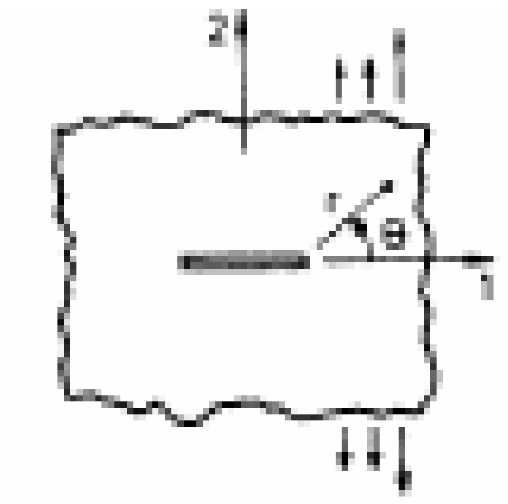
LEFM—Linear Elastic Fracture Mechanics

Crack-tip fields for plane stress and plane strain: Mode I and Mode II

Linear, isotropic elastic solids with Young's modulus, E, and Poisson's ratio, ν.

The in-plane stresses in the crack tip fields are the same in plane stress and plane strain, however, $\sigma_{33}=0$ in plane stress and $\sigma_{33}=\nu(\sigma_{11}+\sigma_{22})$ in plane strain



Mode I - -Symmetric stresses & strains fields at tip.

Universal behavior (*all problems*!) as r approaches crack tip:

$$\sigma_{\alpha\beta}=\frac{K_I}{\sqrt{2\pi r}}\tilde{\sigma}^I_{\alpha\beta}(\theta)$$

$$u_\alpha=u^0_\alpha+\frac{K_I}{\bar{E}}\sqrt{\frac{r}{2\pi}}\,\tilde{u}^I_\alpha(\theta)$$

$\bar{E}=E$ in plane stress

$=\dfrac{E}{1-\nu^2}$ in plane strain

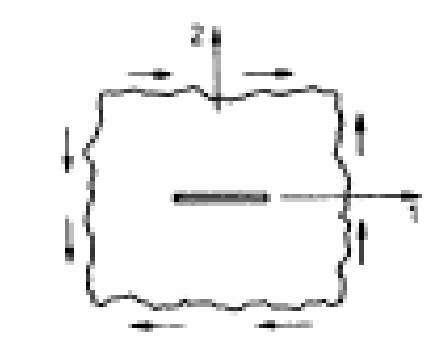
The distributions, $\tilde{\sigma}^I_{\alpha\beta}$, are given in most texts on LEFM

Mode I-- On the plane ahead of the tip: $\sigma_{22}=\dfrac{K_I}{\sqrt{2\pi r}}$, $\sigma_{12}=0$ (standard definition)

where K_I is called the Mode I stress intensity factor

Mode II - -Anti - symmetric stresses & strains fields at tip.

Universal behavior (*all problems*!) as r approaches crack tip ($\tilde{\sigma}_{\alpha\beta}^{II}$ in the texts):



$$\sigma_{\alpha\beta} = \frac{K_{II}}{\sqrt{2\pi r}} \tilde{\sigma}_{\alpha\beta}^{II}(\theta)$$

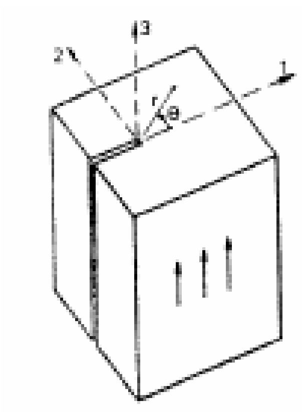
$$u_\alpha = u_\alpha^0 + \frac{K_{II}}{\bar{E}} \sqrt{\frac{r}{2\pi}}\, \tilde{u}_\alpha^{II}(\theta)$$

On the plane ahead of the tip: $\sigma_{12} = \dfrac{K_{II}}{\sqrt{2\pi r}}, \ \sigma_{22} = 0$ (standard definition)

where K_{II} is called the Mode II stress intensity factor

Mode III - -Out - of - plane shearing (also called "anti - plane shear").

