represent the energy release rate in terms of the field in the body:

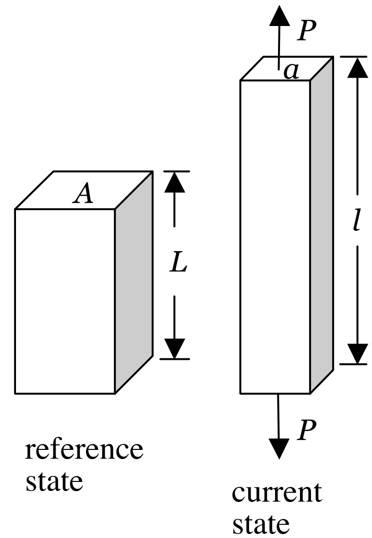
$$G = \int \left(WN_1 - T_i F_{i1} \right) dL.$$

The integral is known as the *J* integral (*J* for James R. Rice). This lecture describes the *J* integral, along with examples of calculation. Uses of the *J* integral are often better appreciated in the context of individual applications, which we will describe in later lectures. In current engineering practice, the field in an elastic body is commonly determined by using finite element method. Once the field is determined, the finite element code calculates the *J* integral, which gives the energy release rate.

Nonlinear Field Theory of Elasticity

The *J* integral can be developed for both linear and nonlinear theory of elasticity. The nonlinear theory will be used in class, and the linear theory of will be used in a homework problem.

For a detailed development of the field theory of elasticity, see http://imechanica.org/node/7794. The main ingredients of the theory are as follows.



Uniaxial deformation. We proceed with our subject incrementally, beginning with the simplest structure: a bar. When the bar is not subject to any force, the cross-sectional area is A and the length is L. We will call this state the reference state. The bar is then subject to an axial force P, and deforms to a new state, cross-sectional area a and length l. We will call this state the current state. The experimentalist records the force as a function of the length. For an elastic body, the force-length curve is reversible upon loading and unloading.

Define the stretch, λ, as the length of the bar in the current state divided by the length of the bar in the reference state:

$$\lambda = \frac{\text{length in current state}}{\text{length in reference state}} = \frac{l}{L}.$$

When dealing with large deformation, we must be specific about the area used in defining the stress. Define the nominal stress, s, as the force applied to the bar in the current state divided by the cross-sectional area of the bar in the reference state:

$$s = \frac{\text{force in the current state}}{\text{area in the reference state}} = \frac{P}{A}.$$

The nominal stress is also known as the engineering stress, or the first Piola-Kirchhoff stress.

When the bar elongates from length l to length $l + \delta l$, the force P does work $P\delta l$. Recall the definitions of stress and strain, $P = sA$ and $l = \lambda L$. Consequently the work done by the force is $P\delta l = ALs\delta\lambda$. Since AL is the volume of the bar in the reference state, we note that

$$s\delta\lambda = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

We say that the nominal stress and the stretch are *work-conjugate*.

We assume that the bar is made of an elastic material. In the current state, denote the elastic energy of the bar by U. The work done by the force equals the elastic energy, $\delta U = P\delta l$. That is, the elastic energy is the area under the force-length curve.

Define the nominal density of elastic energy, W, as the elastic energy in the bar in the current state divided by the volume of the bar in the reference state:

$$W = \frac{\text{elastic energy in the current state}}{\text{volume in the reference state}} = \frac{U}{AL}.$$

The nominal density of elastic energy is a function of the stretch, $W(\lambda)$.

For the bar made of an elastic material, the work done by the force equals the elastic energy, $\delta U = P\delta l$. Dividing both sides by the volume of the bar in the reference state, AL, we obtain that

$$\delta W = s\delta\lambda .$$

According to calculus, this expression is equivalent to

$$s = \frac{dW(\lambda)}{d\lambda} .$$

Once we experimentally determined the force-length curve, we divide the force by A and divide the length by L. This procedure scales the force-length curve to the stress-stretch curve. The area under the stress-stretch curve gives the nominal density of elastic energy, $W(\lambda)$.

