Lagrangian approach to origami vertex analysis: Kinematics – Supplementary Information

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I. PREAMBLE

First recall the compatibility condition [bH02]

$$\mathbf{F}_N\left(\phi_1, \ldots, \phi_N\right) = \prod_{i=1}^{N} \mathbf{Q}_{\hat{\mathbf{b}}_i}\left(\phi_i\right) = \mathbf{I}, \tag{1}$$

where the rotation transformations can be expressed using

$$\mathbf{Q}_{\hat{\mathbf{u}}}\left(\varphi\right) = \exp\left(\varphi \mathbf{A}_{\hat{\mathbf{u}}}\right) = \left(1 - \cos\varphi\right) \hat{\mathbf{u}} \otimes \hat{\mathbf{u}} + \cos\varphi \mathbf{I} + \sin\varphi \mathbf{A}_{\hat{\mathbf{u}}}. \tag{2}$$

Although the primary focus of this work has been in obtaining and utilizing closed-form solutions, we show here that the Lagrangian approach also facilitates exploiting symmetry for reduced order computational folding.

II. REDUCED ORDER COMPUTATIONAL FOLDING

Inspired by the computational folding approach first developed by Tachi [Tac09], we show how the reduced order compatibility conditions can be linearized and approximated numerically. To this end, consider a folded origami with flat state crease vectors, $\hat{\mathbf{b}}_i$, fold angles, ϕ_i, and folded configuration crease vectors, $\hat{\mathbf{c}}_i$ for $i = 1, \ldots, N$. Let ψ_i denote a perturbation to the fold angle, ϕ_i, such that $\phi_i \to \phi_i + \psi_i$. A natural question is: "given the folded state of the origami, what are the allowable perturbations, ψ_i, $i = 1, \ldots, N$?". Substituting the perturbed fold angles into (1), and assuming $\prod_{i=1}^{N} \mathbf{Q}_{\hat{\mathbf{b}}_i}\left(\phi_i\right) = \mathbf{I}$, one obtains

$$\prod_{i=1}^{N} \mathbf{Q}_{\hat{\mathbf{b}}_i}\left(\phi_i + \psi_i\right) = \prod_{i=1}^{N} \mathbf{Q}_{\hat{\mathbf{c}}_i}\left(\psi_i\right) = \mathbf{I} \tag{3}$$

the significance of which is clear once one realizes that there is no requirement that the flat state be the choice of reference configuration. One can, in principle, just as easily describe the deformation of the structure relative to a given folded configuration, in which case, (3) is the analogous compatibility condition. However, (3) is of the same form as (1) and, hence, no easier to solve. To make further progress, we will assume the fold angle perturbations are "small enough". To this end, expanding (2) to linear order about zero angle of rotation, we obtain the infinitesimal rotation tensor

$$\mathbf{Q}_{\hat{\mathbf{u}}}\left(\psi\right) = \mathbf{I} + \psi \mathbf{A}_{\hat{\mathbf{u}}} + \mathcal{O}\left(\psi^2\right). \tag{4}$$

Let $\epsilon = \max_i \left|\psi_i\right|$. Then, to linear order in ϵ, the left hand side of (3) takes the form

$$\prod_{i=1}^{N} \mathbf{Q}_{\hat{\mathbf{c}}_i}\left(\psi_i\right) = \prod_{i=1}^{N} \left(\mathbf{I} + \psi_i \mathbf{A}_{\hat{\mathbf{c}}_i} + \mathcal{O}\left(\psi_i^2\right)\right) = \mathbf{I} + \sum_{i=1}^{N} \psi_i \mathbf{A}_{\hat{\mathbf{c}}_i} + \mathcal{O}\left(\epsilon^2\right). \tag{5}$$

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Dropping higher order terms and setting this equal to the identity, we arrive at the linearized compatibility equation

$$\sum_{i=1}^{N} \psi_i \mathbf{A}_{\hat{\mathbf{c}}_i} = \mathbf{0}, \tag{6}$$

which can readily be solved using numerical methods for finding matrix nullspaces. Computational discovery of a folding trajectory can be accomplished iteratively by, at each step, solving for possible perturbations to the fold angles, updating the fold angles via numerical integration and crease vectors via the deformation map, and then repeating the process about each new folded configuration [Tac09]. This process is analogous to the *updated Lagrangian* formulation in continuum solid mechanics.

