Motivation for Christoffel symbols:

Suppose we have a symmetric, +ve definite matrix field defined from a given (compatible) deformation $\underset{\sim}{x}$ corresponding to the coordinates $\{\xi^i\}$

So $$h_{ij} := \frac{\partial \underset{\sim}{x}}{\partial \xi^i} \cdot \frac{\partial \underset{\sim}{x}}{\partial \xi^j} \equiv \underset{\sim}{a}_i \cdot \underset{\sim}{a}_j .$$

Suppose we ask the question of writing $\underset{\sim}{a}_{i,j} = {}_{(\xi)}\Lambda_{ijk}\, \underset{\sim}{a}^k$. in terms of the field h_{ij}.

Then $\underset{\sim}{a}_{i,j} \cdot \underset{\sim}{a}_k = {}_{(\xi)}\Lambda_{ijk}$.

Now
$$(\underset{\sim}{a}_i \cdot \underset{\sim}{a}_k)_{,j} = \boxed{\underset{\sim}{a}_{i,j} \cdot \underset{\sim}{a}_k} + \underset{\sim}{a}_i \cdot \underset{\sim}{a}_{k,j} .$$
$$(\underset{\sim}{a}_j \cdot \underset{\sim}{a}_k)_{,i} = \boxed{\underset{\sim}{a}_{j,i} \cdot \underset{\sim}{a}_k} + \underset{\sim}{a}_j \cdot \underset{\sim}{a}_{k,i}$$
$$(\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j)_{,k} = \underset{\sim}{a}_{i,k} \cdot \underset{\sim}{a}_j + \underset{\sim}{a}_i \cdot \underset{\sim}{a}_{j,k}$$

Now because $\underset{\sim}{a}_i = \dfrac{\partial \underset{\sim}{x}}{\partial \xi^i}$

$$\underline{\underline{\underset{\sim}{a}_{i,j} = \underset{\sim}{a}_{j,i}}}$$

$$\therefore \quad \underset{\sim}{a}_{i,j} \cdot \underset{\sim}{a}_k = {}_{(\xi)}\Lambda_{ijk} = \frac{1}{2}\left[h_{ik,j} + h_{jk,i} - h_{ij,k} \right]$$
$$= \frac{1}{2}\left[(\underset{\sim}{a}_i \cdot \underset{\sim}{a}_k)_{,j} + (\underset{\sim}{a}_j \cdot \underset{\sim}{a}_k)_{,i} - (\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j)_{,k} \right]$$

So given a body R with prescribed RCG field $\underline{C}$ & parametrization for R $\{x^i\}$ with natural basis $\{\underset{\sim}{E}_i\}$ ${}_{(x)}\Lambda_{ijk} = \frac{1}{2}\left[(\underset{\sim}{E}_i \cdot \underline{C}\, \underset{\sim}{E}_k)_{,j} + (\underset{\sim}{E}_j \cdot \underline{C}\, \underset{\sim}{E}_k)_{,i} - (\underset{\sim}{E}_i \cdot \underline{C}\, \underset{\sim}{E}_j)_{,k} \right]$

Compatibility of Finite Strain

Suppose we have a one-to-one mapping

$$(x^1, x^2, x^3) \longmapsto (y^1, y^2, y^3)$$

with two positive definite symmetric matrix fields h_{ij} & $g_{\alpha\beta}$ specified. Because of the one-to-one nature we can consider each of these fields as functions of either $\{x^i\}$ or $\{y^i\}$.

Let these fns at each $\{x^i\}$ (& hence $\{y^i(x)\}$) satisfy

$$h_{ij} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} g_{\alpha\beta}.$$

then

$$\frac{\partial h_{ij}}{\partial y^k} = \left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} \frac{\partial x^\beta}{\partial y^j} + \frac{\partial x^\alpha}{\partial y^i} \frac{\partial^2 x^\beta}{\partial y^j \partial y^k} \right) g_{\alpha\beta}$$

$$+ \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \underbrace{\frac{\partial x^\gamma}{\partial y^k} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}}_{\frac{\partial g_{\alpha\beta}}{\partial y^k}}.$$

because $g_{\alpha\beta} = g_{\beta\alpha}$

$$\frac{\partial h_{ij}}{\partial y^k} = \left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} \frac{\partial x^\beta}{\partial y^j} + \frac{\partial x^\beta}{\partial y^i} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} \right) g_{\alpha\beta}$$

$$+ \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}.$$

$$\frac{\partial h_{ik}}{\partial y^j} = \left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} \frac{\partial x^\beta}{\partial y^k} + \frac{\partial x^\beta}{\partial y^i} \frac{\partial^2 x^\alpha}{\partial y^k \partial y^j} \right) g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} \frac{\partial x^\gamma}{\partial y^j} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$$

$$\left(\frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\gamma}{\partial y^k} \frac{\partial x^\beta}{\partial y^j} \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} \right)$$

$$\frac{\partial h_{jk}}{\partial y^i} = \left(\frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} \frac{\partial x^\beta}{\partial y^k} + \frac{\partial x^\beta}{\partial y^j} \frac{\partial^2 x^\alpha}{\partial y^k \partial y^i} \right) g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial y^j} \frac{\partial x^\beta}{\partial y^k} \frac{\partial x^\gamma}{\partial y^i} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$$

$$\left(\frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right)$$

Define $\displaystyle {}^{(y)}\Gamma_{ijk} := \frac{1}{2}\left[\frac{\partial h_{ik}}{\partial y^j} + \frac{\partial h_{jk}}{\partial y^i} - \frac{\partial h_{ij}}{\partial y^k} \right]$