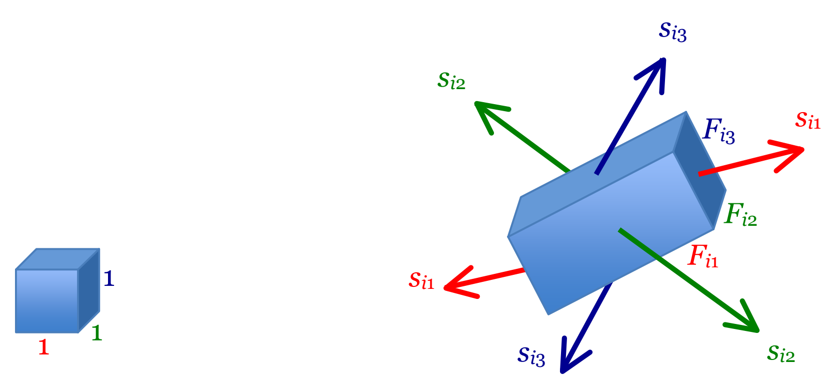
A body is a sum of many small parts. We next generalize the above procedure to a body of an arbitrary shape and subject arbitrary load. The field theory regards the body as a sum of many small pieces. Each small piece is called a material particle, and undergoes a homogeneous deformation. The deformation of the body is in general inhomogeneous—that is, the deformation varies from one part of the body to another part.

In what follows, we will first describe the homogeneous deformation of an individual small piece, and then describe the inhomogeneous deformation of a body.

Homogeneous deformation in three dimensions. We first consider a small part of the body. We list the three ingredients of the theory: geometry of deformation, balance of forces, and conditions of thermodynamic equilibrium.

Geometry of deformation. In the undeformed state, the part is a unit cube, with edges coinciding with the coordinates. Subject to forces on the faces of the part, the part undergoes a homogeneous deformation. In the deformed state, the part deforms into a parallelepiped.

The deformation maps the three orthogonal edges of the cube to three edges of the parallelepiped. The edges of the parallelepiped are three vectors, noted as F_{i1}, F_{i2}, and F_{i3}. The first subscript indicates the component of each vector, and the second subscript differentiates the three edges. The nine quantities F_{iK} together are known as the deformation gradient.



undeformed state deformed state

Balance of forces. Acting on the three pairs of parallel faces of the parallelepiped are forces s_{i1}, s_{i2}, and s_{i3}. The first subscript indicates the component of each vector, and the second subscript differentiates the three pairs of parallel faces. The nine quantities s_{iK} together are known as the nominal stress.

Let **N** be the unit vector normal to an element of an area in the undeformed body. Define the nominal traction **T** by the force acting on the element of the surface in the deformed body divided by the area of the element in the undeformed body. The balance of forces relates the nominal traction to the nominal stress:

$$T_i = s_{iK} N_K .$$

Condition of thermodynamic equilibrium. We consider a small part of the body in thermodynamic equilibrium with external forces and with an environment of constant temperature. We will drop temperature from the list of variables. Let W be the Helmholtz free energy of the parallelepiped. We say a material is elastic if the Helmholtz free energy is a function of the deformation gradient, $W(\mathbf{F})$. This function is an input to the field theory of elasticity. W is also known as the nominal density of elastic energy.

When the part is in a state of thermodynamic equilibrium, the nominal stress relates to the deformation gradient as

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} .$$

This equation of states generalizes Hooke's stress-strain relation.

Inhomogeneous deformation. We call each small part of the body a material particle. As the body deforms, each material particle moves in space. Now focus on a particular material particle in the body. When the body is in the undeformed state, the material particle is at a place in the space, of coordinate **X**. When the body is in a deform state, the same material particle moves to a difference place in space, of coordinate **x**. The function $\mathbf{x}(\mathbf{X})$ fully describes the deformation of the body. The deformation gradient relates to the deformation as

$$F_{iK} = \frac{\partial x_i(\mathbf{X})}{\partial X_K} .$$

When the deformation of the body is homogeneous, the deformation gradient is the same for all material particles in the body. In general, the deformation of the body is inhomogeneous, so that the deformation gradient varies from one material particle to another. That is, the deformation gradient as a function of material particle, $F_{iK}(\mathbf{X})$.

In general, the stress in the deformed body is inhomogeneous, and we write the stress as a function of material particles, $s_{iK}(\mathbf{X})$. The balance of forces requires that the field of stress satisfy

$$\frac{\partial s_{iK}(\mathbf{X})}{\partial X_K} = 0.$$

Here we neglect body force and inertia.

Elastic energy of the body. When a body is in equilibrium with a set of applied forces, the field in the body is in general inhomogeneous. The Helmholtz free energy of the body is the sum of energy of all material particles:

$$U = \int W dV.$$

The energy density is of the body in the deformed state, but the integration is carried over the volume of the body in the reference state.