Computational folding while imposing a reflection symmetry. Given the overview of an algorithm for computational folding, let us now return our attention to the reduced order formulation for reflection symmetries (section IV of the main text). Here we consider the folded origami, Ω, as our new choice of reference configuration, where the $\hat{\mathbf{c}}_i$ satisfy have a reflection symmetry about the plane orthogonal to $\hat{\mathbf{e}}_2$. Let the deformation map be given by

$$\mathbf{\Phi}'(\mathbf{y}) = \begin{cases} \mathbf{Q}_{\hat{\mathbf{c}}_1}\left(\frac{\psi_1}{2}\right)\mathbf{y}, & \mathbf{y} \in \Omega^{(1)} \\ \mathbf{Q}_{\hat{\mathbf{c}}_1}\left(\frac{\psi_1}{2}\right)\prod_{i=2}^{j}\mathbf{Q}_{\hat{\mathbf{c}}_i}(\psi_i)\,\mathbf{y}, & \mathbf{y} \in \Omega^{(j)}, j \neq 1 \end{cases}. \tag{7}$$

Using arguments similar to those given in the main text, we arrive at

$$\mathbf{F}'_M \coloneqq \mathbf{Q}_{\hat{\mathbf{c}}_1}\left(\frac{\psi_1}{2}\right)\prod_{i=2}^{M}\mathbf{Q}_{\hat{\mathbf{c}}_i}(\psi_i), \tag{8a}$$

$$\mathbf{F}'_N = \mathbf{Q}_{\hat{\mathbf{c}}_1}\left(-\frac{\psi_1}{2}\right), \tag{8b}$$

$$\left(\mathbf{F}'_{N/2-1}\hat{\mathbf{c}}_{N/2}\right)\cdot\hat{\mathbf{e}}_2 = 0, \tag{8c}$$

as the definitions and conditions for $\psi_i, i = 1, \dots N$ to describe a compatible configuration with a reflection symmetry about the plane orthogonal to $\hat{\mathbf{e}}_2$. Substituting in the rotation expansion, (4), and dropping higher order terms, we obtain the linear, symmetry reduced order condition

$$\begin{aligned} \left(\left(\mathbf{I} + \frac{\psi_1}{2}\mathbf{A}_{\hat{\mathbf{c}}_1}\right)\left(\mathbf{I} + \psi_2\mathbf{A}_{\hat{\mathbf{c}}_2}\right)\dots\left(\mathbf{I} + \psi_{N/2-1}\mathbf{A}_{N/2-1}\right)\hat{\mathbf{c}}_{N/2}\right)\cdot\hat{\mathbf{e}}_2 + \mathcal{O}\left(\epsilon^2\right) = 0, \\ \left(\left(\mathbf{I} + \frac{\psi_1}{2}\mathbf{A}_{\hat{\mathbf{c}}_1} + \sum_{i=2}^{N/2-1}\psi_i\mathbf{A}_{\hat{\mathbf{c}}_i}\right)\hat{\mathbf{c}}_{N/2}\right)\cdot\hat{\mathbf{e}}_2 = 0, \end{aligned} \tag{9}$$

A numerical solution can be obtained by iteratively

1. choosing $\psi_1, ..., \psi_{N/2-1}$ such that (9) is satisfied and $\sqrt{\psi_1^2 + \dots \psi_{N/2-1}^2} = 1$,
2. letting $\psi_{N/2+1} = \psi_{N/2-1}, \dots, \psi_N = \psi_2$,
3. performing an integration step (e.g. Euler integration: $\phi_i \to \phi_i + \eta\psi_i$ where η is the size of the Euler step), and
4. updating the crease unit vectors (i.e. $\hat{\mathbf{c}}_i \to \mathbf{F}_i(\phi_1, \dots, \phi_N)\,\hat{\mathbf{b}}_i$).

While forward Euler integration is simple to implement, we remark that, based on numerical experiments, 4th order Runge-Kutta is very well suited for this approach, both in terms of efficiency and stability.

Computational folding while imposing reflections and rotations. Consider next the computational folding of vertices with reflection and rotational symmetries. Let a folded state be given such that $\varphi = 2\pi/2h, h \in \mathbb{N}$, $\mathcal{G} = \mathcal{D}_{2h}$, $\hat{\mathbf{c}}_k = \mathbf{Q}_{\hat{\mathbf{e}}_3}(\varphi/2)\,\hat{\mathbf{c}}_1$ (i.e. $\boldsymbol{\sigma}_{\hat{\mathbf{c}}_k \times \hat{\mathbf{e}}_3} \in \mathcal{G}$), and $\Omega^{(c)} = \bigcup_{i=1}^{k-1} \Omega^{(i)}$, where $\Omega^{(c)} \subset \Omega$ is the unit cell to be perturbed such that $\Omega = \mathrm{Orb}_{\mathcal{G}}\,\Omega^{(c)}$. The perturbation in the elevation angle is denoted by τ. Then the deformation map with respect to the folded configuration of interest is given by

$$\boldsymbol{\Phi}'(\mathbf{y}) = \begin{cases} \mathbf{Q}_{\hat{\mathbf{e}}_1}(\tau)\,\mathbf{Q}_{\hat{\mathbf{c}}_1}\left(\frac{\psi_1}{2}\right)\mathbf{y}, & \mathbf{y} \in \Omega^{(1)} \\ \mathbf{Q}_{\hat{\mathbf{e}}_1}(\tau)\,\mathbf{Q}_{\hat{\mathbf{c}}_1}\left(\frac{\psi_1}{2}\right)\prod_{i=2}^{j}\mathbf{Q}_{\hat{\mathbf{c}}_i}(\psi_i)\,\mathbf{y}, & \mathbf{y} \in \Omega^{(j)}, j \neq 1 \end{cases}. \tag{10}$$

