Alternatively, if defined thru the RCG field on R,

$$ {}_{(y)}\Lambda_{ijk} = \frac{1}{2}\left[(\underset{\sim}{E}_i \cdot \underset{\sim}{C}\, \underset{\sim}{E}_k)_{,j} + (\underset{\sim}{E}_j \cdot \underset{\sim}{C}\, \underset{\sim}{E}_k)_{,i} - (\underset{\sim}{E}_i \cdot \underset{\sim}{C}\, \underset{\sim}{E}_j)_{,k} \right] $$

where $\underset{\sim}{E}_i = \dfrac{\partial \underset{\sim}{X}}{\partial y^i}$ $\quad \underset{\sim}{X} \in R$

(R is reference configuration on which $\underset{\sim}{C}$ is specified)

& $h_{ik} = \underset{\sim}{E}_i \cdot \underset{\sim}{C}\, \underset{\sim}{E}_k$

so that ${}_{(y)}\Lambda_{ijk} = \frac{1}{2}\left[\partial h_{ik}/\partial y^j + \partial h_{jk}/\partial y^i - \partial h_{ij}/\partial y^k \right]$

$$ {}_{(x)}\Lambda_{\alpha\beta\gamma} = \frac{1}{2}\left[(\underset{\sim}{E}_\alpha \cdot \underset{\sim}{C}\, \underset{\sim}{E}_\gamma)_{,\beta} + (\underset{\sim}{E}_\beta \cdot \underset{\sim}{C}\, \underset{\sim}{E}_\gamma)_{,\alpha} - (\underset{\sim}{E}_\alpha \cdot \underset{\sim}{C}\, \underset{\sim}{E}_\beta)_{,\gamma} \right] $$

$g_{\alpha\beta} = \underset{\sim}{E}_\alpha \cdot \underset{\sim}{C}\, \underset{\sim}{E}_\beta$. & $\underset{\sim}{E}_\alpha = \dfrac{\partial \underset{\sim}{X}}{\partial x^\alpha}$

${}_{(x)}\Lambda_{\alpha\beta\gamma} = \frac{1}{2}\left[\partial g_{\alpha\gamma}/\partial x^\beta + \partial g_{\beta\gamma}/\partial x^\alpha - \partial g_{\alpha\beta}/\partial x^\gamma \right]$

& we have $h_{ik} = \underset{\sim}{E}_i \cdot \underset{\sim}{C}\, \underset{\sim}{E}_k$

$$ = \underset{\sim}{E}_\alpha \frac{\partial x^\alpha}{\partial y^i} \cdot \underset{\sim}{C}\, \underset{\sim}{E}_\beta \frac{\partial x^\beta}{\partial y^k}. $$

$$ \Rightarrow \boxed{h_{ik} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} g_{\alpha\beta}}. $$

Compatibility for Right Cauchy-Green Strain Field.

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} = \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right)^T \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right)$$

Qn: Given a +ve, definite symmetric tensor field $\underset{\sim}{C}$ on the reference configuration R marked generically by the point $\underset{\sim}{x}$, under what conditions on the tensor field $\underset{\sim}{C}$ does $\exists$ a position field $\underset{\sim}{y}(\underset{\sim}{x})$

s.t. $\left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right)^T \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right) = \underset{\sim}{C} \quad \forall \underset{\sim}{x} \in R.$

Necessity:

Suppose $\exists$ such a field $\underset{\sim}{y}$ defining a configuration B. Introducing Rectangular Cartesian coordinates for R & B

$$\frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^i}{\partial x^\beta} = C_{\alpha\beta}.$$

$$\Rightarrow \quad \frac{\partial y^i}{\partial x^\alpha} \delta_{ij} \frac{\partial y^j}{\partial x^\beta} = C_{\alpha\beta} \qquad g_{\alpha\beta}$$

h_{ij}

Define $\frac{\partial y^i}{\partial x^\alpha} =: u^i_\alpha$; we have $_{(y)}\Gamma^i_{jk} = 0.$

Then (from page 4)

$$\frac{\partial u^i_\alpha}{\partial x^\beta} = u^i_\gamma \, {}_{(x)}\Gamma^\gamma_{\alpha\beta} = u^i_\gamma \, C^{\gamma\mu} \frac{1}{2}\left[C_{\alpha\mu,\beta} + C_{\beta\mu,\alpha} - C_{\alpha\beta,\mu}\right]$$