A composite system that does not receive work from the rest of the world. Consider a body subject to a constant force P, with Δ being the displacement of the force. We can always picture the constant force as a hanging weight. The potential energy of the weight is $-P\Delta$. The body and the weight together form a composite system. The composite exchanges with the environment by heat, but not by work. The Helmholtz free energy of the composite is

$$\Pi = U - P\Delta.$$

This quantity is known as the potential energy of the body in mechanics.

The general idea is to form a composite system that does not receive any work from the rest of the world. We have considered two examples:

1. When a body is loaded by a fixed displacement, the body itself is composite system, and the free energy of the system is U.
2. When a body is loaded by a hanging weight, the body and the weight together form a composite system, and the free energy of the system is $U - P\Delta$.

We can also consider yet another example. When a body in series with a spring is loaded by a fixed displacement, the body and the spring together form a composite system, and the free energy of the system is the sum of that of the body and that of the spring.

Energy Release Rate

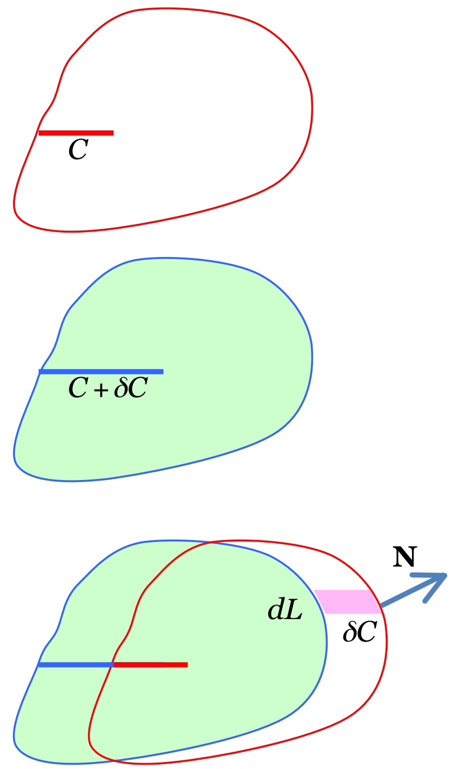
Express energy release rate in terms of field variables. Model a body with the nonlinear theory of elasticity. Under the plane-strain or plane-stress conditions, we represent the body by a region in the plane, and the boundary of the body by a curve in the plane. Consider a sample of unit thickness. Each point on the boundary is subject to constant nominal traction T_i. The potential energy per unit thickness of the body is

$$\Pi = \int W(\mathbf{F}) dA - \int T_i(\mathbf{X}) x_i(\mathbf{X}) dL$$

The first integral extends over the body, and the second integral extends over the boundary of the body.

Now compare two specimens, one having a crack of length C in the undeformed state, and other having a crack of length $C + \delta C$ in the undeformed state. The cracks are in the direction of coordinate X_1. The two specimens are of the same general shape. In the deformed states, the cracks in the two specimens are traction-free, and external boundaries of the specimens are subject to the same tractions. The two specimens have different potential energies, $\Pi(C)$ and $\Pi(C+\delta C)$.

In the undeformed state, we translate one specimen relative to the other by a distance δC, so that the tips of the cracks in the two specimens coincide. Consider an element of the boundary dL. Let $\mathbf{N}$ be the unit vector normal to the element. As one specimen translates relative to the other by a distance δC, the element dL sweeps across a region in the shape of a parallelogram. The area of the parallelogram is $\delta C N_1 dL$. In the deformed states of the two specimens, the parallelogram contributes a difference in the elastic energies in the two specimens, $-WN_1\delta C dL$.



Consider a set of material particles. In the undeformed state, the set of particles forms a straight segment δC. In the deformed state, the set of material particles forms a segment represented by vector $F_{i1}\delta C$. The applied force acting on the element is $T_i dL$. Consequently, the applied force acting on the element dL gives a difference in the potential energies of the two specimens, $F_{i1}\delta C T_i dL$.

Combining the above contributions, we obtain the difference in the potential energies of the two specimens:

$$\Pi(C+\delta C)-\Pi(C)=-\int WN_1\,\delta C\,dL+\int T_iF_{i1}\,\delta C\,dL .$$

The first term is due to the elastic energy of the body, and the second term is due to the potential energy of the applied traction.