Universal behavior (*all problems*!) as r approaches crack tip:



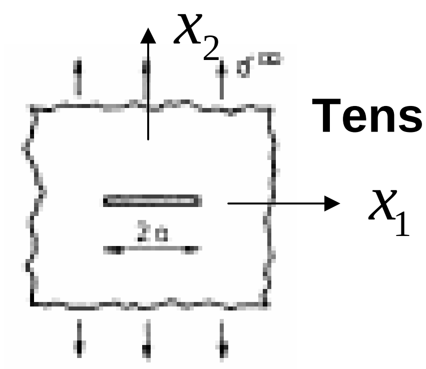
$$\left(\sigma_{13}, \sigma_{23}\right) = \frac{K_{III}}{\sqrt{2\pi r}} \left(\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right) \right)$$ ($\sigma_{12} = K_{III} / \sqrt{2\pi r}$ on plane ahead of tip)

$$u_3 = \frac{2K_{III}}{G} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right)$$ ($G \equiv E/(2(1+\nu))$ is shear modulus)

Derivation of the Mode III fields given in class

By superposition (linear elasticity), conditions at any crack tip can always be represented as a sum of the three modes.

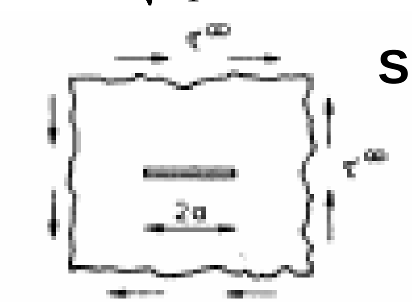
Some Basic Solutions



Tensile crack

Exact solution: $K_I = \sigma^\infty \sqrt{\pi a}$

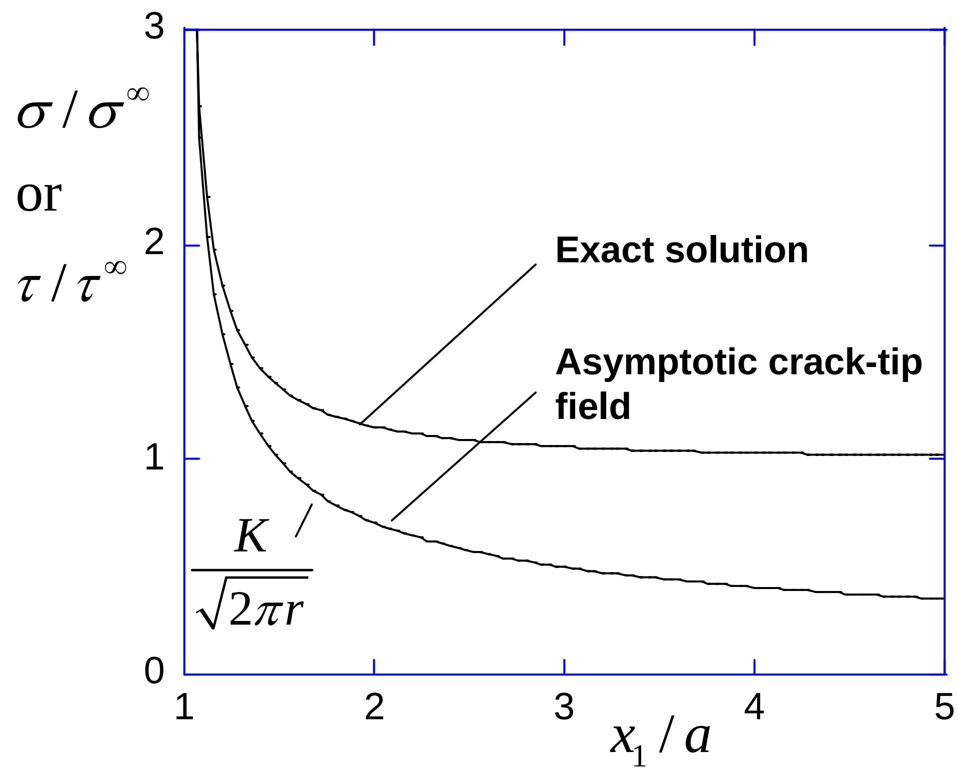
$$\sigma_{22} = \frac{\sigma^\infty x_1}{\sqrt{x_1^2 - a^2}} \quad (x_1 > a,\ x_2 = 0)$$