Similarly $\displaystyle {}^{(x)}\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left[\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right]$

(Christoffel symbols)

$$\therefore \quad {}^{(y)}\Gamma_{ijk} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k} \, {}^{(x)}\Gamma_{\alpha\beta\gamma}$$

$$- \frac{1}{2} \frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} \frac{\partial x^\beta}{\partial y^j} g_{\alpha\beta} \; (*) \; - \frac{1}{2} \frac{\partial x^\beta}{\partial y^i} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} g_{\alpha\beta} \; (**)$$

$$+ \frac{1}{2} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} \frac{\partial x^\beta}{\partial y^k} g_{\alpha\beta} \; (***) \; + \frac{1}{2} \frac{\partial x^\beta}{\partial y^i} \frac{\partial^2 x^\alpha}{\partial y^k \partial y^j} g_{\alpha\beta} \; (**)$$

$$+ \frac{1}{2} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} \frac{\partial x^\beta}{\partial y^k} g_{\alpha\beta} \; (***) \; + \frac{\partial x^\beta}{\partial y^j} \frac{\partial^2 x^\alpha}{\partial y^k \partial y^i} g_{\alpha\beta} \; (*)$$

$$\boxed{ {}^{(y)}\Gamma_{ijk} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k} \, {}^{(x)}\Gamma_{\alpha\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} \frac{\partial x^\beta}{\partial y^k} g_{\alpha\beta} }$$

Now define (Christoffel symbols of second kind)

$$ {}_{(y)}\Gamma^{k}_{ij} := h^{km}\, {}_{(y)}\Gamma_{ijm}. $$

where $h^{km} = g^{\alpha\beta} \dfrac{\partial y^k}{\partial x^\alpha} \dfrac{\partial y^m}{\partial x^\beta}$

$[h_{ij}]^{-1} = h^{ij}$

& $[g_{\alpha\beta}]^{-1} = g^{\alpha\beta}$

Then

$$ {}_{(y)}\Gamma^{k}_{ij} = h^{km} g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} \frac{\partial x^\beta}{\partial y^m} + h^{km} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^m} {}_{(x)}\Gamma_{\alpha\beta\gamma}. $$

$$ = \frac{\partial y^k}{\partial x^\rho} g^{\rho\nu} \frac{\partial y^m}{\partial x^\nu} g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} \frac{\partial x^\beta}{\partial y^m} + \frac{\partial y^k}{\partial x^\rho} \frac{\partial y^m}{\partial x^\nu} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^m} g^{\rho\nu} {}_{(x)}\Gamma_{\alpha\beta\gamma}. $$

$$ = g_{\alpha\beta} g^{\beta\rho} \frac{\partial y^k}{\partial x^\rho} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} + \delta^{\gamma}_{\nu} \frac{\partial y^k}{\partial x^\rho} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} g^{\rho\nu} {}_{(x)}\Gamma_{\alpha\beta\gamma} $$

$$ \boxed{ {}_{(y)}\Gamma^{k}_{ij} = \frac{\partial y^k}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial y^j \partial y^i} + \frac{\partial y^k}{\partial x^\rho} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} {}_{(x)}\Gamma^{\rho}_{\alpha\beta} } $$

$$\therefore \quad \frac{\partial^2 x^\mu}{\partial y^j \partial y^i} = \frac{\partial x^\mu}{\partial y^k}(y)\Lambda^k_{ij} - \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}(x)\Lambda^\mu_{\alpha\beta}$$

Of Course then

$$\boxed{\frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} = \frac{\partial y^i}{\partial x^\gamma}(x)\Lambda^\gamma_{\alpha\beta} - \frac{\partial y^j}{\partial x^\alpha}\frac{\partial y^k}{\partial x^\beta}(y)\Lambda^i_{jk}}$$

Now, let $\{y^i\}$ be Cartesian coordinates on deformed body & $\{x^i\}$ " " on undeformed.

so deformed points $\underset{\sim}{y} = y^i \underset{\sim}{e}_i$

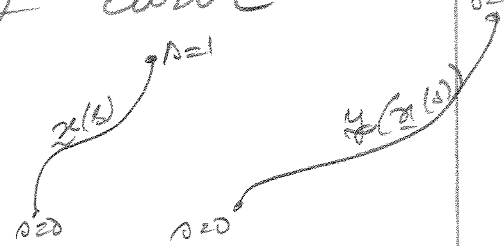
undeformed " $\underset{\sim}{x} = x^i \underset{\sim}{e}_i$.