By definition, the energy release rate is

$$G=-\frac{\Pi(C+\delta C)-\Pi(C)}{\delta C}.$$

The partial derivative means that we vary the length of the crack, but keep the loading conditions fixed. Comparing the above two expressions, we obtain that

$$G=\int\left(WN_1-T_iF_{i1}\right)dL .$$

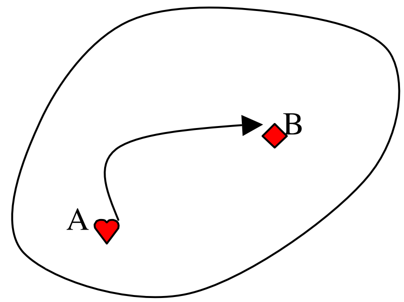
This equation expresses the energy release rate in terms of field variables.

Properties of the J Integral

The *J* integral. Let us focus on the integral:

$$J=\int\left(WN_1-T_iF_{i1}\right)dL .$$

The path of integration is a curve, drawn in the body in the undeformed state, connecting two material particles. The unit vector $\mathbf{N}$ is normal to the path of integration.



Recall the relations in the theory of elasticity: $F_{iK} = \partial x_i(\mathbf{X}) / \partial X_K$, $s_{iK} = \partial W(\mathbf{F}) / \partial F_{iK}$ and $T_i = s_{iK} N_K$. The J integral can also be written as

$$J = \int \left[W N_1 - N_K \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \frac{\partial x_i(\mathbf{X})}{\partial X_1} \right] dL.$$

The integral is a functional of $\mathbf{x}(\mathbf{X})$, the field that describes the deformation of the body. The dummy variable of integration is $\mathbf{X}$, the coordinates of material particles in the undeformed body.

Yet another alternative expression of the *J* integral. A material particle occupies a place of coordinate $\mathbf{X}$ in the undeformed state, and occupies a place of coordinate $\mathbf{x}$ in the deformed state. The displacement of the material particle is

$$\mathbf{u} = \mathbf{x}(\mathbf{X}) - \mathbf{X}.$$

Thus

$$\frac{\partial u_i(\mathbf{X})}{\partial X_1} = \frac{\partial x_i(\mathbf{X})}{\partial X_1} - \delta_{i1}.$$

Inserting this expression into the definition of the J integral, we obtain that

$$J = \int \left(W N_1 - T_i \frac{\partial u_i(\mathbf{X})}{\partial X_1} \right) dL.$$

In reaching this expression, we have used the following relation:

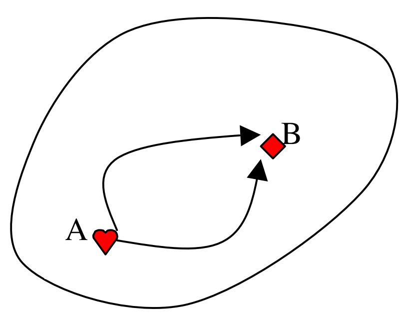
$$\int T_1 dL = 0.$$

This relation results from the balance of forces.

Divergence theorem. In the following development, we will need a result in calculus: the divergence theorem. Consider a region in a plane. Let $f(\mathbf{X})$ be a smooth function defined in the region, and N_K be the unit vector normal to the curve around the region. The divergence theorem is

$$\int \frac{\partial f(\mathbf{X})}{\partial X_K} dA = \int f N_K dL.$$

The integral on the left is over the area of a region, and integral on the right is over the curve surrounding the region.



The *J* integral is path-independent. The two material particles A and B can be connected by numerous paths. For any two paths, so long as no singularity exists between them, the J integrals along the two paths are identical.

To prove this statement, reverse the direction of one path, so that the two paths form a closed contour. This closed contour is used as a path of integration. We need to prove that the *J* integral vanishes if the closed contour encloses no singularity. The proof invokes the divergence theorem, as well as the field theory of elasticity. We show that the second part of the *J* integral equals the first part:

$$\int T_i F_{i1}\, dL = \int N_K s_{iK} F_{i1}\, dL \qquad \text{recall } T_i = s_{iK} N_K$$

$$= \int \frac{\partial \left(s_{iK} F_{i1} \right)}{\partial X_K} dA \qquad \text{divergence theorem}$$

$$= \int \left(\frac{\partial s_{iK}}{\partial X_K} F_{i1} + s_{iK} \frac{\partial F_{i1}}{\partial X_K} \right) dA$$

$$= \int s_{iK} \frac{\partial F_{i1}}{\partial X_K} dA \qquad \text{recall } \frac{\partial s_{iK}}{\partial X_K} = 0$$