Shear crack

Exact solution: $K_{II} = \tau^\infty \sqrt{\pi a}$

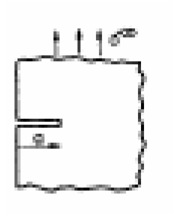
$$\sigma_{12} = \frac{\tau^\infty x_1}{\sqrt{x_1^2 - a^2}} \quad (x_1 > a,\ x_2 = 0)$$



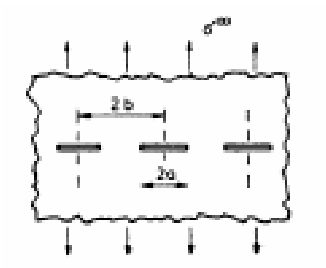
For these two problems the K-field gives and accurate estimate of the stress ahead of the crack for r/a<1/4

There are several excellent compilations of solutions. We will make use of Tada, Paris & Irwin

Some Basic Solutions, continued

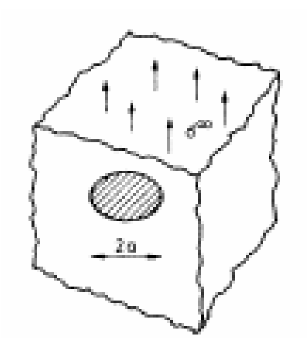


tensile edge-crack : $K_I = 1.122\sigma^{\infty}\sqrt{\pi a}$



periodic tensile cracks :

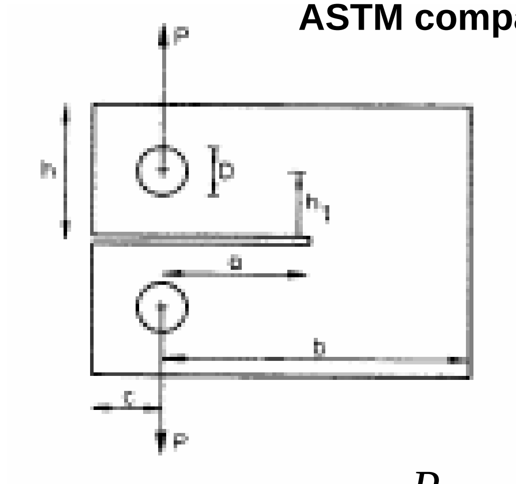
$$K_I = \sigma^{\infty}\sqrt{\pi a}\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]^{1/2}$$



Mode I at all points
of crack edge

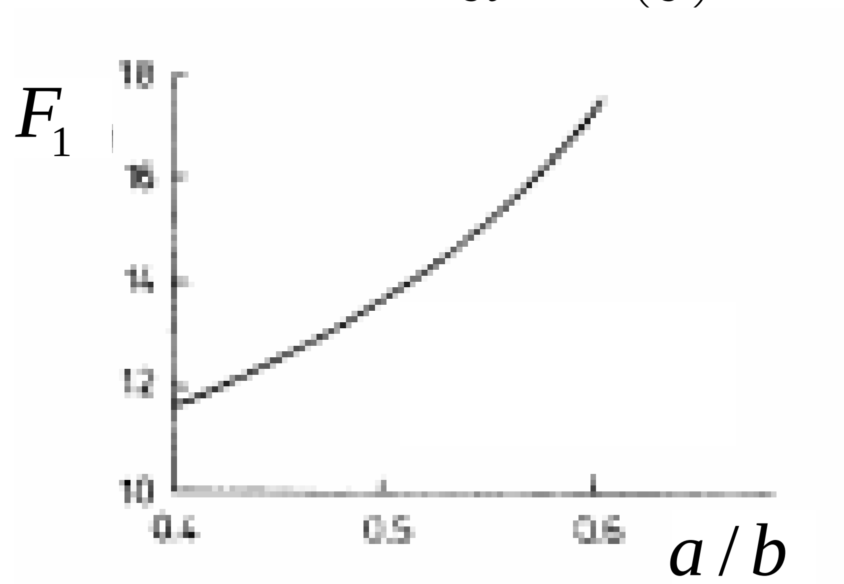
penny shaped crack : $K_I = \frac{2}{\pi}\sigma^{\infty}\sqrt{\pi a}$

ASTM compact tensile specimen



$$h = 0.6b$$
$$h_1 = 0.275b$$
$$c = D = 0.25b$$
$$thickness \equiv t = b/2$$

$$K_I = \frac{P}{bt}\sqrt{a}\ F_1\left(\frac{a}{b}\right)$$



(see Tada, et al page???)