Take a curve in undeformed configuration parametrized as $s \longmapsto \underset{\sim}{x}(s)$ $\quad s \in [0,1]$

Its image is $s \longmapsto \underset{\sim}{y}(\underset{\sim}{x}(s))$.

Undeformed length of curve

$$\int_{s=0}^{1} \left| \frac{d\underset{\sim}{x}}{ds} \cdot \frac{d\underset{\sim}{x}}{ds} \right| ds$$



Deformed length $\displaystyle\int_{s=0}^{s=1} \left| \frac{d\underset{\sim}{y}}{ds} \cdot \frac{d\underset{\sim}{y}}{ds} \right| ds = \int_{s=0} \left| \frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}} \frac{d\underset{\sim}{x}}{ds} \cdot \frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}} \frac{d\underset{\sim}{x}}{ds} \right|$

$$= \int_{s=0}^{s=1} \left| \frac{d\underset{\sim}{x}}{ds} \cdot \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right)^T \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right) \frac{d\underset{\sim}{x}}{ds} \right| ds.$$

$\therefore$ Length (& angle) changes characterized by

$$\underline{C} := \left(\frac{\partial y}{\partial x}\right)^T \left(\frac{\partial y}{\partial x}\right) \qquad \underline{F}^T \underline{F}$$

↔ Right Cauchy-Green Tensor.

* Suppose two deformations $y(x)$ & $z(x)$ have the same Right Cauchy Green Tensors at all x. Then they differ at most by a rigid deformation.

$$\frac{\partial y_i}{\partial x_j}\frac{\partial y_i}{\partial x_k} = \frac{\partial z_m}{\partial x_j}\frac{\partial z_m}{\partial x_k} = C_{jk} \ \forall x.$$

$$\Rightarrow \frac{\partial y_i}{\partial z_m}\frac{\partial z_m}{\partial x_j}\frac{\partial y_i}{\partial z_n}\frac{\partial z_n}{\partial x_k} = \frac{\partial z_m}{\partial x_j}\frac{\partial z_m}{\partial x_k}.$$

$$\Rightarrow \frac{\partial z_m}{\partial x_j}\left(\frac{\partial y_i}{\partial z_m}\frac{\partial y_i}{\partial z_n}\right)\frac{\partial z_n}{\partial x_k} = \frac{\partial z_m}{\partial x_j}\frac{\partial z_m}{\partial x_k}.$$

$$\frac{\partial y_i}{\partial z_p}\frac{\partial y_i}{\partial z_q} = \frac{\partial x_j}{\partial z_p}\frac{\partial z_m}{\partial x_j}\frac{\partial z_m}{\partial x_k}\frac{\partial x_k}{\partial z_q}$$

$$\Rightarrow \frac{\partial y_i}{\partial z_p}\frac{\partial y_i}{\partial z_q} = \delta_{mp}\delta_{mq}$$

$$= \delta_{pq}$$

$$\Rightarrow \left(\frac{\partial y}{\partial z}\right)^T\left(\frac{\partial y}{\partial z}\right) = \underline{I}$$

Now
$$\frac{\partial y_i}{\partial z_p}\frac{\partial y_i}{\partial z_q} = \delta_{pq}$$

$$\Rightarrow \frac{\partial y_m}{\partial z_p}\delta_{mn}\frac{\partial y_n}{\partial z_q} = \delta_{pq}$$

We had:
$$\frac{\partial x^\alpha}{\partial y^i} g_{\alpha\beta}\frac{\partial x^\beta}{\partial y^j} = h_{ij}$$

if we make the associations

$g_{\alpha\beta} \rightsquigarrow \delta_{mn}$

$y_m \rightsquigarrow x^\alpha$

$z_p \rightsquigarrow y^i$

$h_{ij} \rightsquigarrow \delta_{pq}$.

Then
$$\frac{\partial^2 y_m}{\partial z_p \partial z_q} = \frac{\partial y_m}{\partial z_k}\,{}_{(z)}\Lambda^k_{pq} - \frac{\partial y_r}{\partial z_p}\frac{\partial y_s}{\partial z_q}\,{}_{(y)}\Lambda^m_{rs}$$

But since $g_{\alpha\beta}$ & h_{ij} are both constants on the domain (Identity matrix)

$$\therefore\ {}_z\Lambda^k_{pq} = 0 \ \& \quad {}_y\Lambda^m_{rs} \equiv 0$$

$$\therefore \frac{\partial^2 y_m}{\partial z_p \partial z_q} = 0 \Rightarrow \frac{\partial y_m}{\partial z_p} = \text{Constant } R_{mp}$$

But $R_{mp}R_{mq} = \delta_{pq}$ $(\underline{R}^T R = I)$ $\therefore$ $\underline{y}(\underline{z}) = \underline{R}(\underline{z}-\underline{z}_0) + \underline{y}_0$