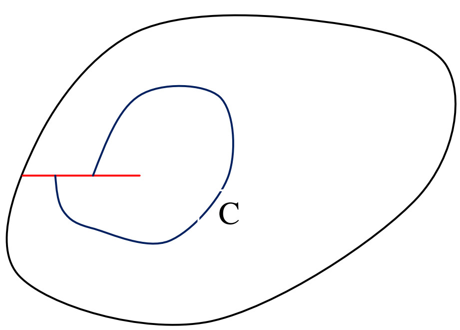
$$= \int s_{iK} \frac{\partial^2 x_i}{\partial X_1 \partial X_K} dA \qquad \text{recall } F_{iK} = \frac{\partial x_i \left(\mathbf{X} \right)}{\partial X_K} \quad .$$

$$= \int \frac{\partial W\left(\mathbf{F} \right)}{\partial F_{iK}} \frac{\partial^2 x_i}{\partial X_1 \partial X_K} dA \qquad \text{recall } s_{iK} = \frac{\partial W\left(\mathbf{F} \right)}{\partial F_{iK}}$$

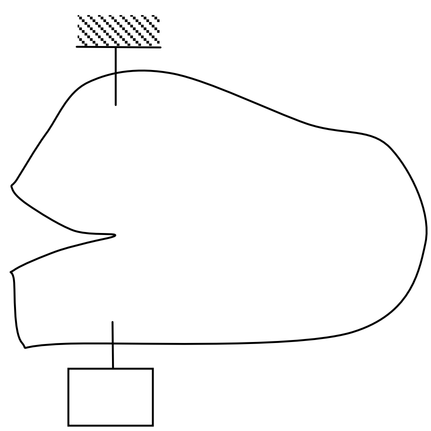
$$= \int \frac{\partial W\left(\mathbf{F} \right)}{\partial F_{iK}} \frac{\partial F_{iK}}{\partial X_1} dA$$

$$= \int \frac{\partial W\left(\mathbf{F} \right)}{\partial X_1} dA$$

$$= \int W\left(\mathbf{F} \right) N_1\, dL \qquad \text{divergence theorem}$$



Undeformed body



Deformed body

The *J* integral associated with a traction-free crack. Consider a crack in a body subject to a load. The faces of the crack are traction-free. In the undeformed body, a path of integration starts from a material particle on the lower face of the crack, surrounds the tip of the crack, and ends at a material particle on the upper face of the crack. Because the path encloses the singular tip of the crack, the J integral does not vanish. We can make two statements:

1. Given the two material particles on the faces of the crack, the J integral is independent of the path.
2. The J integral is the same independent of the choice of the material particles on the two faces of the crack.

Use the *J* integral to calculate energy release rate in the finite-element method. For a crack in an elastic body subject to a load, the energy release rate is defined as the decrease in the elastic energy of the body associated with extension of the crack by unit area, while the load is held fixed. We have used this definition directly to calculate the energy release rate for a few cases. In each case, we somehow obtain the elastic energy in the body as a function of the area of the crack, and then take the partial derivative of this function with respect to the area of the crack.

This procedure, however, is difficult to apply when we solve the boundary-value problem by using the finite-element method. Let us imagine how we might go about it. For a given area of the crack, we use the finite-element method to determine the field in the body, and then integrate to obtain the elastic energy stored in the body. We then change the area of the crack slightly, solve a new boundary-value problem, and then obtain the elastic energy in the body. We then take the ratio between the difference in the elastic energy and the difference in the area of crack. Here are two obvious difficulties:

- In each boundary-value problem, the field is singular around the front of the crack.
- The elastic energies in the two bodies are large quantities compared to their difference. To calculate the energy release rate this way would require us to determine the field very accurately.

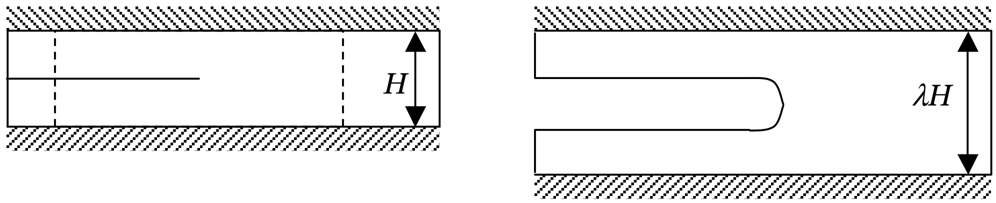
Both of the above difficulties are avoided if we use the J integral to calculate the energy release rate. The integrand involves the field in the body of a fixed crack. The path of integration needs to enclose the tip of crack, but can be chosen to be far away from the tip of the crack.