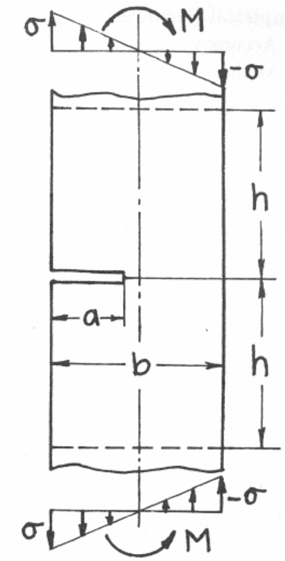
An example of results in the
Stress Analysis of Cracks Handbook
by H. Tada, P.C. Paris and G.R. Irwin
ASME Press, New York, NY, 2000

An edge-notched infinite beam under pure bending. Pages 55-57 of the Handbook

Results are presented on the this page and the next for the stress intensity factor (a mode I problem), the crack opening displacement at the surface, and the additional rotation due to the presence of the crack.

THE PURE BENDING SPECIMEN

A. Stress Intensity Factor

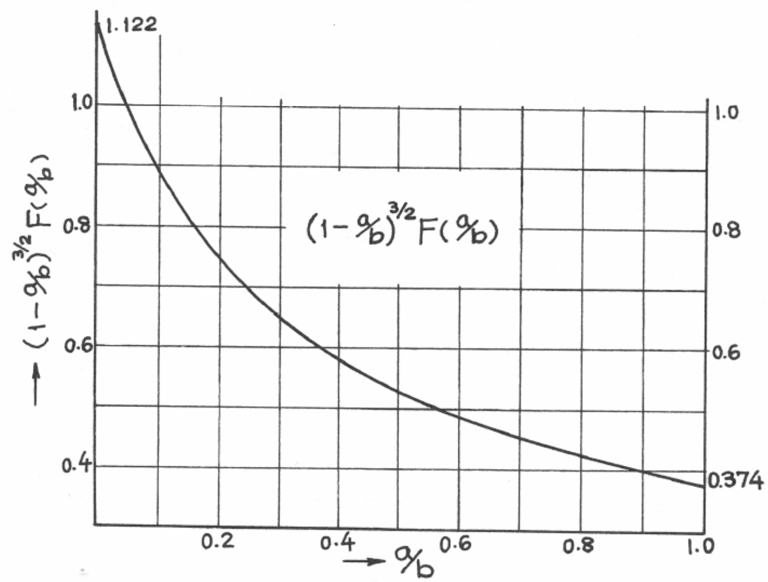


$$\sigma = \frac{6M}{b^2}$$

$$K_I = \sigma\sqrt{\pi a}\, F(a/b)$$

Numerical Values of $F(a/b)$

The curve in the following figure was drawn based on the results having better than 0.5% accuracy. Also used for four-point bending.



Methods and References
1. Singular Integral Equation, **Bueckner 1960**
2. Boundary Collocation Method $\left(h/b \geq 2\right)$, **Gross 1965a**
3. Weight Function Method, **Bueckner 1970, 1971**
4. Green's Function Method $\left(h/b \geq 1.5\right)$, **Emery 1969**
5. Asymptotic Approximation, **Benthem 1972**