Now $\frac{\partial u^i_\alpha}{\partial x^\beta \partial x^\rho} = \frac{\partial u^i_\alpha}{\partial x^\rho \partial x^\beta}$

$$\Rightarrow \quad u^i_{\gamma,\rho}(x)\Lambda^\gamma_{\alpha\beta} + u^i_\gamma(x)\Lambda^\gamma_{\alpha\beta,\rho} - u^i_{\gamma,\beta}(x)\Lambda^\gamma_{\alpha\rho} - u^i_\gamma(x)\Lambda^\gamma_{\alpha\rho,\beta} = 0.$$

$$\Rightarrow \quad u^i_\mu(x)\Lambda^\mu_{\gamma\rho}(x)\Lambda^\gamma_{\alpha\beta} + u^i_\mu(x)\Lambda^\mu_{\alpha\beta,\rho} - u^i_\mu(x)\Lambda^\mu_{\gamma\beta}(x)\Lambda^\gamma_{\alpha\rho} - u^i_\mu(x)\Lambda^\mu_{\alpha\rho,\beta} = 0$$

$$\Rightarrow u^i_\mu\left[\Lambda^\mu_{\alpha\beta,\rho} - \Lambda^\mu_{\alpha\rho,\beta} + \Lambda^\mu_{\gamma\rho}\Lambda^\gamma_{\alpha\beta} - \Lambda^\mu_{\gamma\beta}\Lambda^\gamma_{\alpha\rho}\right] = 0.$$

& because $u^i_\mu = \frac{\partial y^i}{\partial x^\mu}$ is invertible

$$\boxed{R^\mu_{\alpha\beta\rho} := \Lambda^\mu_{\alpha\beta,\rho} - \Lambda^\mu_{\alpha\rho,\beta} + \Lambda^\mu_{\gamma\rho}\Lambda^\gamma_{\alpha\beta} - \Lambda^\mu_{\gamma\beta}\Lambda^\gamma_{\alpha\rho}} = 0.$$

(mixed components of Riemann-Christoffel curvature tensor.)

Sufficiency:

Let $R^{\mu}_{\alpha\beta\rho} \equiv 0$.

The set of eqns.

$$\frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha$$

& $$\frac{\partial u^i_\alpha}{\partial x^\beta} = u^i_\gamma \, \Lambda^\gamma_{\alpha\beta}$$

(where Λ is constructed from the components of C in the primary basis for $\{x^\beta\}$)

have a solution $\{y^i(x)\}$ & $\{u^i_\alpha(x)\}$ if

$$\boxed{{}_{(x)}\Lambda^\gamma_{\alpha\beta} = {}_{(x)}\Lambda^\gamma_{\beta\alpha}}$$

{true by defn. of ${}_{(x)}\Lambda^\gamma_{\beta\alpha}$}.

and $R^{\mu}_{\alpha\beta\rho} \equiv 0$

{by a theorem of T. Y. Thomas 1934, Annals of Mathematics, 35, 730-734}

Sufficient Condition for compatibility of a +ve def. symm. 2nd order tensor field C with a defmn with the $\left(\frac{\partial y}{\partial x}\right)^T \left(\frac{\partial y}{\partial x}\right) = C$.

Proof: Form the components $C_{ij} \equiv \underset{\sim}{e}_i \cdot \underset{\sim}{C}\underset{\sim}{e}_j \equiv g_{ij}$,

where $\{\underset{\sim}{e}_i\}$ is the natural basis on the ref. configuration R for the coord. system $\{x^i\}$.

Now, consider the system of 'm' eqns

[Sokolnikoff, p. 93, 39.4 (2)]
$$\frac{\partial u_i^m}{\partial x^j} = \Gamma_{ij}^{r} u_r^m \qquad (m, i, j, r = 1 \text{ to } n)$$

From Thomas, we know that the existence of a unique u_i^m, $m, i = 1$ to n, is guaranteed on a simply connected domain iff

$$R^{r}_{\cdot kij} = 0 \quad \& \quad \Gamma_{ij}^{r} = \Gamma_{ji}^{r}.$$

Also, in such a case $u_i^m \in C^2(R)$ & hence

$$\frac{\partial u_i^m}{\partial x^j \partial x^k} = \frac{\partial u_i^m}{\partial x^k \partial x^j}, \qquad — (1)$$

Now, Consider the system of eqn

$$\frac{\partial y^m}{\partial x^i} = u_i^m. \qquad — (2)$$

Clearly, because of (1) & simple connected

$\exists\ y^m(x^k)$ (corresponding to some $y_0^m(x_0^k)$) on R

such that (2) is satisfied.

Given that $\{y^m\}$ are generated from the $\{u_i^m\}$, they satisfy the important property $\det\left|\frac{\partial y^m}{\partial x^k}\right| \neq 0$ as is shown next.

Consider $\frac{\partial}{\partial x^m}\left(g^{kp} v_k w_p\right)$, where v_k & w_p are solns to (0) with i.cs. $v_k(x_0^r) = V_k^0$ & $w_k(x_0^r) = w_k^0$.