Examples

A long crack in a strip of a material pulled by two rigid grips. Rivlin and Thomas (1953) described this experimental setup. They obtained the energy release rate directly from its definition. The result is

$$G = HW(\lambda),$$

where H is the width of the undeformed strip, and λH is the width of the deformed stripe. The quantity $HW(\lambda)$ can be determined from experimentally measured load-displacement curve of a strip with no crack, pulled by two rigid grips.



The same result can be obtained by using the J integral along the dotted lines indicated in the undeformed strip (Rice, 1968). Use the J integral in the form

$$J = \int \left(WN_1 - T_i \frac{\partial u_i(\mathbf{X})}{\partial X_1} \right) dL .$$

The only nonzero contribution comes from the vertical line in the strip ahead of the crack.

Layered materials. In the above, the material is taken to be homogeneous. When the two material particles are of the same deformation gradient, they have the same free energy function—that is, the nominal density of free energy is a function the deformation gradient:

$$W = W(\mathbf{F}).$$

The function does not depend on $\mathbf{X}$ explicity.

For a layered material, which is homogenous in the X_1-direction, but inhomogeneous in the X_2-direction, the energy density takes the form

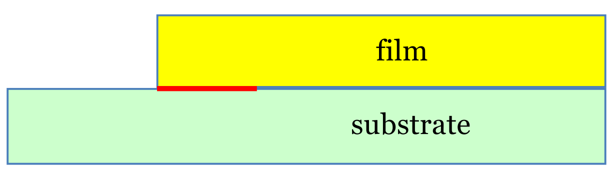
$$W = W(\mathbf{F}, X_2).$$

Going through the same steps, you can confirm that the J integral is still path-independent.

Finally consider a generally inhomogeneous material, for which the energy density takes the form

$$W = W(\mathbf{F}, X_1, X_2).$$

Going through the same steps, you will find that the J integral is path-dependent.



Detaching highly stretchable materials. Jinda Tang and Jianyu Li are studying the delamination of highly stretchable materials, such as elastomers and gels, using a specific experimental setup. Attach a film to a substrate, and leave a detached region as a pre-existing crack. Both materials are elastic and highly stretchable. When a force pulls the substrate, the tip of the crack blunts. When the force reaches some critical value, the crack runs. The length of the crack is large compared to the thicknesses of the film and the substrate, so that the crack runs in a steady state, and the energy release rate is independent of the length of the crack. We wish to relate the energy release rate of the crack to the applied force.

First measure the stress-stretch relations of the film and the substrate separately. When a uniaxial force pulls the film, the nominal stress S_f in the film relates to the stretch λ through a curve $S_f(\lambda)$, and the area under this curve is the nominal density of the free energy in the film as a function of the stretch, $W_f(\lambda)$. Similarly, when a uniaxial force pulls the substrate, the nominal stress S_s in the substrate relates to the stretch λ through a curve $S_s(\lambda)$, and the area under this curve is the nominal density of the free energy in the film as a function of the stretch, $W_s(\lambda)$.

When the laminate is in the undeformed state, the thickness of the film is H_f, the thickness of the substrate is H_s, and the width of the film and substrate is *B*. In the deformed state, a force *P* pulls the substrate. Far behind the tip of the crack, the film is stress-free, and the substrate is in a state of uniaxial stress, with the stretch λ' determined by

$$H_s S_s(\lambda') = P .$$

Far ahead the tip of the crack, the film and substrate are attached and both have the same stretch λ'', determined by

$$H_f S_f(\lambda'') + H_s S_s(\lambda'') = P .$$

Now compare two specimens. Both specimens have the total length *L* in the reference state. One specimen has a crack of length C in the undeformed state, and has potential energy

$$\Pi(C) = C\left[BH_s W_s(\lambda') - P\lambda'\right] + (L-C)\left[BH_f W_f(\lambda'') + BH_s W_s(\lambda'') - P\lambda''\right]$$

in the deformed state.