Empirical Formulas
a. Accuracy
b. Method, reference

$$F(a/b) = 1.122 - 1.40(a/b) + 7.33(a/b)^2 - 13.08(a/b)^3 + 14.0(a/b)^4$$

a. 0.2% for $a/b \le 0.6$
b. Least squares fitting **(Brown 1966)**

$$F(a/b) = \sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}\;\frac{0.923 + 0.199\left(1 - \sin\frac{\pi a}{2b}\right)^4}{\cos\frac{\pi a}{2b}}$$

a. Better than 0.5% for any a/b
b. **Tada 1973**

B. Displacements

Crack Opening at Edge

$$\delta = \frac{4\sigma a}{E'} V(a/b)$$

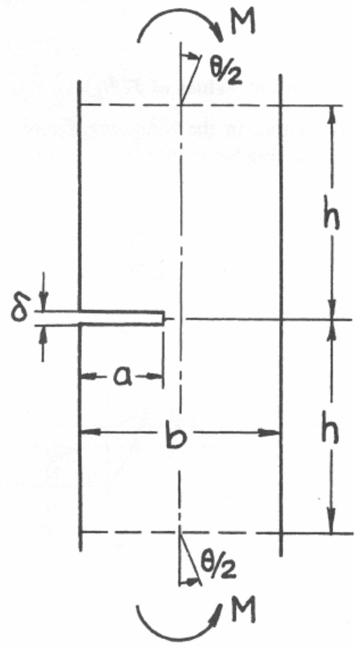
Gross' results (**Gross 1967**, Boundary Collocation Method) are expected to have 0.5% accuracy for $0.2 \le a/b \le 0.7$. An empirical formula with 1% accuracy for any a/b is (**Tada 1973**)

$$V(a/b) = 0.8 - 1.7(a/b) + 2.4(a/b)^2 + \frac{0.66}{(1 - a/b)^2}$$

Additional Remote Point ($h/b > 2$) Displacement (Rotation) Due to Crack

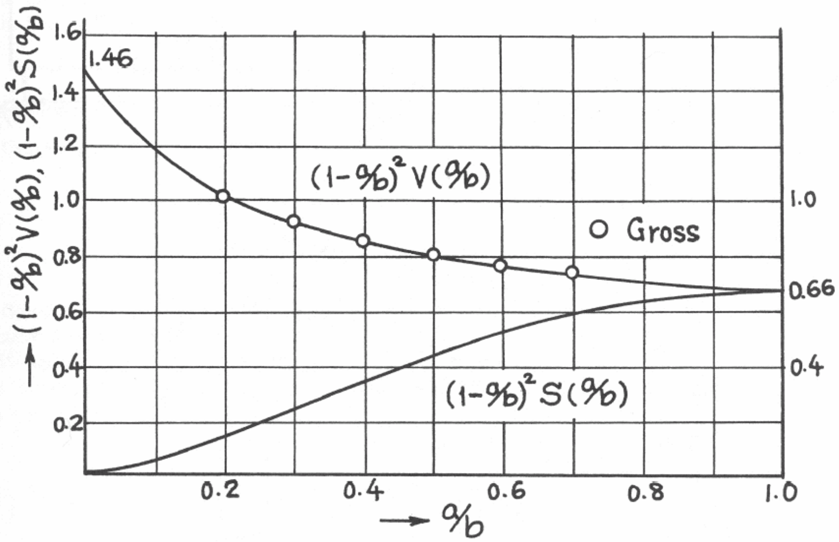
$$\theta_{crack} = \theta_{total} - \theta_{no\,crack}$$

$$\theta_{crack} = \frac{4\sigma}{E'} S(a/b)$$



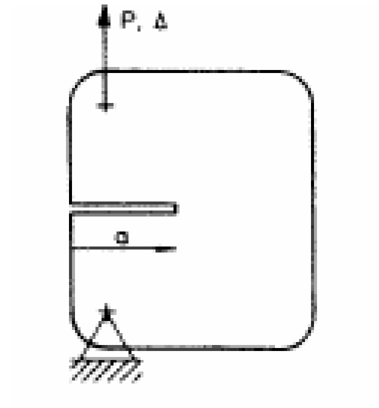
The following formula has better than 1% accuracy for any a/b.

$$S(a/b) = \left(\frac{a/b}{1 - a/b}\right)^2 \left\{5.93 - 19.69(a/b) + 37.14(a/b)^2 - 35.84(a/b)^3 + 13.12(a/b)^4\right\}$$



Method: Paris' Equation (**Paris 1957**) (See **Appendix B.**)
Reference: **Tada 1973**
(See also **pages 2.16, 2.27, 9.1** etc., for related solutions.)