The analog of the rotation-reflection compatibility condition (equation 5.15 of the main text) where the "reference" configuration is a folded state is given by:

$$\frac{\left(\mathbf{F}'_{k-1}(\tau, \psi_1, \ldots, \psi_{k-1})\,\hat{\mathbf{c}}_k\right)\cdot\hat{\mathbf{e}}_2}{\left(\mathbf{F}'_{k-1}(\tau, \psi_1, \ldots, \psi_{k-1})\,\hat{\mathbf{c}}_k\right)\cdot\hat{\mathbf{e}}_1} = \frac{\hat{\mathbf{c}}_k\cdot\hat{\mathbf{e}}_2}{\hat{\mathbf{c}}_k\cdot\hat{\mathbf{e}}_1}. \tag{11}$$

Substituting in (4) and rearranging, one obtains

$$\begin{aligned} &\left[\left(\left(\mathbf{I} + \tau\mathbf{A}_{\hat{\mathbf{e}}_1} + \sum_{i=1}^{k-1}\psi_i\mathbf{A}_{\hat{\mathbf{c}}_i}\right)\hat{\mathbf{c}}_k\right)\cdot\hat{\mathbf{e}}_2\right](\hat{\mathbf{c}}_k\cdot\hat{\mathbf{e}}_1) \\ &\quad - \left[\left(\left(\mathbf{I} + \tau\mathbf{A}_{\hat{\mathbf{e}}_1} + \sum_{i=1}^{k-1}\psi_i\mathbf{A}_{\hat{\mathbf{c}}_i}\right)\hat{\mathbf{c}}_k\right)\cdot\hat{\mathbf{e}}_1\right](\hat{\mathbf{c}}_k\cdot\hat{\mathbf{e}}_2) = 0, \end{aligned} \tag{12}$$

which, again, is linear in the unknowns. Thus, it can iteratively be solved, numerical integration can be used to update the fold angles, and the deformation map can be used to update the crease vectors.

III. SUMMARY LIST OF ANIMATIONS

- **Animation #1:** `6-fold vertex_alpha1-pi_6.avi` and `.gif`

  Folding animation of a 6-fold vertex with $\alpha_1 = \pi/6$ along the trajectory highlighted by the location in the kinematic domain denoted by the moving star (see figure 3).

- **Animation #2:** `6-fold vertex_alpha1-pi_4.avi` and `.gif`

  Folding animations of a 6-fold vertex with $\alpha_1 = \pi/4$ folding along ($\phi_2 = 0 \to -\pi, \phi_3 = \pi$) and then ($\phi_2 = -\pi, \phi_3 = \pi \to 0$) (see figure 4b).

- **Animation #3:** `6-fold vertex_alpha1-5pi_12.avi` and `.gif`

  Folding animations of a 6-fold vertex with $\alpha_1 = 5\pi/12$ folding along ($\phi_2 = 0 \to -\pi, \phi_3 = \pi$) and then ($\phi_2 = -\pi, \phi_3 = \pi \to 0$). Animations #3 and #4 are included to highlight why the admissible regions of configuration space are different between these two cases (see figure 4b).

- **Animation #4:** `6-fold vertex_alpha1-11pi_24.avi` and `.gif`

  Animation of the hinge-like behavior observed in the 6-fold vertex when α_1 approaches $\pi/2$, as shown in this 6-fold, $\alpha_1 = 11\pi/24$ vertex (see figure 4a and 4b).

- **Animation #5:** `8-fold vertex_alpha1-pi_6.avi` and `.gif`

  Folding animation of an 8-fold vertex with $\alpha_1 = \pi/6$ and $\mathcal{D}_2$ symmetry. To highlight the different regions in configuration space, the vertex is folded into admissible regions, regions where no solution exists, and regions where contact occurs (see figure 9).

[bH02] sarah belcastro and Thomas C Hull. Modelling the folding of paper into three dimensions using affine transformations. *Linear Algebra and its applications*, 348(1-3):273–282, 2002.

[Tac09] Tomohiro Tachi. Simulation of rigid origami. *Origami*, 4(08):175–187, 2009.