The other specimen has a crack of length $C+\delta C$ in the undeformed state, and has potential energy

$$\Pi(C+\delta C) = (C+\delta C)\left[BH_s W_s(\lambda') - P\lambda'\right] + (L-C-\delta C)\left[BH_f W_f(\lambda'') + BH_s W_s(\lambda'') - P\lambda''\right]$$

in the deformed state.

By definition, the energy release rate is the reduction in the potential energy associated with unit increase of the crack area:

$$G = -\frac{\Pi(C+\delta C) - \Pi(C)}{B\delta C},$$

giving that

$$G = H_f W_f(\lambda'') + H_s W_s(\lambda'') - H_s W_s(\lambda') + \frac{P}{B}(\lambda' - \lambda'').$$

In writing the potential energy of each specimen, we have assumed that the detached substrate is in a homogeneous state of stretch λ', and the attached laminate is in a homogeneous state of λ''. In the region around the tip of the crack, however, the field is inhomogeneous. Because the crack is in a steady state, the inhomogeneous field in this region is identical in the two specimens. Consequently, this inhomogeneous field does not affect the calculation of the energy release rate.

We can also use the *J* integral to obtain the above expression for the energy release rate.

These relations together relate the energy release rate to the applied force P. We measure the stress-stretch curves of the two materials, $S_f(\lambda)$ and $S_s(\lambda)$. The areas under these curves give the nominal densities of energy of the two materials, $W_f(\lambda)$ and $W_s(\lambda)$. The applied force P determines the stretch λ' in the detached substrate by $H_s S_s(\lambda') = P$, and determines the stretch λ'' in the attached laminate by $H_f S_f(\lambda'') + H_s S_s(\lambda'') = P$.

A thin, compliant film on a substrate. In a limiting case that the substrate is much stiffer than the substrate, the attached film does not constrain the substrate, so that the stretch in the detached substrate is nearly the same as the stretch in the attached laminate, $\lambda' \approx \lambda''$. In this limiting case, the energy release rate reduces to

$$G = H_f W_f(\lambda).$$

When the stretch is of order unity, the free energy density is on the order of the elastic modulus, E. For a hydrogel, a representative value is $E = 10^4$ Pa. For the film of thickness H_f = 0.1 mm, the energy release rate is on the order of 1 J/m^2. For given materials, the laminate will not detach if the film is thin and stretch is small.

A stiff film on a compliant substrate. In this limiting case, the attached laminate does not deform, λ'' = 1, so that the energy release rate is entirely due to the detached substrate:

$$G = H_s\left[S_s(\lambda')\lambda' - W_s(\lambda')\right].$$

For experimental details, see Jingda Tang, Jianyu Li, Joost J. Vlassak, Zhigang Suo. Adhesion between highly stretchable materials. Soft Matter 12, 1093-1099 (2016). http://www.seas.harvard.edu/suo/papers/350.pdf

Historical Notes

Eshelby studied the change in the potential energy associated with the movement of a singularity. He called this change the force on the singularity, and expressed this force as an integral. He did not relate his idea to the Griffith theory of cracks. Rice and Cherepanov independently discovered the J integral within the context of fracture mechanics.

- J.D. Eshelby, The force on an elastic singularity. Phil. Trans. Roy. Soc. London A 244, 87-111 (1951).
- J.D. Eshelby, The continuum theory of lattice defects. Solid State Physics 3, 79-144 (1956).
- J.R. Rice, A path independent integral and the approximate analysis of strain concentration by notches and cracks. J. Appl. Mech. 35, 379-386 (1968). http://esag.harvard.edu/rice/015_Rice_PathIndepInt_JAM68.pdf
- J.R. Rice, Mathematical analysis in the mechanics of fracture, Chapter 3 of Fracture: An Advanced Treatise (Vol. 2, Mathematical Fundamentals) (ed. H. Liebowitz), Academic Press, N.Y., 1968, pp. 191-311. http://esag.harvard.edu/rice/018_Rice_MathAnalMechFract_68.pdf

- G.P. Cherepanov, Crack propagation in continuous media. Journal of Applied Mathematics and Mechanics 31, 503 (1967).
- G.P. Cherepanov, Cracks in solids. International Journal of Solids and Structures 4, 811-831 (1968).
- S.J. Chang, Path-independent integral for rupture of perfectly elastic materials. Journal of Applied Mathematics and Physics 23, 149-152 (1972).