Energy Release Rate, Prescribed Load vs. Prescribed Displacement, and Relation to Stress Intensity Factors



Generic cracked body

Prescribed load/thickness, P. Load-point displacement $= \Delta$

$G =$ energy release rate (J/m^2)

$SE =$ strain energy of the system/thickness

$P\Delta =$ potential energy/thickness of load

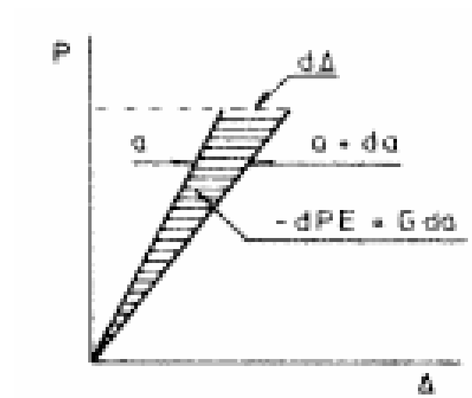
$PE =$ energy of the system/thickness

Note that for any linear elastic system, $SE = P\Delta/2,$ and thus for **prescribed load**:

$PE = SE - P\Delta = -P\Delta/2$

The energy release rate for **prescribed load** is defined as

$$G = -\left(\frac{\partial PE}{\partial a}\right)_P \quad \Rightarrow \quad G = \frac{1}{2}\frac{\partial(P\Delta)}{\partial a}\bigg|_P = \frac{P}{2}\left(\frac{\partial \Delta}{\partial a}\right)_P$$



Graphical interpretation of energy release rate due to crack advance

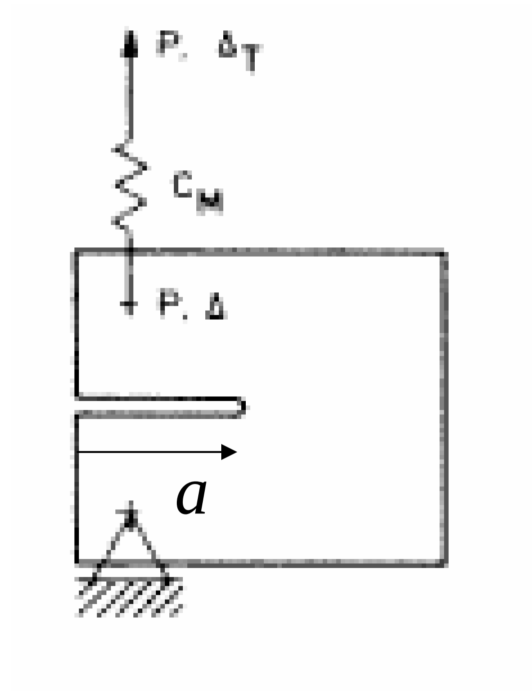
Define **compliance**, $\quad C = \dfrac{\Delta}{P}$

which depends only on geometry, including, a, E and ν. Thus,

$$\left(\frac{\partial \Delta}{\partial a}\right)_P = P\frac{dC}{da} \Rightarrow \boxed{G = \frac{1}{2}P^2\frac{dC}{da}}$$

Energy Release Rate, continued: Role of compliance and loading conditions

As indicated in the figure, introduce a linear spring with compliance C_M in series with the cracked body. Let Δ_T be the total displacement through which P works. Now, let Δ_T **be prescribed**.



$$\Delta_T = \Delta + C_M P = \Delta + (C_M / C)\Delta$$

$$PE = SE + \frac{1}{2}C_M P^2 = \frac{1}{2}C^{-1}\Delta^2 + \frac{1}{2}C_M{}^{-1}\left(\Delta_T - \Delta\right)^2$$

and a straight-forward calculation (see pg.8 of notes) again gives

$$G = \frac{1}{2}P^2\frac{dC}{da}$$

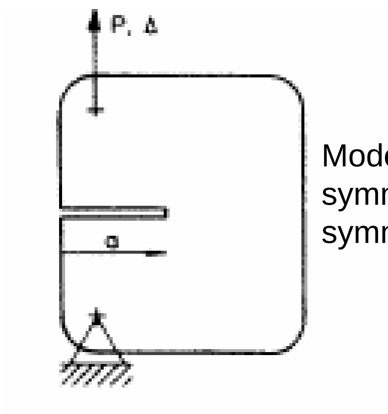
Note! The energy release rate does not depend on C_M.