Now, $\frac{\partial}{\partial x^m}\left(g^{kp} v_k w_p\right) = \frac{\partial}{\partial x^m}(g^{kp}) v_k w_p + g^{rp}\Gamma^k_{rm} v_k w_p + g^{kr}\Gamma^p_{rm} v_k w_p$

$$= \left[\frac{\partial g^{kp}}{\partial x^m} + g^{rp}\Gamma^k_{rm} + g^{kr}\Gamma^p_{rm}\right] v_k w_p.$$

But $\frac{\partial g^{kp}}{\partial x^m} = -g^{rk}\Gamma^p_{rm} - g^{rp}\Gamma^k_{rm}$ (Sokolnikoff; p.77 (31.8))

$$= -g^{kr}\Gamma^p_{rm} - g^{rp}\Gamma^k_{rm}.$$

$\therefore \frac{\partial}{\partial x^m}\left(g^{kp} v_k w_p\right) = 0$

& $g^{kp} v_k w_p$ is constant on R (connectedness of R).

Now, generate the n covariant vector fields u_i^m, indexed by m, from i.c.s

$$g^{is}\,\mathring{u}_i^m\,\mathring{u}_s^n = \delta^{mn}.$$ (Can be done by Gram Schmitt)

Then at all $\{x^k\}$

$$g^{kp}\,u_k^m\,u_p^n = \delta^{mn}.$$ [Where δ^{mn} are not tensor components.]

but $u_k^m = \dfrac{\partial y^m}{\partial x^k}$

$$\therefore \quad \frac{\partial y^m}{\partial x^k}\,g^{kp}\,\frac{\partial y^n}{\partial x^p} = \delta^{mn}.$$

Since g^{kp} is +ve def. symmetric $\exists$ U^{kp} s.t. $g^{kp} = \sum_{s=1}^{n} U^{ks}U^{sp}$ with

$\det|U^{ks}| \neq 0$.

$$\therefore \sum_{s=1}^{n}\left\{\left(\frac{\partial y^m}{\partial x^k}\,U^{ks}\right)\left(U^{sp}\,\frac{\partial y^n}{\partial x^p}\right)\right\} = \delta^{mn}$$ [Where summation on 'upstairs-downstairs' repeated indices remains valid.]

$$\Rightarrow \det\left|\frac{\partial y^m}{\partial x^k}\,U^{ks}\right| \neq 0$$

$$\Rightarrow \det\left|\frac{\partial y^m}{\partial x^k}\right| \neq 0 \quad \text{at all } \{x^k\}$$

Also $$\frac{\partial y^m}{\partial x^k} g^{kp} \frac{\partial y^n}{\partial x^p} = \delta^{mn}$$

$$\Rightarrow \frac{\partial x^r}{\partial y^m} \frac{\partial y^m}{\partial x^k} g^{kp} \frac{\partial y^n}{\partial x^p} = \frac{\partial x^r}{\partial y^m} \delta^{mn}$$ (Can de because of invert.

$$\Rightarrow g^{rp} \frac{\partial y^n}{\partial x^p} = \frac{\partial x^r}{\partial y^n}$$

$$\Rightarrow g^{rp} \frac{\partial y^n}{\partial x^p} \frac{\partial x^t}{\partial y^n} = \frac{\partial x^t}{\partial y^n} \frac{\partial x^r}{\partial y^n}$$

$$\Rightarrow g^{rt} = \frac{\partial x^t}{\partial y^n} \frac{\partial x^r}{\partial y^n}$$

[these eqns are for matrices & no tensorial meaning ~~need~~ be attached to them at this pt.]

Let $$\frac{\partial y^i}{\partial x^k} \equiv A^i_{\ k}$$

$$\Rightarrow \frac{\partial x^t}{\partial y^n} = A^{-1\,t}_{\ \ n} \ ; \ \frac{\partial x^r}{\partial y^n} = A^{-1\,r}_{\ \ n}$$

$$\therefore \frac{\partial x^t}{\partial y^n} \frac{\partial x^r}{\partial y^n} = A^{-1\,r}_{\ \ n} A^{-T\,n}_{\ \ \ t} = A^{-1} A^{-T}$$

$$\therefore g_{rt} = A^{T\,n}_{\ \ r} A^n_{\ t} = \frac{\partial y^n}{\partial x^r} \frac{\partial y^n}{\partial x^t}$$

$$\therefore \boxed{C_{rt} = \frac{\partial y^n}{\partial x^r} \frac{\partial y^n}{\partial x^t}}$$

Now, choose any coordinate system, for $\mathcal{E}_3$ and assign the region $\mathcal{Y}$ of $\mathcal{E}_3$ that corresponds to the region y^n, in this coord system, as the image of the reference. Let $\frac{\partial \underset{\sim}{y}}{\partial y^n} = \underset{\sim}{E}_n$.

Clearly then $\frac{\partial y^n}{\partial x^r} = \underset{\sim}{E}^n \cdot \frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}} \underset{\sim}{e}_r$

& because $\det\left|\frac{\partial y^n}{\partial x^r}\right| \neq 0 \quad \therefore \quad \frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}$ is invertible

Hence, the constructed $\underset{\sim}{y}$ satisfies

$$\left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right)^T \left(\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right) = \underset{\sim}{C}.$$