In particular, G is the same for both prescribed load and prescribed displacement.

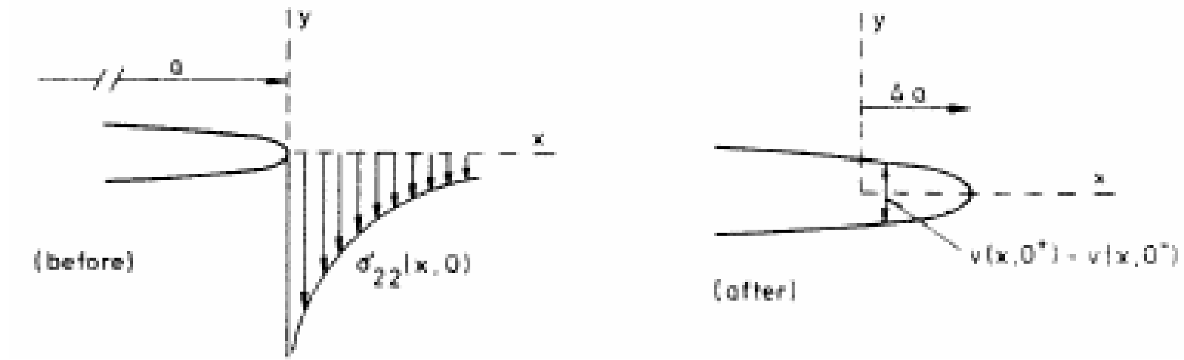
This is not intutive.

Most results for G are obtained either from analytical or numerical calculations (see later). However, the above formula permits experimental evaluation of G by experimentally measuring the compliance at two nearly equal crack lengths, a and a+da.

Energy Release Rate, continued: Relation between G and K's



Mode I situation: symmetric body and symmetric loading.



Since G is independent of loading, consider a body under prescribed Δ.

Consider two configurations of the cracked body one with a and the other with a+Δa.

Because the system is elastic, the energy released in advancing the crack Δa is the same as the work done in closing the crack from a+Δa to a (see figure). Because Δ is prescribed, no work is done by applied loads in closing crack. The work to close the crack is

$$\Delta W = G\Delta a = \frac{1}{2}\int_0^{\Delta a} \sigma_{22}(x,0)\left[u_2(x,0^+) - u_2(x,0^-)\right]dx$$

$$= \frac{2}{\pi \bar{E}} K_I(a) K_I(a+\Delta a) \int_0^{\Delta a} \sqrt{\frac{\Delta a - x}{x}}\,dx$$

$$= \frac{1}{\bar{E}} K_I(a) K_I(a+\Delta a)\Delta a$$

$$\sigma_{22}(x,0) = K_I(a)/\sqrt{2\pi x}$$

$$u_2(x,0^+) - u_2(x,0^-) = K_I(a+\Delta a)\frac{8}{\bar{E}}\sqrt{\frac{\Delta a - x}{2\pi}}$$

$\bar{E} = E$ in plane stress, $\bar{E} = E/(1-\nu^2)$ in plane strain

$$\Rightarrow \boxed{G = \frac{1}{\bar{E}} K_I^{\,2} \quad \text{(Irwin's universal relation!)}}$$

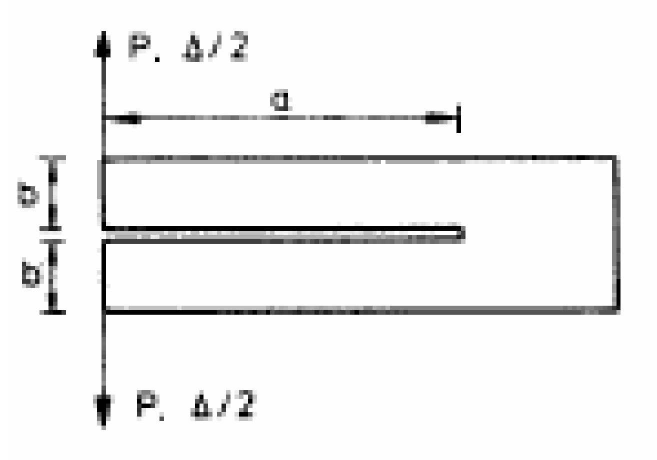
Under all three modes, one finds

$$G = \frac{1}{\bar{E}}\left(K_I^{\,2} + K_{II}^{\,2}\right) + \frac{1}{2G} K_{III}^{\,2}$$

Discuss units

Energy Methods for Determining Energy Release Rate

Double cantilever beam specimen



Compute compliance of specimen treating each arm as a cantilever beam, and consider the specimen to have unit thickness and P is force/thickness

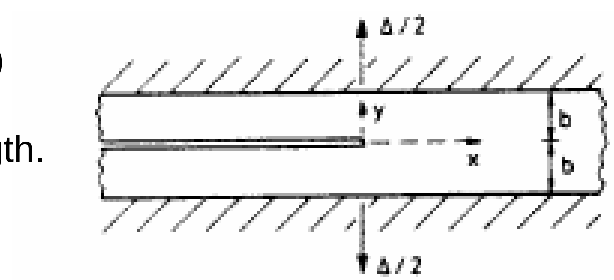
$$\frac{\Delta}{2} = \frac{Pa^3}{3\bar{E}I} = \frac{4Pa^3}{\bar{E}b^3} \quad \Rightarrow \quad C \equiv \frac{\Delta}{P} = \frac{8a^3}{\bar{E}b^3}$$

$$G = \frac{1}{2}P^2\frac{dC}{da} = \frac{12P^2a^2}{\bar{E}b^3}$$

Since this is a mode I problem (by symmetry), Irwin's relation gives

$$K_I = 2\sqrt{3}\frac{Pa}{b^{3/2}}$$

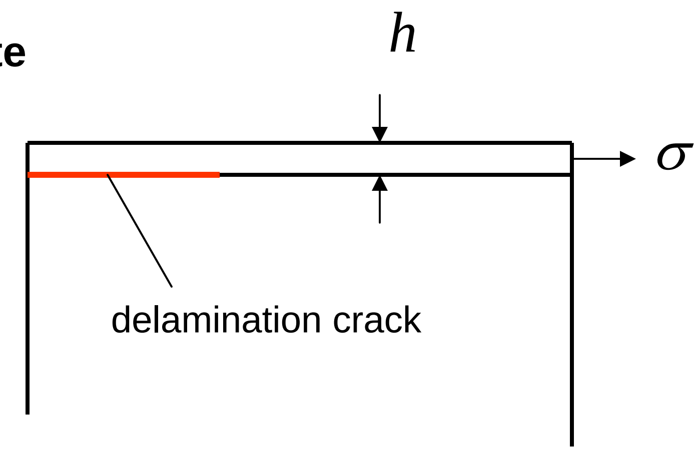
See notes pg. 10 for the problem of a thin strip (plane stress) with a semi-Infinite crack subject to rigid grips.
This is an exact solution. Note it is independent of crack length.



$$G = \frac{E\Delta^2}{4(1-\nu^2)b}, \quad K_I = \frac{1}{2\sqrt{1-\nu^2}}\frac{E\Delta}{\sqrt{b}}$$

Energy Methods for Determining Energy Release Rate, continued

Delamination of stressed thin film on elastic substrate



1D analysis (uniaxial stress in film) σ

strain energy/area in film far ahead of tip $= \dfrac{1}{2}\dfrac{\sigma^2 h}{E}$

strain energy/area in film far behind tip $= 0$

energy released/area due to crack advance:

$$G = \frac{1}{2}\frac{\sigma^2 h}{E}$$

which is independent of crack length.

This problem does not have symmetry and it is an example where the crack is a combination of mode I and II. Later in the course we will determine the two stress intensity factors.

The simple result for G is valid when the crack is long enough such that steady-state conditions apply. In practice this means the crack length has to be long compared to the film thickness.